University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 9 ('Final Exam') – Solutions

Due 12:30pm, Wednesday, December 12, 2018

1. **Brain-warmer.** Explain why the Wilson-Fisher fixed point in $2 < d < 4$ has only two relevant operators, which may be associated with ϕ and ϕ^2. What happened to ϕ^3?

   (For example, this was important in our discussion of scaling, where we were able to relate the whole zoo of greek letters (critical exponents) to only *two* numbers y_t, y_h, the scaling exponents for the two relevant operators, one of which breaks the symmetry.)

   Clearly the Ising model critical point has two relevant perturbations (one of which breaks the $\mathbb{Z}_2$ symmetry): the magnetic field and the temperature. How can this be described by the ϕ^4 field theory, which looks like it has three relevant operators (ϕ, ϕ^2, ϕ^3), two of which break the $\mathbb{Z}_2$ symmetry?

   ϕ is just an integration variable in $Z = \int [D\phi] e^{-S[\phi]}$. The ϕ^3 term can be removed by a field redefinition $\phi \to \phi + a$.

2. **Rotation invariance as an emergent symmetry.**

   Give an RG analysis which explains why the critical behavior of lattice magnets (which do not have continuous spatial rotation symmetry) can be described by rotation-invariant field theories.

   To be more precise about what I am asking: consider a hypercubic lattice, and a magnet with an $\mathsf{O}(n)$ symmetry, so that there is an n-component order parameter. As in the problem on HW 8, analyze what perturbations of the rotation-symmetric action preserve lattice rotations but not continuous rotations, and decide what are their scaling dimensions at the Wilson-Fisher fixed point.

   [I am re-posting this important problem from last homework, since there was universal confusion about the fact that I was asking about *spatial* rotation symmetry, as opposed to rotations in spin space.

   Please note that a priori the spin rotation symmetry

   $$S(r)^a \to R^{ab} S^b(r), \ R \in \mathsf{O}(n)$$

   is completely independent of the spatial rotation symmetry

   $$S^a\left(r^i\right) \to S^a\left(\Lambda^{ij} r^j\right), \ \Lambda \in \mathsf{O}(d).$$

Spin-orbit couplings break the product of these two groups down to a diagonal subgroup; such couplings are present in Lorentz-invariant field theories, and in lattice models involving large-Z atoms, but are often negligible.] No need to repeat if you answered this question last time.]

Looking at the dispersion relation for a (hyper)cubic lattice is instructive. The fourier transform of the adjacency matrix is

$$\tilde{A}(k) = \sum_{\mu=1}^{d} (1 - \cos k_\mu a) \stackrel{ka \ll 1}{=} k_\mu k^\mu + \mathcal{O}\left(k^4\right) \ .$$

The leading term in the small-wavenumber expansion is the familiar rotation-invariant Laplacian term. The next term, which breaks rotation invariance, is $\mathcal{O}(k^4)$. In general, the answer is that the terms which break rotation invariance (like $\partial_x^2\phi\partial_y^2\phi$ or $\phi\sum_i \partial_i^4\phi$) are irrelevant operators.

Now you could ask me: what about a term like $\partial_x\phi\partial_y\phi$? Or more generally suppose we perturb $(\nabla\phi)^2$ by $A_{ij}\partial_i\phi\partial_j\phi$. Here the answer is that I can diagonalize the matrix $\delta_{ij} + A_{ij}$ to put the kinetic term in the form $a_i\left(\partial_i\phi\right)^2$. Then, at the cost of a change of coordinates $x_i \to x_i/\sqrt{a_i}$ I can restore the familiar rotation-invariant form of the kinetic term.

A symmetry of a fixed point which can be broken only by irrelevant operators is called an emergent symmetry (by condensed matter physicists) or an accidental symmetry (by high energy physicists). This difference of tone is very revealing.

3. **OPE.** Consider the Gaussian fixed point with $\mathsf{O}(n)$ symmetry. Compute the OPE coefficients for the operators $\mathcal{O}_2 \equiv: \phi_a\phi_a :$, $\mathcal{O}_4 \equiv: (\phi_a\phi_a)^2 :$, and the identity operator (here $a = 1..n$ and the repeated index is summed). Use this information to compute the beta function, find the Wilson-Fisher fixed point and the correlation length critical exponent ν there.

   Define normal-ordered operators as before – leave out the self-contractions. Eliding the spatial dependence for brevity, as we did for the Ising model ($n = 1$), we find the following algebra for the symmetric and relevant operators at the Gaussian fixed point:

$$\mathcal{O}_2\mathcal{O}_2 \sim 2n\mathbb{1} + 4\mathcal{O}_2 + \mathcal{O}_4 + \cdots \quad (1)$$
$$\mathcal{O}_2\mathcal{O}_4 \sim 0\mathbb{1} + (4n+8)\mathcal{O}_2 + 8\mathcal{O}_4 + \cdots \quad (2)$$
$$\mathcal{O}_4\mathcal{O}_4 \sim (8n^2+16)\mathbb{1} + (32n+64)\mathcal{O}_2 + (8n+64)\mathcal{O}_4 + \cdots \quad (3)$$
$$(4)$$

   Notice that setting $n = 1$ this reduces to the expression on page 121 of the lecture notes. (Anticipating the scaling $u_\star \sim \epsilon, r_\star \sim \epsilon^2$ we actually only really need C_{422} and C_{442}.)

Using the result of conformal perturbation theory,

$$\beta_r = 2r - 4r^2 - 2(4n+8)ru - (32n+64)u^2 + ..., \quad \beta_u = \epsilon u - r^2 - 16ru - (8n+64)u^2 + \cdots$$

and the WF fixed point occurs at

$$u_\star = \frac{\epsilon}{8(n+8)}, \quad r_\star = \frac{n+2}{4(n+8)^2}\epsilon^2$$

(notice that this parameter r differs from the one around eqn 6.32 of the notes by the shift we made when we defined normal-ordered operators). Linearizing about the fixed point gives

$$\frac{dr}{d\ell} = 2r - 2(4n+8)ru_\star + \mathcal{O}(\epsilon^2) = (2 - 2\frac{4n+8}{8(n+8)}\epsilon)r + \cdots$$

which is solved by

$$r \sim \left(e^\ell\right)^{1/\nu}$$

with

$$\nu = \frac{1}{2} - \frac{1}{4}\frac{n+2}{n+8}\epsilon.$$

4. **A proof of the Mermin-Wagner-Hohenberg-Coleman theorem.**

   (a) **Bogoliubov-Schwartz inequality.** Convince yourself that

$$\langle AA^\star\rangle\langle BB^\star\rangle \geq |\langle AB^\star\rangle|^2 \tag{5}$$

   where A, B are anything, and $\langle ...\rangle$ means some statistical average.

   For any x, $0 \leq \langle (A+xB)(A+xB)^\star\rangle$. Expand out the RHS and choose $x = -\frac{\langle AB^\star\rangle}{BB^\star}$.

Now consider a microscopic realization of an XY model in $d = 2$ on a lattice with N total sites. At each site we place a 2-component unit-normalized spin $\vec{S}_i = (\cos\phi_i, \sin\phi_i)$. The Hamiltonian is

$$-H = \sum_{\langle ij\rangle} J\cos(\phi_i - \phi_j) + h\sum_i \cos\phi_i \ .$$

The magnetization is $m = \langle\cos\phi_i\rangle$. The claim is that for $d \leq 2$, $\lim_{h\to 0} m$ must vanish in the thermodynamic limit – no spontaneous breaking of the continuous symmetry $\phi_i \to \phi_i + \alpha$.

To apply (5), we will (with cold-blooded foresight) choose

$$A \equiv \frac{1}{N}\sum_j e^{-\mathbf{i}q\cdot r_j}\sin\phi_j, \quad B \equiv \frac{1}{N}\sum_k e^{-\mathbf{i}q\cdot r_k}\partial_{\phi_k} H.$$

(b) Show that

$$\sum_q \langle AA^\star \rangle = \frac{1}{N}\sum_j \langle \sin^2\phi_j \rangle \le 1.$$

(c) Show that

$$\langle AB^\star \rangle = \frac{Tm}{N}.$$

Here's a hint:

$$0 = \frac{1}{Z}\int_0^{2\pi} \prod_i d\phi_i \partial_{\phi_k}\left(e^{-H/T}\sin\phi_j\right).$$

(More generally $0 = \int_0^{2\pi}\prod_i d\phi_i \partial_{\phi_k}\left(f(\phi)\right)$ as long as $f(\phi)$ is a periodic function, $f(\phi) = f(\phi + 2\pi)$. This kind of relation is sometimes called a Ward identity.)

(d) Show that

$$\langle BB^\star \rangle = \frac{T}{N^2}\sum_{ij} e^{-\mathbf{i}q(r_i - r_j)}\langle \partial_{\phi_i}\partial_{\phi_j} H\rangle.$$

Hint: use the same trick as in the previous part.

The required Ward identity is

$$0 = \frac{1}{Z}\int_0^{2\pi} \prod_i d\phi_i \partial_{\phi_k}\left(e^{-H/T}\partial_{\phi_j} H\right).$$

(e) Show that (I have in mind a hypercubic lattice with coordination number $z = 2d$)

$$\langle BB^\star \rangle \le \frac{T}{N}\left(h + J\left(z - 2\sum_{\mu=1}^{d}\cos(q_\mu a)\right)\right) \le \frac{T}{N}\left(h + J\vec{q}^{\,2}\right)$$

(f) Conclude that

$$1 \ge \sum_q \langle AA^\star \rangle \ge \frac{Tm^2}{N^2}\sum_q \frac{1}{h + Jq^2}.$$

Take the thermodynamic limit, and argue that the resulting inequality requires

$$\lim_{h\to 0} m = 0 \text{ for } d \le 2.$$

Notice that the crucial icepick to the forehead here is the same infrared logarithmic divergence of $\oint_{BZ}\frac{\mathrm{d}^2 q}{m+q^2}$ that arose in the discussion of fluctuation corrections to the magnetization.

(g) [optional bonus question] Generalize the argument to the $\mathsf{O}(n)$ model.

Parametrize the n-component spin as

$$\vec{S}_i = (\cos\phi_i, \sin\phi_i \hat{n})$$

where $\hat{n}$ is an $(n-1)$-component unit vector. The magnetization is $m = \langle \cos\phi_i \rangle$. From here everything goes through until part (e) where we need to take derivatives of

$$H = -J \sum_{\langle kl \rangle} \vec{S}_k \cdot \vec{S}_l = -J \sum_{\langle kl \rangle} (\cos\phi_k \cos\phi_l + \sin\phi_k \sin\phi_l \hat{n}_k \cdot \hat{n}_l).$$

The derivatives are

$$\partial_{\phi_i} \partial_{\phi_j} H = -2J \left(\delta_{\langle ij \rangle} \tilde{S}_i \cdot \tilde{S}_j - \sum_{\langle i|\ell \rangle} \delta_{ij} \vec{S}_i \cdot \vec{S}_l \right) + h \cos\phi_i \delta_{ij}.$$

Here $\vec{S}_k \cdot \vec{S}_l = \cos\phi_k \cos\phi_l + \sin\phi_k \sin\phi_l \hat{n}_k \cdot \hat{n}_l$ as in H above and

$$\tilde{S}_k \cdot \tilde{S}_l \equiv (\sin\phi_k \sin\phi_l + \cos\phi_k \cos\phi_l \hat{n}_k \cdot \hat{n}_l).$$

This is the inner product of the spins after the replacement $\phi_i \to \pi/2 - \phi_i$. Then

$$\sum_{ij} e^{-\mathbf{i}\vec{q}\cdot\vec{r}_{ij}} \partial_{\phi_i} \partial_{\phi_j} H = h \sum_i \cos\phi_i + \sum_i \sum_{\langle i|j \rangle} \left(\tilde{S}_i \cdot \tilde{S}_j - e^{-\mathbf{i}\vec{q}\cdot\vec{r}_{ij}} \vec{S}_i \cdot \vec{S}_j \right).$$

Unlike the $n = 2$ case, the two terms are not of exactly the same form.

5. **Long-range interactions and the lower critical dimension.** Consider perturbing an $\mathsf{O}(n)$ model by long-range interaction of the form

$$\Delta H = g \int đ^d q \; \Phi_a(q) |q|^r \Phi_a(-q).$$

(a) [optional] What does ΔH look like in position space? What does

$$\int d^d x d^d y \sum_a^n \frac{(\Phi_a(x) - \Phi_a(y))^2}{|x-y|^{d+r}}$$

look like in momentum space?

(b) Find the lower critical dimension as a function of r.

As in the lecture notes, we can consider the fluctuation corrections to a presumed expectation value in the $a = 1$ direction:

$$\langle S_1 \rangle = 1 - \frac{1}{2} \langle \sigma^2 \rangle + \cdots = 1 - (n-1) \frac{1}{2} \oint \frac{đ^d k}{K k^2 + g k^r} + \cdots$$

If $r < 2$, then $k^r \gg k^2$ near the IR limit $k \to 0$ of the integral. Therefore

$$\oint \frac{đ^d k}{K k^2 + g k^r} \overset{r<2}{\simeq} \int_{1/L} \frac{đ^d k}{g k^r} \sim \left(\frac{1}{L} \right)^{d-r}.$$

When $r > 2$ then the g term doesn't affect the IR region of the integral. So $d_c = \min(r, 2)$. (If $r < 0$ the integral is actually finite for all d, but $r < 0$ means the interaction strength. actually *grows* with distance.)

6. **Perturbative RG for worldsheet (Edwards-Flory) description of SAWs.**

In this problem we make quantitative the analogy

unrestricted RW:SAW::gaussian fixed point:WF fixed point.

Parametrize a continuous-time random walk in d dimensions by a trajectory $\vec{r}(t)$. Consider the Edwards hamiltonian

$$H[\vec{r}] = \frac{K}{2}\int_0^L ds\,\left(\frac{d\vec{r}}{ds}\right)^2 + \frac{u}{2}\int_{|s_1-s_2|>a} ds_1 ds_2\;\delta^d[\vec{r}(s_1)-\vec{r}(s_2)]$$

with a self-avoidance coupling $u > 0$, a short-distance cutoff a, and an IR cutoff L. We would like to understand the large-L scaling of the polymer size, R,

$$R^2(L) \equiv \langle|\vec{r}(L)-\vec{r}(0)|^2\rangle \sim L^{2\nu}.$$

(a) Consider the probability density for two points a distance $|s_1-s_2|$ along the chain to be separated in space by a displacement $\vec{x}$,

$$P(\vec{x}; s_1-s_2) = \langle\delta^d[\vec{r}(s_1)-\vec{r}(s_2)-\vec{x}]\rangle.$$

Show that the polymer size R can be obtained from its fourier transform $\tilde{P}(\vec{q}; s)$ by the relation

$$R^2(L) = -\nabla_q^2 \tilde{P}(\vec{q}; L)|_{q=0}. \tag{6}$$

(So far this does not involve a choice of hamiltonian.)

$$-\vec{\nabla}_q^2\tilde{P}(\vec{q};L)|_{q=0} = -\vec{\nabla}_q^2\int d^dx e^{-\mathbf{i}\vec{q}\cdot\vec{x}}P(\vec{x};L)|_{q=0} = \int d^dx\vec{x}^2 P(\vec{x};L) = R^2(L).$$

(b) For the free case $u = 0$, compute the equilibrium polymer size $R_0(L)$ in terms of d, L, K. It may be helpful to derive a relation of the form

$$\langle e^{\mathbf{i}\int_0^L ds\vec{k}(s)\cdot\vec{r}(s)}\rangle_0 = e^{-\frac{1}{2K}\int_0^L dsds'\vec{k}(s)\cdot\vec{k}(s')G(s-s')}. \tag{7}$$

$$\begin{aligned}
\tilde{P}(\vec{q};L) &= \int d^dx e^{\mathbf{i}\vec{q}\cdot\vec{x}}\langle\delta^d(r(L)-r(0)-x)\rangle && (8)\\
&= \langle e^{\mathbf{i}\vec{q}\cdot(\vec{r}(L)-\vec{r}(0))}\rangle && (9)\\
&= \langle e^{\mathbf{i}\vec{q}\cdot\int_0^L ds\frac{d\vec{r}}{ds}}\rangle && (10)\\
&= \frac{1}{Z}\int[D\vec{r}]e^{-\int_0^L ds\left(\frac{K}{2}\left(\frac{d\vec{r}}{ds}\right)^2-\mathbf{i}\vec{q}\cdot\frac{d\vec{r}}{ds}\right)} && (11)\\
&= e^{-\frac{L}{2K}\vec{q}^2}. && (12)
\end{aligned}$$

In the last step we completed the square. Therefore (using (6)

$$R(L)^2 = \frac{Ld}{K}.$$

(c) Develop an expansion of $\tilde{P}(\vec{q}; L)$ to first order in u, using the cumulant expansion as in §6.6 of the lecture notes. You should find an expression of the form $R^2(L) = R_0^2(L)\left(1 + \delta R_1^2(L) + \mathcal{O}(u^2)\right)$ with

$$\delta R_1^2(L) = \frac{u}{L}\left(\frac{K}{2\pi}\right)^{d/2} \int_0^L ds_1 \int_{s_1+a}^L ds_2 \frac{A^2(s_1, s_2; L)}{|s_1 - s_2|^{\frac{d-2}{2}}}.$$

Letting $\mathcal{U} \equiv -\frac{u}{2}\int ds_1 \int ds_2 \delta^d(r(s_1) - r(s_2))$, and denoting $\langle ...\rangle_0$ averages with $u = 0$, we have

$$P(\vec{R}; L) = \langle \delta^d(r_L - r_0 - R)\rangle \tag{13}$$

$$= \langle \delta^d(r_L - r_0 - R)e^{\mathcal{U}}\rangle_0 = \langle \delta^d(r_L - r_0 - R)\rangle_0 + \langle \delta^d\left(\vec{r}(L) - \vec{r}(0) - \vec{R}\right)\mathcal{U}\rangle_0 + \mathcal{O}(u^2) \tag{14}$$

$$\equiv P_0(\vec{R}; L) - \frac{u}{2}\int ds_1 \int ds_2 \langle \delta^d\left(\vec{r}(L) - \vec{r}(0) - \vec{R}\right)\delta^d(\vec{r}(s_1) - \vec{r}(s_2))\rangle_0 + \mathcal{O}(u^2) \tag{15}$$

$$= P_0(\vec{R}; L)\left(1 - \frac{u}{2}\int đ^d q \int đ^d p \int ds_1 \int ds_2 \langle e^{\mathbf{i}\vec{p}\cdot(\vec{r}(L)-\vec{r}(0)-\vec{R})} e^{\mathbf{i}\vec{q}\cdot(\vec{r}(s_1)-\vec{r}(s_2))}\rangle_0 + \mathcal{O}(u^2)\right) \tag{16}$$

$$\stackrel{(7)}{=} P_0(R; L) - \frac{u}{2}\int đ^d q \int ds_1 \int ds_2 e^{-\frac{1}{2K}\int ds \int ds' \vec{k}_s\cdot\vec{k}_{s'} G(s-s')} e^{\mathbf{i}\vec{q}\cdot\vec{R}} \tag{17}$$

where we defined

$$\vec{k}_s \equiv \vec{p}\left(\delta(s - s_1) - \delta(s - s_2)\right) + \vec{q}\left(\delta(s - L) - \delta(s)\right).$$

Here $G(s) = -\frac{1}{\partial_s^2}{}_{s,0} = -\frac{1}{2}|s|$ is the massless propagator in one dimension

$$-\partial_s^2 G(s) = \delta(s).$$

The exponent is

$$-\int ds \int ds' \vec{k}_s \cdot \vec{k}_{s'} G(s - s') = q^2 L + p^2|s_1 - s_2| + \vec{p}\cdot\vec{q}\aleph$$

with $\aleph \equiv |s_1 - L| - |s_2 - L| + |s_2| - |s_1| = 2(s_2 - s_1)$. Doing the Gaussian integrals over $\vec{p}$ gives

$$P(\vec{R}; L) = P_0(\vec{R}; L) - \int đ^d q e^{-\mathbf{i}\vec{q}\cdot\vec{R}} \frac{u}{2}\int ds_1 \int ds_2 \left(\frac{2\pi K}{|s_1 - s_2|}\right)^{d/2} e^{-q^2\left(\frac{L}{4K} - \frac{|s_1-s_2|}{2K}\right)} + \mathcal{O}(u^2).$$

This expression is explicitly of the form $\delta P(\vec{R};L) = \int đ^d q e^{-\mathbf{i}\vec{q}\cdot\vec{R}}\delta\tilde{P}(\vec{q}lL)$ (I should have just started from (9)) and we can read off

$$\tilde{P}(\vec{q};L) = \tilde{P}_0(\vec{q};L) + \delta\tilde{P}(\vec{q};L) \tag{18}$$

$$= e^{-\frac{Lq^2}{2K}} - \frac{u}{2}\int ds_1 \int ds_2 \left(\frac{2\pi K}{|s_1-s_2|}\right)^{d/2} e^{-q^2\left(\frac{L}{4K}-\frac{|s_1-s_2|}{2K}\right)} + \mathcal{O}(u^2)\ . \tag{19}$$

Therefore

$$\frac{\delta R_1^2}{R_0^2} = -\nabla_q^2|_{q=0}\delta\tilde{P}(\vec{q};L) \stackrel{?}{=} +\frac{u}{2}\int ds_1\int ds_2 \left(\left(\frac{2\pi K}{|s_1-s_2|}\right)^{d/2-1} 2\pi - \left(\frac{2\pi K}{|s_1-s_2|}\right)^{d/2} L\right). \tag{20}$$

What gives with the second (more singular!) term? To be more precise, (9) says (in the interacting case)

$$\tilde{P}(\vec{q};L) = \langle e^{\mathbf{i}\vec{q}\cdot(\vec{r}(L)-\vec{r}(0))}\rangle = \frac{\langle e^{\mathbf{i}\vec{q}\cdot(\vec{r}(L)-\vec{r}(0))}e^{\mathcal{U}}\rangle_0}{\langle e^{\mathcal{U}}\rangle_0}$$

whose expansion in powers of u is

$$\tilde{P}(\vec{q};L) = \frac{\langle e^{\mathbf{i}\vec{q}\cdot(\vec{r}(L)-\vec{r}(0))}\rangle_0 - \langle e^{\mathbf{i}\vec{q}\cdot(\vec{r}(L)-\vec{r}(0))}\mathcal{U}+\cdots\rangle_0}{1-\langle\mathcal{U}\rangle_0+\cdots} \tag{21}$$

$$= \langle e^{\mathbf{i}\vec{q}\cdot(\vec{r}(L)-\vec{r}(0))}\rangle_0 - \langle e^{\mathbf{i}\vec{q}\cdot(\vec{r}(L)-\vec{r}(0))}\mathcal{U}\rangle_0 + \underbrace{\langle e^{\mathbf{i}\vec{q}\cdot(\vec{r}(L)-\vec{r}(0))}\rangle_0\langle\mathcal{U}\rangle_0}_{\text{disconnected}} + \mathcal{O}(u^2). \tag{22}$$

The term labelled 'disconnected' precisely cancels the second term in (20). Therefore

$$\delta R_1^2 = +\frac{Kd}{L}\frac{u}{2}\int ds_1\int ds_2\left(\frac{2\pi K}{|s_1-s_2|}\right)^{d/2-1} 2\pi$$

which has the advertised form.

(d) Show that the integrals in the previous part diverge as $a/L \to 0$ below a certain dimension d_c. More precisely, by changing variables to $s = s_1 - s_2$ and $\bar{s} = (s_1+s_2)/2$ (and ignoring stuff at the upper limit of integration, as appropriate for $L \gg a$) show that

$$\delta R_1^2(L) \simeq u\left(\frac{K}{2\pi}\right)^{d/2}\int_a^L ds s^{\frac{\epsilon}{2}-1}$$

with $\epsilon = d_c - d$.

The region of the integration where $s = s_1 - s_2 \ll L$ looks like

$$\int_0^L d\bar{s}\int_a ds s^{1-d/2}$$

which diverges for $d \le 4$ (and is finite for $d > 4$). Hence $d = d_c = 4$ is the upper critical dimension.

(e) How does $\vec{r}$ scale with $s \mapsto bs$ if we demand that the free hamiltonian ($u = 0$) is a fixed point? What is ν at the free fixed point?

Defining $\vec{r} \mapsto b^\nu \vec{r}$, we require

$$\frac{K}{2} \int ds \, (\partial_s \vec{r})^2 = \frac{K}{2} b^{2\nu - 1} \int ds \, (\partial_s \vec{r})^2$$

which says $\nu = \frac{1}{2}$ (as for the Gaussian fixed point of the $\mathsf{O}(n)$ model).

(f) Find d_c by power counting.

Let's absorb K into $\vec{r}$ (or treat it as dimensionless), and therefore $[r] = \frac{1}{2}$, and $0 = [u] + 2 - d[r]$ which says $[u] = \frac{d}{2} - 2$, and hence u is dimensionless at $d = d_c = 4$.

(g) We wish to integrate out the short distance fluctuations with wavelengths between a and ba, to find an effective Hamiltonian governing the remaining degrees of freedom:

$$\tilde{H}[\vec{r}] = \frac{\tilde{K}}{2} \int_0^L ds \, \left(\frac{d\vec{r}}{ds} \right)^2 + \frac{\tilde{u}}{2} \int_{|s_1 - s_2| > ba} ds_1 ds_2 \delta^d[\vec{r}(s_1) - \vec{r}(s_2)]$$

Using the first-order-in-u result for δR above, show that for small ϵ and small $\log b$, the coarse-grained 'stiffness' parameter is of the form

$$\tilde{K} = K(1 - \bar{v} \log b)$$

and find $\bar{v}$.

Let's take a high-energy point of view: R is a physical observable, we let the couplings run with scale to preserve the physics. So we must have (the Callan-Symanzik equation)

$$R^2(L) = R_0^2(1 + \delta R_1^2) = \tilde{R}_0^2(1 + \delta \tilde{R}_1^2)$$

where the RHS is what we would get if we computed with parameters $\tilde{u}, \tilde{K}$ and cutoff ba. Here

$$\delta \tilde{R}_1^2 = \tilde{u} \left(\frac{\tilde{K}}{2\pi} \right)^{d/2} \int_{ba}^L s^{\epsilon/2 - 1}.$$

Using $\tilde{R}_0^2 = \frac{Ld}{\tilde{K}} = \frac{Ld}{(K + \delta K)} = R_0^2 \left(1 - \frac{\delta K}{K} + \cdots \right)$, and anticipating that $\tilde{u} = u + \mathcal{O}(u^2)$ we have

$$R_0^2 \left(1 + u \left(\frac{K}{2\pi} \right)^{d/2} \int_a^L ds s^{\epsilon/2 - 1} \right) \stackrel{!}{=} R_0^2 \left(1 - \frac{\delta K}{K} + u \left(\frac{K}{2\pi} \right)^{d/2} \int_{ba}^L ds s^{\epsilon/2 - 1} \right) \tag{23}$$

$$= R_0^2 \left(1 - \frac{\delta K}{K} - u \left(\frac{K}{2\pi} \right)^{d/2} \int_a^{ba} ds s^{\epsilon/2 - 1} + \left(\frac{K}{2\pi} \right)^{d/2} \int_a^L ds s^{\epsilon/2 - 1} \right) \tag{24}$$

from which we conclude

$$\tilde{K} = K - Ku\left(\frac{K}{2\pi}\right)^{d/2}\int_a^{ba} ds s^{\epsilon/2-1} + \mathcal{O}(u^2) = K\left(1 - u\left(\frac{K}{2\pi}\right)^{d/2}(\log b + \mathcal{O}(\epsilon))\right) + \mathcal{O}(u^2).$$

That is, $\bar{v} = u\left(\frac{K}{2\pi}\right)^{d/2}$.

(h) A similar calculation yields the running of the interaction strength of the form $\tilde{u} = u(1 - 2\bar{v}\log b)$. Do the rescaling step of the RG procedure, redefining s by a factor of $b = 1 + \ell + \mathcal{O}(\ell^2)$ and rescaling the $\vec{r} \to Z(b)\vec{r}$ to put the Hamiltonian back in the original form with the original cutoff and renormalized parameters K', u'.

Including the rescaling step (and defining the transformation of $\vec{r}$ to be $\vec{r}' = b^{\nu}\vec{r}$), the coupling u runs like:

$$u' = b^{2-\nu d}\tilde{u} = (1 + (2-\nu d)\log b + \cdots)u(1 - 2\bar{v}\log b) = u\left(1 + (2 - \nu d - 2\bar{v})\log b\right).$$

Similarly,

$$K' = b^{2\nu-1}\tilde{K} = (1 - (2\nu - 1)\log b)K(1 - \bar{v}\log b) = K(1 - (2\nu - 1 - \bar{v})\log b).$$

(i) Find the beta functions for $K(\ell)$ and $u(\ell)$. Find ν to first order in ϵ at the nontrivial fixed point.

The beta functions are then

$$\partial_\ell u = (2\nu - 1 - \bar{v})u, \;\; \partial_\ell K = -u\bar{v}.$$

The scaling of K can be absorbed into a redefinition of the field $\vec{r}$ (just like the coefficient of $(\vec{\nabla}\phi)^2$ in the $\mathsf{O}(n)$ model). Therefore we demand K is independent of ℓ and hence

$$\nu = \frac{1+\bar{v}}{2}. \tag{25}$$

Using this in the β function equation for u gives

$$\partial_\ell u = 2 - \frac{d}{2} - \left(\frac{d}{2} + 2\right)\bar{v} = \frac{\epsilon}{2}u - \left(\frac{K}{2\pi}\right)^2 u^2 + \ldots$$

There is a fixed point at $u_\star = \left(\frac{K}{2\pi}\right)^{-d/2}\frac{\epsilon}{2}$. At the fixed point, K scales as

$$\tilde{K} = K\left(1 - u_\star\left(\frac{K}{2\pi}\right)^2\log b\right) = K\left(1 - \frac{\epsilon}{8}\log b\right) \simeq b^{-\epsilon/8}K.$$

If we set $K = 1$ (*i.e.* absorb K into $\vec{r}$), the exponent ν, defined as $R \sim L^\nu$ is determined by the scaling of $\vec{r}$ needed to preserve the physics:

$$\vec{r}' = b^\nu \vec{r}$$

which we've found in (25) to be $\nu = \frac{1+\bar{v}}{2}$; at the fixed point, $\bar{v} = u_\star \left(\frac{K}{2\pi}\right)^{d/2} = \epsilon/8$. (Alternatively,

$$\vec{r}' = b^\nu \vec{r} = b^{-1/2}\sqrt{\frac{\tilde{K}}{K}}\vec{r} \simeq b^{-\frac{1}{2}}\sqrt{\frac{b^{-\epsilon/8}K}{K}}\vec{r} = b^{-\left(\frac{1}{2}+\frac{\epsilon}{16}\right)}.\ )$$

Therefore

$$\nu = \frac{1}{2} + \frac{\epsilon}{16},$$

which agrees with the $\mathsf{O}(n)$ model exponent for $n \to 0$.

7. **Self-avoiding membranes?**

[Optional, slightly open-ended.] Consider redoing the Edwards-Flory analysis for a theory of membranes. The fields are now $\vec{r}(\sigma_1, \sigma_2, \cdots, \sigma_D)$, vectors parametrizing the embedding of a D-dimensional object into $\mathbb{R}^d$. We might consider perturbing the Gaussian action

$$S_0[r] = \int d^D\sigma \sum_{\alpha=1}^{D} \left(\partial_{\sigma_\alpha}\vec{r}\right)^2$$

by a self-avoidance term

$$S_u[r] = \int d^D\sigma \int d^D\sigma' \delta^d(\vec{r}(\sigma - \sigma')).$$

For various d and D, what does the Flory argument predict for the scaling exponent of the brane size with the linear size L of the base space? For which values is the excluded-volume term relevant?

Dimensional analysis gives $d_u = \frac{4D}{D-2}$ for the upper critical dimension.

Are there other terms we should consider in the action?

Try to resist googling before you think about this question.