University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 7 – Solutions

Due 12:30pm Wednesday, November 21, 2018

1. **Brain-warmer.** Why is $\int d^d x \vec{\nabla}\phi_<(x) \cdot \vec{\nabla}\phi_>(x) = 0$?

2. **Practice with systematically ignoring small things.**

   In doing perturbative RG, as we are going to do for the next few weeks, it is very useful to be systematic about ignoring corrections which are of the same size as corrections we are not computing. To do this, it is useful to introduce (or keep track of) an expansion parameter whose powers count orders of perturbation theory. By 'of the same size' I mean corrections that come with the same number of powers of g.

   In lecture, we found the leading correction to the mean field free energy (down by one power of g), and found that the inverse susceptibility was

   $$\chi^{-1} = r_0 + g_0\delta(r_0) + \mathcal{O}(g_0^2)$$

   where $\delta(r_0) \sim \int \frac{đ^d q}{q^2+r_0}$ is some known function of the bare coupling r_0.

   We assume that the parameter $r_0(T)$ is analytic in the temperature near T_c. This is the conservative assumption: the thing we are trying to explain is how physics can become non-analytic in T at some finite T; we don't want to put it in from the beginning. More precisely: we can rule out singular dependence of r_0 on T because non-analyticity requires the thermodynamic limit, and the microscopic couplings are properties of finite chunks of the system.

   The definition of the critical temperature T_c is the value of T where the correlation length blows up. Use the susceptibility sum rule (you proved this on the last homework) to relate this condition to $\chi(T_c)$.

   Use the previous two pieces of input to prove the expression I claimed in lecture which eliminates r and relates χ^{-1} directly to the deviation from the critical temperature $t \equiv \frac{T-T_c}{T_c}$:

   $$\chi^{-1}(t) = c_1 t(1 + \partial_t\delta(t)) + \mathcal{O}(g^2)$$

   where c_1 is a non-universal constant. You will have to ignore all errors of order g^2. Determine the function $\partial_t\delta$.

3. **An example of the power of the RG logic.**

Consider quantum mechanics of a single particle in d dimensions, with Hamiltonian

$$H = \frac{p^2}{2m} + V(q), \quad [q,p] = i.$$

Consider the (say, euclidean) path integral for this problem,

$$Z = \int [dq] e^{-S[q]}, \quad S[q] = \int dt \left(\frac{m}{2}\dot{q}^2 - V(q) \right).$$

To be more precise, with periodic boundary conditions, $Z(\beta) = \int_{q(t+\beta)=q(t)} [dq] e^{-S[q]} = \mathrm{tr} e^{-\beta H}$ is the thermal partition function. Alternatively, instead of Z, we could consider the Green's function $G(q_1, t_1; q_2, t_2) = \int_{q(t_1)=q_1}^{q(t_2)=q_2} [dq] e^{-S[q]}$.

Working by analogy with our treatment of field theory, show that any smooth[1] potential V is a *relevant* perturbation of the free particle, *i.e.* the Gaussian fixed point with $H = \frac{p^2}{2m}$.

Hint: change variables to $\phi(t) \equiv \sqrt{m} q(t)$.

Use this to explain in words why the high energy asymptotics of the density of states

$$N(E) \equiv \{\# \text{ of eigenvalues of } H \text{ less than } E\}$$

is given by the *Weyl formula* (even for $V(q) \neq 0$):

$$N(E) = E^{d/2} K_d L^d + \ldots$$

where $K_d = \frac{\Omega_{d-1}}{(2\pi)^d}$ as usual, and L is the linear size of the box in which we put the particle (an IR cutoff).

Hint: we can represent the density of states by a path integral using an inverse Laplace transform:

$$\mathrm{tr} \frac{1}{\omega - H} = \int d\beta \; e^{\beta\omega} Z(\beta)$$

and the relation

$$\mathrm{Im} \frac{1}{\omega + \mathbf{i}\epsilon - H} = \pi \delta(\omega - H).$$

The fact that $V(q)$ is smooth means we can Taylor expand it

$$V(q) = \sum_n g_n q^n.$$

[1]Some singular potentials are also relevant perturbations. If $V(q) \sim q^{-\alpha}$, how big can α be for my statement to remain true? Thanks to Brian Vermilyea for reminding me that a singular enough potential will cause trouble.

We may as well consider each q^n separately; it will be interesting to allow n to be negative. Repeating our analysis of coarse-graining the fast modes, and rescaling to fix the kinetic term (really just specializing to $d = 1$), we have

$$S_{\text{eff}}[\tilde{q}] = \int dt \left(\frac{1}{2}\dot{\tilde{q}}^2 + \sum_n b^{\frac{n+2}{2}} (g_n + \delta g_n)\tilde{q}^n \right)$$

where $\tilde{q} \equiv b^{-\frac{1}{2}} q_<$ in order to preserve the kinetic term. So

$$g'_n = b^{\frac{n+2}{2}} (g_n + \delta g_n).$$

We see that as long as $n > -2$, this grows as b grows. (In perturbation theory, the fluctuation effects δg_n can't change this conclusion.) Thus we conclude that any smooth potential is a relevant perturbation of the gaussian fixed point.

Therefore, the effect of any smooth potential gets smaller at short times or high energies – to get a given value of a_n at energy scale E/b, at scale E it had to be smaller by a factor of $b^{\frac{n+2}{2}}$. To make this more precise, represent the density of states in terms of the path integral as in the hint. At large E, the integral over β is dominated by the saddle point at $\beta_\star$, which sets $E = -\partial_\beta \log Z|_{\beta=\beta_\star}$. As long as the average energy grows with temperature, this means that we need the high-temperature behavior of $Z(\beta)$, which is dominated by the fastest modes, described by taking $b \ll 1$ in the above scaling.

We can see this more directly in the path integral representation of Z just by changing variables: $\tilde{q} = b^{-1/2} q, \tilde{t} = t/b, \tilde{\beta} = \beta/b$, so that

$$Z(\beta) = \int [Dq] e^{-\int_0^\beta dt \left(\frac{1}{2}\dot{q}^2 + V(q)\right)} = \int [D\tilde{q}] e^{-\int_0^{\tilde{\beta}/b} dt \left(\frac{1}{2}\dot{\tilde{q}}^2 + \sum_n g_n b^{n+2/2} \tilde{q}^n\right)} .$$

Now, if we want to evaluate this at asymptotically small β, we can just choose $b \ll 1$, in which case we can treat all the potential terms as perturbations. Once we know that the answer is given by the free particle answer, we can either inverse-Laplace transform the free-particle partition function $Z(\beta) = \left(\frac{2\pi m L^2}{\beta}\right)^{d/2}$, or directly count states

$$N(E) = L^d \int_{\sum k^2 < 2mE} đ^d k = L^d K_d (2mE)^{d/2}/d,$$

which is the Weyl formula.