University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 8 – Solutions

Due 12:30pm, Monday, December 3, 2018

1. **Brain-warmer.**

   Get Mathematica to analyze the integral

   $$I_d(r, b) \equiv \int_{\Lambda/b}^{\Lambda} \frac{đ^d q}{(q^2 + r)^2}$$

   as $d \to 4, r \to 0, b \to \infty$, in various orders. If you prefer, instead of Mathematica, you can use your brain, or some other algebra software or wetware. Make sure you agree with the statements made in lecture about it.

2. **RG analysis of less symmetric spin systems.**

   Suppose that we break the rotation symmetry of the $O(n)$ model to the subgroup of $\pi/2$ rotations, *i.e.* the cubic symmetry, (for example, for $n = 2$, $(s_1, s_2) \to (s_2, -s_1)$.) If the spins live on the cubic lattice, a spin-orbit coupling could do this.

   (a) Don't look at the next part of the problem yet! What functions of an n-vector ϕ_a that are invariant under $\pi/2$ rotations, but not general rotations?

   $\vdots$

   $\vdots$

   $\vdots$

   $\vdots$

   $\vdots$

   $\vdots$

(b) Show that in addition to the usual $O(n)$-symmetric interaction

$$\int d^d x \sum_{a,b=1}^{n} u\phi_a^2 \phi_b^2,$$

the LG free energy should include a term of the form

$$\int d^d x \sum_{a=1}^{n} v\phi_a^4.$$

Argue that this is the only new term (preserving cubic symmetry but not the full $O(n)$ symmetry) which can be a relevant perturbation of the Gaussian fixed point near $d = 4$.

(c) Treating $\mathcal{O}(u) = \mathcal{O}(v) = \mathcal{O}(\epsilon)$, redo the analysis of the running couplings in $d = 4 - \epsilon$ dimensions to derive beta functions for u and v up to corrections of order $\mathcal{O}(u^3) = \mathcal{O}(v^2 u) = ... = \mathcal{O}(\epsilon^3)$.

(d) Your answer to the previous part will be of the form

$$
\begin{aligned}
-b\partial_b u &= -\epsilon u + A_1 u^2 + A_2 uv + A_3 v^2 + \mathcal{O}(u^3) \\
-b\partial_b v &= -\epsilon u + B_1 u^2 + B_2 uv + B_3 v^2 + \mathcal{O}(u^3).
\end{aligned}
\tag{1}
$$

You should find that $A_3 = B_1 = 0$. Find four fixed points:

- The gaussian fixed point.
- A fixed point where only $u \neq 0$.
- A fixed point where only $v \neq 0$. Describe the physics of this fixed point. (Hint: the action is a sum of n terms.)
- A fixed point where both (u, v) are nonzero.

(In every case, the assumption of $u \sim v \sim \epsilon$ is self-consistent.)

Cardy, page 108.

(e) Analyze the stability of these fixed points (by computing the matrix of derivatives of the beta functions at each fixed point). Draw the phase diagram. Which fixed point dominates the critical behavior? You will want to consider different cases depending on whether $n > 4$ or $n < 4$.

(f) When $n > 4$ you may find that v wants to become negative. This means that the effective potential for m becomes unbounded, within our approximation. What have we left out that will restore sanity? What does this mean for the order of the phase transition? (Notice that mean field theory predicts a continuous transition, so any change in this conclusion is a dramatic effect of the fluctuations, more dramatic than just changing the values of critical exponents by a little.)

This problem is solved in §5.8 of Cardy's book.

3. **Rotation invariance as an emergent symmetry.** Give an RG analysis which explains why the critical behavior of lattice magnets (which do not have continuous spatial rotation symmetry) can be described by rotation-invariant field theories.

Looking at the dispersion relation for a (hyper)cubic lattice is instructive. The fourier transform of the adjacency matrix is

$$\tilde{A}(k) = \sum_{\mu=1}^{d} (1 - \cos k_\mu a) \overset{ka \ll 1}{=} k_\mu k^\mu + \mathcal{O}\left(k^4\right) \ .$$

The leading term in the small-wavenumber expansion is the familiar rotation-invariant Laplacian term. The next term, which breaks rotation invariance, is $\mathcal{O}(k^4)$. In general, the answer is that the terms which break rotation invariance (like $\partial_x^2 \phi \partial_y^2 \phi$) are irrelevant operators.

Now you could ask me: what about a term like $\partial_x \phi \partial_y \phi$? Or more generally suppose we perturb $(\nabla\phi)^2$ by $A_{ij}\partial_i\phi\partial_j\phi$. Here the answer is that I can diagonalize the matrix $\delta_{ij} + A_{ij}$ to put the kinetic term in the form $a_i (\partial_i \phi)^2$. Then, at the cost of a change of coordinates $x_i \to x_i/\sqrt{a_i}$ I can restore the familiar rotation-invariant form of the kinetic term.

A symmetry of a fixed point which can be broken only by irrelevant operators is called an emergent symmetry (by condensed matter physicists) or an accidental symmetry (by high energy physicists). This difference of tone is very revealing.