University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 6 – Solutions

Due 12:30pm Wednesday, November 14, 2018

1. **Convexity of the free energy.**

   (a) Prove that the free energy of the Ising model is anti-convex in h, the external magnetic field.

   (b) Prove that the free energy is anti-convex in T.

2. **When the higher-derivative terms in the LG free energy are important.** [Based on Goldenfeld Exercise 5-3]

   Consider a system whose Landau-Ginzburg free energy looks like

$$F_{\rm LG}[m] = \int dx \left(atm^2 + \frac{1}{2}bm^4 + \frac{1}{2}\gamma \left(\partial_x m\right)^2 + \frac{1}{2}\sigma \left(\partial_x^2 m\right)^2 \right).$$

   For simplicity take the system to be one-dimensional and put it in a box of size L; we'll use periodic boundary conditions, $m(x+L) = m(x)$. Suppose that the constants $\sigma > 0, b > 0, a > 0$ but allow that t and (here's the interesting bit) γ can be of either sign. The problem is to work out the phase diagram in the $t - \gamma$ plane.

   (a) Fourier expand the magnetization

$$m(x) = \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{\mathbf{i}q_n x} m_n, \quad q_n = \frac{2\pi n}{L}.$$

   Find the inverse transform for m_n in terms of $m(x)$ and check that all the (Kronecker and Dirac) delta functions work out.

   (b) Write $F_{\rm LG}[m]$ in terms of the fourier modes m_n.

   Taking the thermodynamic limit $L \to \infty$,

$$F_{\rm LG}[m] = \int đ^d k \left(at + \frac{1}{2}\gamma k^2 + \frac{1}{2}\sigma k^4 \right) m_k m_{-k} + \int đ^d k đ^d p đ^d q \frac{1}{2} b m_k m_p m_q m_{-k-p-q}.$$

   Note that since m is real, $m_q = m^\star_{-q}$.

(c) By minimizing with respect to m_n for all n, show that the system exhibits three possible phases:
(1) a paramagnetic phase where $m = 0$
(2) a ferromagnetic phase where $m \neq 0$ but is uniform, and
(3) a *spatially modulated* phase where $m_q \neq 0$ for some nonzero wavenumber $q \neq 0$.
[Note that (because $b > 0$), when one mode m_q condenses, it becomes unfavorable for others to do so – so you should assume you only need one Fourier component.]

The eom is

$$0 = m_q(at + \frac{1}{2}\gamma q^2 + \frac{1}{2}\sigma q^4) + 4\int đ^d p đ^d k \; m_p m_k m_{-p-k+q}.$$

If we keep just one mode, it says

$$0 = m_q^\star \left((at + \frac{1}{2}\gamma q^2 + \frac{1}{2}\sigma q^4) + 4|m_q|^2 \right)$$

which has solutions

$$(1) \quad m_q = 0, \forall q, \quad F[(1)] = 0 \tag{1}$$

$$(2) \quad m_0 = -\frac{at}{2b}, \quad F[(2)] = -\frac{(at)^2}{4b} \tag{2}$$

$$(3) \quad |m_q|^2 = -\frac{at + \frac{1}{2}\gamma q^2 + \frac{1}{2}\sigma q^4}{2b}, \quad F[(3)] = -\frac{(at + \frac{1}{2}\gamma q^2 + \frac{1}{2}\sigma q^4)^2}{4b}. \tag{3}$$

In solution (3), the phase of m_q merely shifts the nodes of the pattern.

(d) What is the wavelength of the modulated mode which condenses? Find the phase boundaries and draw the phase diagram. What is the order of the transition at the various phase boundaries?

(Actually (2) is a special case of (3), but hey it makes a difference if translation symmetry is broken.) The solution with the lowest free energy wins. The extremum over q of $at + \frac{1}{2}\gamma q^2 + \frac{1}{2}\sigma q^4$ occurs at $q_\star^2 = -\frac{\gamma}{2\sigma}$, where it takes the value $at - \frac{\gamma^2}{8\sigma}$. $q_\star$ is the wavevector at which m will condense; note that $q_\star$ is only real for $\gamma \leq 0$. Solution (3) only exists when $\gamma \leq 0$ and

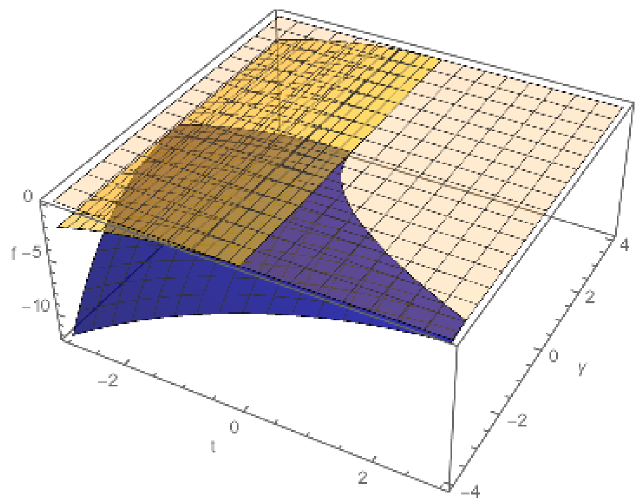
$$0 \leq |m_{q_\star}|^2 = -\frac{at + \frac{1}{2}\gamma q_\star^2 + \frac{1}{2}\sigma q_\star^4}{2b} = -(at - \frac{\gamma^2}{8\sigma})$$

which requires $at < \frac{\gamma^2}{8\sigma}$. The free energies of (2) and (3) cross at $\gamma = 0$ (where they are the same because $q_\star = 0$ when $\gamma = 0$), and at

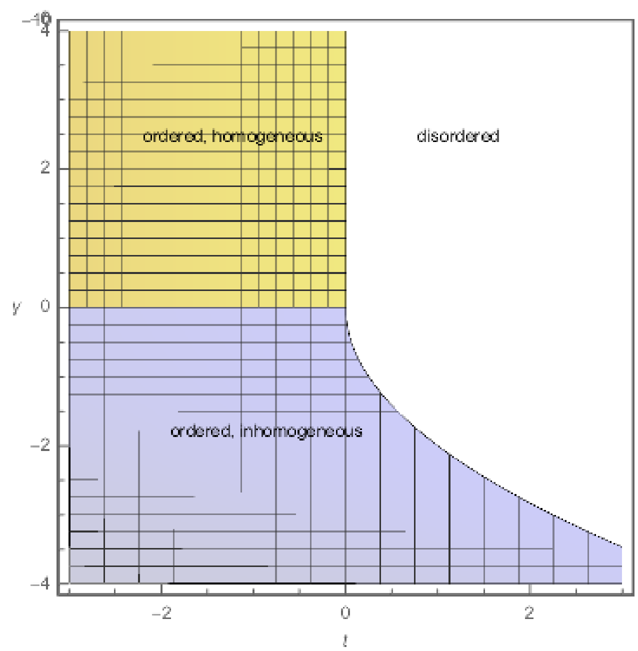
$$t = \frac{\gamma^2}{16a\sigma},$$

but the latter is at positive t where solution (2) is not real.

I find that the free energies of solutions (1) (in transparent yellow), (2) (in orange) and (3) (in blue) – when they exist – look like:



Therefore it seems to me that the inhomogeneous solution (3) wins in the region $\gamma < 0, at < \frac{\gamma^2}{8\sigma}$, the homogeneous, ordered solution (2) wins when $t < 0, \gamma > 0$, and the disordered solution (1) wins when $t > 0, at > \frac{\gamma^2}{8\sigma}$:



The transition at $at = \frac{\gamma^2}{8\sigma}$ is first order – solution (1) and (3) are not continuously related (though of course their free energies agree along this curve). The transition line at $\gamma = 0, t < 0$ is continuous, since as $\gamma \to 0$ from below, $q_\star \to 0$, and the inhomogeneous solution becomes the homogeneous one.

3. **Effective action.**

   (a) Show that the Legendre transform $\Gamma[m] = \left(F[h] - \sum hm\right)|_{m=\partial_h F/V}$ was done correctly in lecture, up to corrections of $\mathcal{O}(g)$.

(b) Show (using the definition of the Legendre transform and the chain rule) that the susceptibility $\chi = \partial_h m$ is the inverse of the curvature of the effective potential $\gamma(m) = \Gamma[m, \text{uniform}]/V$:

$$\chi^{-1} = \partial_m^2 \gamma.$$

Conclude that the correlation length diverges when $\partial_m^2 \gamma \to 0$. Think about this in terms of the spectrum of normal modes of the system.