University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 7

Due 12:30pm Wednesday, November 21, 2018

1. **Brain-warmer.** Why is $\int d^dx \vec{\nabla}\phi_<(x) \cdot \vec{\nabla}\phi_>(x) = 0$?

2. **Practice with systematically ignoring small things.**

   In doing perturbative RG, as we are going to do for the next few weeks, it is very useful to be systematic about ignoring corrections which are of the same size as corrections we are not computing. To do this, it is useful to introduce (or keep track of) an expansion parameter whose powers count orders of perturbation theory. By 'of the same size' I mean corrections that come with the same number of powers of g.

   In lecture, we found the leading correction to the mean field free energy (down by one power of g), and found that the inverse susceptibility was

   $$\chi^{-1} = r_0 + g_0\delta(r_0) + \mathcal{O}(g_0^2)$$

   where $\delta(r_0) \sim \int \frac{đ^d q}{q^2 + r_0}$ is some known function of the bare coupling r_0.

   We assume that the parameter $r_0(T)$ is analytic in the temperature near T_c. This is the conservative assumption: the thing we are trying to explain is how physics can become non-analytic in T at some finite T; we don't want to put it in from the beginning. More precisely: we can rule out singular dependence of r_0 on T because non-analyticity requires the thermodynamic limit, and the microscopic couplings are properties of finite chunks of the system.

   The definition of the critical temperature T_c is the value of T where the correlation length blows up. Use the susceptibility sum rule (you proved this on the last homework) to relate this condition to $\chi(T_c)$.

   Use the previous two pieces of input to prove the expression I claimed in lecture which eliminates r and relates χ^{-1} directly to the deviation from the critical temperature $t \equiv \frac{T-T_c}{T_c}$:

   $$\chi^{-1}(t) = c_1 t(1 + \partial_t\delta(t)) + \mathcal{O}(g^2)$$

   where c_1 is a non-universal constant. You will have to ignore all errors of order g^2. Determine the function $\partial_t\delta$.

3. **An example of the power of the RG logic.**

Consider quantum mechanics of a single particle in d dimensions, with Hamiltonian

$$H = \frac{p^2}{2m} + V(q), \quad [q, p] = i.$$

Consider the (say, euclidean) path integral for this problem,

$$Z = \int [dq] e^{-S[q]}, \quad S[q] = \int dt \left(\frac{m}{2}\dot{q}^2 - V(q) \right).$$

To be more precise, with periodic boundary conditions, $Z(\beta) = \int_{q(t+\beta)=q(t)} [dq] e^{-S[q]} = \mathrm{tr} e^{-\beta H}$ is the thermal partition function. Alternatively, instead of Z, we could consider the Green's function $G(q_1, t_1; q_2, t_2) = \int_{q(t_1)=q_1}^{q(t_2)=q_2} [dq] e^{-S[q]}$.

Working by analogy with our treatment of field theory, show that any smooth[1] potential V is a *relevant* perturbation of the free particle, *i.e.* the Gaussian fixed point with $H = \frac{p^2}{2m}$.

Hint: change variables to $\phi(t) \equiv \sqrt{m} q(t)$.

Use this to explain in words why the high energy asymptotics of the density of states

$$N(E) \equiv \{\# \text{ of eigenvalues of } H \text{ less than } E\}$$

is given by the *Weyl formula* (even for $V(q) \neq 0$):

$$N(E) = E^{d/2} K_d L^d + ...$$

where $K_d = \frac{\Omega_{d-1}}{(2\pi)^d}$ as usual, and L is the linear size of the box in which we put the particle (an IR cutoff).

Hint: we can represent the density of states by a path integral using an inverse Laplace transform:

$$\mathrm{tr} \frac{1}{\omega - H} = \int d\beta \; e^{\beta\omega} Z(\beta)$$

and the relation

$$\mathrm{Im} \frac{1}{\omega + \mathbf{i}\epsilon - H} = \pi \delta(\omega - H).$$

[1]Some singular potentials are also relevant perturbations. If $V(q) \sim q^{-\alpha}$, how big can α be for my statement to remain true? Thanks to Brian Vermilyea for reminding me that a singular enough potential will cause trouble.