University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 9 ('Final Exam')

Due 12:30pm, Wednesday, December 12, 2018

1. **Brain-warmer.** Explain why the Wilson-Fisher fixed point in $2 < d < 4$ has only two relevant operators, which may be associated with ϕ and ϕ^2. What happened to ϕ^3?

   (For example, this was important in our discussion of scaling, where we were able to relate the whole zoo of greek letters (critical exponents) to only *two* numbers y_t, y_h, the scaling exponents for the two relevant operators, one of which breaks the symmetry.)

2. **Rotation invariance as an emergent symmetry.**

   Give an RG analysis which explains why the critical behavior of lattice magnets (which do not have continuous spatial rotation symmetry) can be described by rotation-invariant field theories.

   To be more precise about what I am asking: consider a hypercubic lattice, and a magnet with an $\mathsf{O}(n)$ symmetry, so that there is an n-component order parameter. As in the problem on HW 8, analyze what perturbations of the rotation-symmetric action preserve lattice rotations but not continuous rotations, and decide what are their scaling dimensions at the Wilson-Fisher fixed point.

   [I am re-posting this important problem from last homework, since there was universal confusion about the fact that I was asking about *spatial* rotation symmetry, as opposed to rotations in spin space.

   Please note that a priori the spin rotation symmetry

   $$S(r)^a \to R^{ab} S^b(r), \;\; R \in \mathsf{O}(n)$$

   is completely independent of the spatial rotation symmetry

   $$S^a\left(r^i\right) \to S^a\left(\Lambda^{ij} r^j\right), \;\; \Lambda \in \mathsf{O}(d).$$

   Spin-orbit couplings break the product of these two groups down to a diagonal subgroup; such couplings are present in Lorentz-invariant field theories, and in lattice models involving large-Z atoms, but are often negligible.] No need to repeat if you answered this question last time.]

3. **OPE.** Consider the Gaussian fixed point with $\mathsf{O}(n)$ symmetry. Compute the OPE coefficients for the operators $\mathcal{O}_2 \equiv : \phi_a \phi_a :$, $\mathcal{O}_4 \equiv : (\phi_a\phi_a)^2 :$, and the identity operator (here $a = 1..n$ and the repeated index is summed). Use this information to compute the beta function, find the Wilson-Fisher fixed point and the correlation length critical exponent ν there.

4. **A proof of the Mermin-Wagner-Hohenberg-Coleman theorem.**

   (a) **Bogoliubov-Schwartz inequality.** Convince yourself that

$$\langle AA^\star\rangle\langle BB^\star\rangle \geq |\langle AB^\star\rangle|^2 \tag{1}$$

   where A, B are anything, and $\langle ...\rangle$ means some statistical average.

Now consider a microscopic realization of an XY model in $d = 2$ on a lattice with N total sites. At each site we place a 2-component unit-normalized spin $\vec{S}_i = (\cos\phi_i, \sin\phi_i)$. The Hamiltonian is

$$-H = \sum_{\langle ij\rangle} J\cos(\phi_i - \phi_j) + h\sum_i \cos\phi_i \ .$$

The magnetization is $m = \langle\cos\phi_i\rangle$. The claim is that for $d \leq 2$, $\lim_{h\to 0} m$ must vanish in the thermodynamic limit – no spontaneous breaking of the continuous symmetry $\phi_i \to \phi_i + \alpha$.

To apply (1), we will (with cold-blooded foresight) choose

$$A \equiv \frac{1}{N}\sum_j e^{-\mathbf{i}q\cdot r_j}\sin\phi_j, \quad B \equiv \frac{1}{N}\sum_k e^{-\mathbf{i}q\cdot r_k}\partial_{\phi_k} H.$$

   (b) Show that

$$\sum_q \langle AA^\star\rangle = \frac{1}{N}\sum_j\langle\sin^2\phi_j\rangle \leq 1.$$

   (c) Show that

$$\langle AB^\star\rangle = \frac{Tm}{N}.$$

   Here's a hint:

$$0 = \frac{1}{Z}\int_0^{2\pi}\prod_i d\phi_i \partial_{\phi_k}\left(e^{-H/T}\sin\phi_j\right).$$

   (More generally $0 = \int_0^{2\pi}\prod_i d\phi_i\partial_{\phi_k}(f(\phi))$ as long as $f(\phi)$ is a periodic function, $f(\phi) = f(\phi + 2\pi)$. This kind of relation is sometimes called a Ward identity.)

   (d) Show that

$$\langle BB^\star\rangle = \frac{T}{N^2}\sum_{ij} e^{-\mathbf{i}q(r_i - r_j)}\langle\partial_{\phi_i}\partial_{\phi_j} H\rangle.$$

   Hint: use the same trick as in the previous part.

(e) Show that (I have in mind a hypercubic lattice with coordination number $z = 2d$)

$$\langle BB^\star \rangle \le \frac{T}{N}\left(h + J\left(z - 2\sum_{\mu=1}^{d} \cos(q_\mu a)\right)\right) \le \frac{T}{N}\left(h + J\vec{q}^2\right)$$

(f) Conclude that

$$1 \ge \sum_q \langle AA^\star \rangle \ge \frac{Tm^2}{N^2}\sum_q \frac{1}{h + Jq^2}.$$

Take the thermodynamic limit, and argue that the resulting inequality requires

$$\lim_{h\to 0} m = 0 \text{ for } d \le 2.$$

(g) [optional bonus question] Generalize the argument to the $\mathsf{O}(n)$ model.

5. **Long-range interactions and the lower critical dimension.** Consider perturbing an $\mathsf{O}(n)$ model by long-range interaction of the form

$$\Delta H = g \int đ^d q \; \Phi_a(q)|q|^r \Phi_a(-q).$$

(a) [optional] What does ΔH look like in position space? What does

$$\int d^d x d^d y \sum_a^n \frac{(\Phi_a(x) - \Phi_a(y))^2}{|x-y|^{d+r}}$$

look like in momentum space?

(b) Find the lower critical dimension as a function of r.

6. **Perturbative RG for worldsheet (Edwards-Flory) description of SAWs.**

In this problem we make quantitative the analogy

unrestricted RW:SAW::gaussian fixed point:WF fixed point.

Parametrize a continuous-time random walk in d dimensions by a trajectory $\vec{r}(t)$. Consider the Edwards hamiltonian

$$H[\vec{r}] = \frac{K}{2}\int_0^L ds \left(\frac{d\vec{r}}{ds}\right)^2 + \frac{u}{2}\int_{|s_1-s_2|>a} ds_1 ds_2 \; \delta^d[\vec{r}(s_1) - \vec{r}(s_2)]$$

with a self-avoidance coupling $u > 0$, a short-distance cutoff a, and an IR cutoff L. We would like to understand the large-L scaling of the polymer size, R,

$$R^2(L) \equiv \langle |\vec{r}(L) - \vec{r}(0)|^2 \rangle \sim L^{2\nu}.$$

(a) Consider the probability density for two points a distance $|s_1 - s_2|$ along the chain to be separated in space by a displacement $\vec{x}$,

$$P(\vec{x}; s_1 - s_2) = \langle \delta^d[\vec{r}(s_1) - \vec{r}(s_2) - \vec{x}] \rangle.$$

Show that the polymer size R can be obtained from its fourier transform $\tilde{P}(\vec{q}; s)$ by the relation

$$R^2(L) = -\nabla_q^2 \tilde{P}(\vec{q}; L)|_{q=0}.$$

(So far this does not involve a choice of hamiltonian.)

(b) For the free case $u = 0$, compute the equilibrium polymer size $R_0(L)$ in terms of d, L, K. It may be helpful to derive a relation of the form

$$\langle e^{\mathbf{i} \int_0^L ds \vec{k}(s) \cdot \vec{r}(s)} \rangle_0 = e^{\frac{1}{2K} \int_0^L ds ds' \vec{k}(s) \cdot \vec{k}(s') G(s-s')}.$$

(c) Develop an expansion of $\tilde{P}(\vec{q}; L)$ to first order in u, using the cumulant expansion as in §6.6 of the lecture notes. You should find an expression of the form $R^2(L) = R_0^2(L)\left(1 + \delta R_1^2(L) + \mathcal{O}(u^2)\right)$ with

$$\delta R_1^2(L) = \frac{u}{L} \left(\frac{K}{2\pi}\right)^{d/2} \int_0^L ds_1 \int_{s_1+a}^L ds_2 \frac{A^2(s_1, s_2; L)}{|s_1 - s_2|^{\frac{d-2}{2}}}.$$

(d) Show that the integrals in the previous part diverge as $a/L \to 0$ below a certain dimension d_c. More precisely, by changing variables to $s = s_1 - s_2$ and $\bar{s} = (s_1 + s_2)/2$ (and ignoring stuff at the upper limit of integration, as appropriate for $L \gg a$) show that

$$\delta R_1^2(L) \simeq u \left(\frac{K}{2\pi}\right)^{d/2} \int_a^L ds s^{\frac{\epsilon}{2} - 1}$$

with $\epsilon = d_c - d$.

(e) How does $\vec{r}$ scale with $s \mapsto bs$ if we demand that the free hamiltonian $(u = 0)$ is a fixed point? What is ν at the free fixed point?

(f) Find d_c by power counting.

(g) We wish to integrate out the short distance fluctuations with wavelengths between a and ba, to find an effective Hamiltonian governing the remaining degrees of freedom:

$$\tilde{H}[\vec{r}] = \frac{\tilde{K}}{2} \int_0^L ds \left(\frac{d\vec{r}}{ds}\right)^2 + \frac{\tilde{u}}{2} \int_{|s_1 - s_2| > ba} ds_1 ds_2 \delta^d[\vec{r}(s_1) - \vec{r}(s_2)]$$

Using the first-order-in-u result for δR above, show that for small ϵ and small $\log b$, the coarse-grained 'stiffness' parameter is of the form

$$\tilde{K} = K(1 - \bar{v} \log b)$$

and find $\bar{v}$.

(h) A similar calculation yields the running of the interaction strength of the form $\tilde{u} = u(1 - 2\bar{v}\log b)$. Do the rescaling step of the RG procedure, redefining s by a factor of $b = 1 + \ell + \mathcal{O}(\ell^2)$ and rescaling the $\vec{r} \to Z(b)\vec{r}$ to put the Hamiltonian back in the original form with the original cutoff and renormalized parameters K', u'.

(i) Find the beta functions for $K(\ell)$ and $u(\ell)$. Find ν to first order in ϵ at the nontrivial fixed point.

7. **Self-avoiding membranes?**

[Optional, slightly open-ended.] Consider redoing the Edwards-Flory analysis for a theory of membranes. The fields are now $\vec{r}(\sigma_1, \sigma_2, \cdots, \sigma_D)$, vectors parametrizing the embedding of a D-dimensional object into $\mathbb{R}^d$. We might consider perturbing the Gaussian action

$$S_0[r] = \int d^D\sigma \sum_{\alpha=1}^{D} (\partial_{\sigma_\alpha}\vec{r})^2$$

by a self-avoidance term

$$S_u[r] = \int d^D\sigma \int d^D\sigma' \delta^d(\vec{r}(\sigma - \sigma')).$$

For various d and D, what does the Flory argument predict for the scaling exponent of the brane size with the linear size L of the base space? For which values is the excluded-volume term relevant?

Are there other terms we should consider in the action?

Try to resist googling before you think about this question.