University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 6

Due 12:30pm Wednesday, November 14, 2018

1. **Convexity of the free energy.**

   (a) Prove that the free energy of the Ising model is anti-convex in h, the external magnetic field.

   (b) Prove that the free energy is anti-convex in T.

2. **When the higher-derivative terms in the LG free energy are important.** [Based on Goldenfeld Exercise 5-3]

   Consider a system whose Landau-Ginzburg free energy looks like

$$F_{\text{LG}}[m] = \int dx \, \left(atm^2 + \frac{1}{2}bm^4 + \frac{1}{2}\gamma \left(\partial_x m\right)^2 + \frac{1}{2}\sigma \left(\partial_x^2 m\right)^2 \right).$$

   For simplicity take the system to be one-dimensional and put it in a box of size L; we'll use periodic boundary conditions, $m(x+L) = m(x)$. Suppose that the constants $\sigma > 0, b > 0, a > 0$ but allow that t and (here's the interesting bit) γ can be of either sign. The problem is to work out the phase diagram in the $t - \gamma$ plane.

   (a) Fourier expand the magnetization

$$m(x) = \frac{1}{L}\sum_{n \in \mathbb{Z}} e^{\mathbf{i} q_n x} m_n, \quad q_n = \frac{2\pi n}{L}.$$

   Find the inverse transform for m_n in terms of $m(x)$ and check that all the (Kronecker and Dirac) delta functions work out.

   (b) Write $F_{\text{LG}}[m]$ in terms of the fourier modes m_n.

   (c) By minimizing with respect to m_n for all n, show that the system exhibits three possible phases:
   (1) a paramagnetic phase where $m = 0$
   (2) a ferromagnetic phase where $m \neq 0$ but is uniform, and
   (3) a *spatially modulated* phase where $m_q \neq 0$ for some nonzero wavenumber $q \neq 0$.
   [Note that (because $b > 0$), when one mode m_q condenses, it becomes unfavorable for others to do so – so you should assume you only need one Fourier component.]

(d) What is the wavelength of the modulated mode which condenses? Find the phase boundaries and draw the phase diagram. What is the order of the transition at the various phase boundaries?

3. **Effective action.**

(a) Show that the Legendre transform $\Gamma[m] = (F[h] - \sum hm)\,|_{m=\partial_h F/V}$ was done correctly in lecture, up to corrections of $\mathcal{O}(g)$.

(b) Show (using the definition of the Legendre transform and the chain rule) that the susceptibility $\chi = \partial_h m$ is the inverse of the curvature of the effective potential $\gamma(m) = \Gamma[m, \text{uniform}]/V$:

$$\chi^{-1} = \partial_m^2 \gamma.$$

Conclude that the correlation length diverges when $\partial_m^2 \gamma \to 0$. Think about this in terms of the spectrum of normal modes of the system.