University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 5 – Solutions

Due 12:30pm Monday, November 5, 2018

1. **Brain-warmer.**

   Prove the static susceptibility sum rule, using calculus, algebra and definitions.

2. **A machine for making field theories from lattice spin models.** [Based on Goldenfeld Exercise 3-3]

   Here is a fourth route to mean field theory. Start with the nearest-neighbor Ising hamiltonian on a graph:

$$-H(s) = -\frac{1}{2}\sum_{i\neq j} J_{ij}s_i s_j + \sum_i h_i s_i$$

   with $J_{ij} = J > 0$ for neighbors, and zero otherwise (this is J times the *adjacency matrix* of the graph).

   (a) Prove the identity[1]

$$\int_{-\infty}^{\infty}\prod_{i=1}^{N}\frac{dx_i}{\sqrt{2\pi}}e^{-\frac{1}{2}x_i A_{ij}x_j + x_i B_i} = \frac{1}{\sqrt{\det A}}e^{\frac{1}{2}B_i\left(A^{-1}\right)_{ij}B_j}$$

   for A a real symmetric, positive (all eigenvalues are positive) matrix, and B is an arbitrary vector. [Hints: complete the square by changing variables to $y_i \equiv x_i - \left(A^{-1}\right)_{ij}B_j$. Then use the fact that the integral over y is basis-independent; choose a convenient basis.]

   It may be necessary to add positive diagonal terms $\Delta H = \sum_i J_{ii}s_i s_i = \sum_i J_{ii}$ in order to make J_{ij} a positive matrix. This only adds a constant to the Hamiltonian (and contributes to the constant C below).

   (b) The result of the previous part can be used to rewrite the Ising Boltzmann weights with exponents linear in s_i, like in the previous homework. Show the the Ising partition function can be written as

$$\sum_s e^{-\beta(H(s)+c)} = \int_{-\infty}^{\infty}\prod_{i=1}^{N}d\psi_i\ e^{-\beta S(\psi,h,J)}$$

[1] sometimes called the fundamental theorem of quantum field theory

with

$$S = \frac{1}{2}(\psi_i - h_i)J^{-1}_{ij}(\psi_j - h_j) - T\sum_i \log\left(2\cosh\beta\psi_i\right)$$

for some c. Find c. The RHS is a discretization of a functional integral, in that the dynamical variable ψ_i approximates a function $\psi(x)$ in the limit of small lattice spacing and large lattice.

Let $A_{ij} = (J^{-1}_{ij}/\beta,\ B_i = s_i - (J^{-1}_{ij}h_j..$

$e^{-\beta c} = e^{+\beta\sum_i J_{ii}}\sqrt{\frac{\det J}{(2\pi)^N}}$

(c) Evaluate the ψ integrals by saddle point:

$$Z \simeq e^{-\beta S[\underline{\psi}]}$$

where $\underline{\psi}$ is a configuration of the integration varaibles which minimizes S. Find the equation determining $\underline{\psi}$ and show that the magnetization

$$m_i \equiv \langle s_i\rangle = -\partial_{h_i}F \simeq -\partial_{h_i}S|_{\underline{\psi}}$$

is given by $m_i = \tanh\beta\underline{\psi}$. Invert this equation to find $h_i(m)$.

(d) Let $\underline{S} \equiv S[\underline{\psi}]$. The mean field free energy is $F_{\text{MF}} = \underline{S}$. Show that

$$\underline{S} = \frac{1}{2}\sum_{ij} J_{ij}m_i m_j - T\sum_i \log\left(\frac{2}{\sqrt{1-m_i^2}}\right).$$

Plugging in the mean field solution will give a function of h. Legendre transform to the fixed-m ensemble:

$$\Gamma[m] = \underline{S} + \sum_i h_i(m)m_i$$

and show that the condition $h_i = \partial_{m_i}\Gamma$ reproduces the correct mean field equation.

$$\Gamma[m] = \frac{1}{2}\sum J_{ij}m_i m_j - T\sum_i \ln\frac{2}{\sqrt{1-m_i^2}} + \sum_i \left(T\text{arctanh}(m_i) - J_{ij}m_k\right)m_i \tag{1}$$

$$= -\frac{1}{2}\sum J_{ij}m_i m_j - T\sum_i \ln\frac{2}{\sqrt{1-m_i^2}} + \sum_i T\frac{m_i}{2}\ln\frac{1-m_i}{1+m_i} \tag{2}$$

$$= -\frac{1}{2}\sum_{ij} J_{ij}m_i m_j - T\sum_i -\left(\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right) \tag{3}$$

$$= E - TS \tag{4}$$

where S is the Shannon entropy for a product of binary distributions with $p_\pm = 1 \pm m_i/2$, and we used the identity from problem 1 from the previous homework.

$$h_i = \partial_{m_i}\Gamma = -J_{ij}m_j + T\left(\frac{1}{2}\ln(1+m_i) - \frac{1}{2}\ln(1-m_i)\right) \tag{5}$$

$$= -J_{ij}m_j + T\text{arctanh}m_i \quad \Leftrightarrow m_i = \tanh\left(\beta\left(J_{ij}m_j + h_i\right)\right). \tag{6}$$

3. Check that the expression for the correlation function obtained from mean field theory gives

$$\int d^d r G(r) = \tilde{G}_{k=0} = \chi_T = \frac{\mu}{bt},$$

independent of the interaction range R, consistent with the susceptibility sum rule. The notation is from the lecture notes. (Part of the assignment is to correct numerical prefactors in the above equalities.)

$$\chi_T = \frac{1}{T}\sum_i G(r_i) = \frac{1}{T}\sum_i \oint đ^d k\ e^{\mathbf{i}kx_i}\tilde{G}(k) = \frac{1}{T}\tilde{G}(k=0) = \frac{1}{T}\frac{1}{\beta zJbt} = \frac{\mu}{bt}$$

if we identify $\mu = \frac{A}{zJ}$.

4. **Tensor network renormalization.** [optional bonus problem. This problem is not due next week. Also I just wrote it so please ask if something is unclear.]

Consider the nearest-neighbor Ising model on the triangular lattice. The reference for the strategy we will follow here is by Levin and Nave. Fortunately, it is pretty terse, so you'll still have to figure it out yourself.

I'll wait a little longer to post my solution to this problem to give you some more time to work on it.

(a) Show that the partition function may be written as the contraction of a *tensor network*:

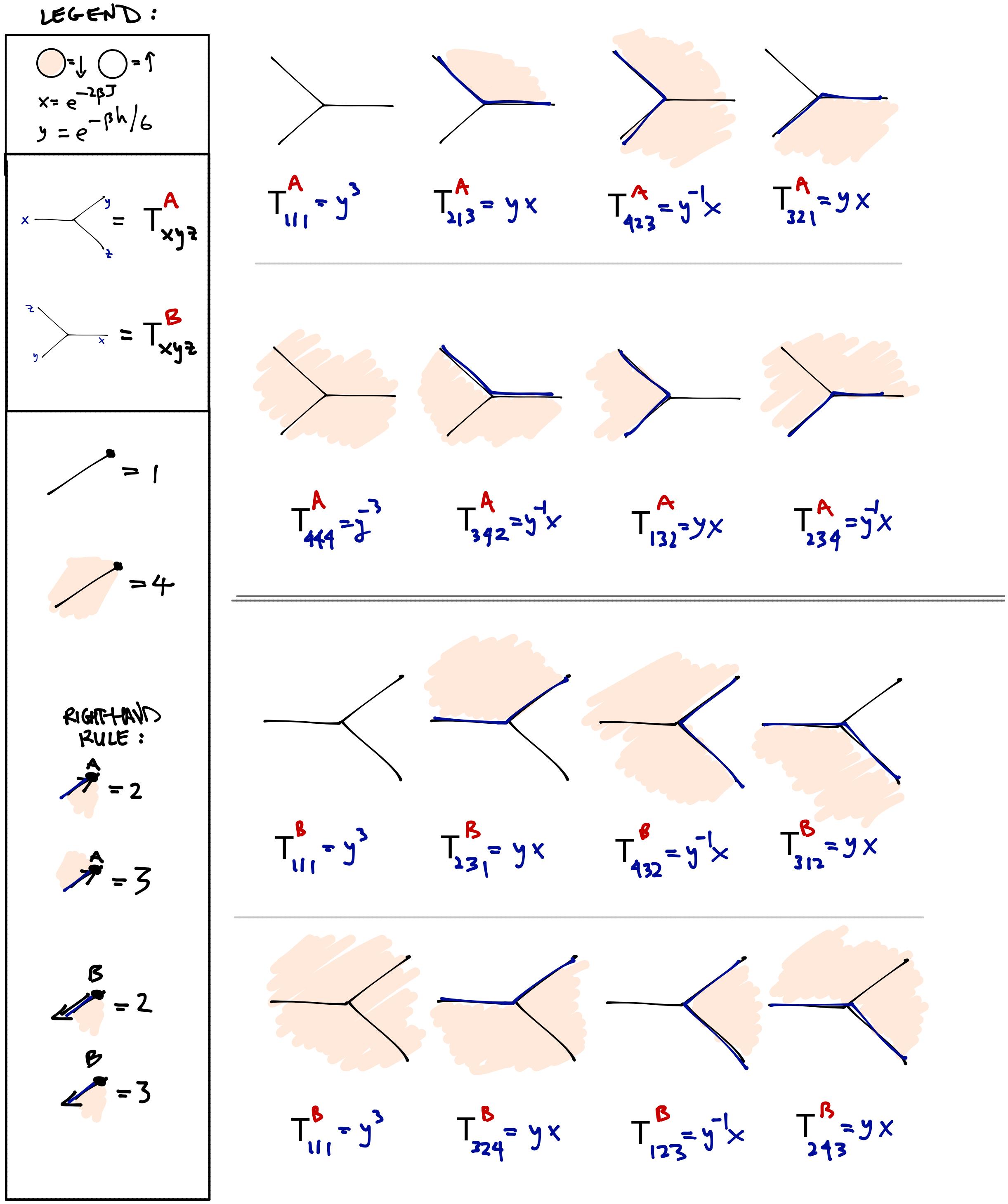
$$Z = \text{tr}e^{-\beta H} = \text{tr}TTTTT\cdots \equiv \sum_{ijklmno\cdots} T_{ijk}T_{klm}T_{mno}\cdots$$

where the tensors T_{ijk} are 3-index objects (tensors) which depend on the couplings, and which are associated with sites of the dual honeycomb lattice. They have one index for each of the incident edges of the honeycomb lattice. Find a set of T_{ijk}, $ijk\cdots = 0, 1$ which makes this equation true, for $h = 0$.

In the absence of a field, we can let the values of the indices signify $0 =$ no domain wall and $1 =$ yes domain wall. And we can enforce the fact that domain walls are closed curves by setting $T_{ijk} \propto \delta_{i+j+k \text{ mod } 2}$.

(b) [slightly harder] Find a set of Ts which works for nonzero h.

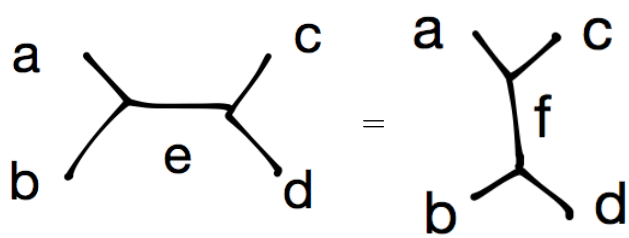
I use different Ts for the two sites in the honeycomb unit cell, which I'll call T_A and T_B. My scheme is explained in the figure. I let the tensor indices run over four values, whose meanings are indicated in the legend at the left of the figure. Whether it's possible to do it with only three I don't know.



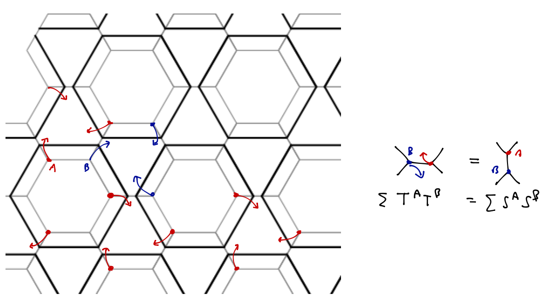
(c) [slightly harder still] Once we've written Z in this form, we can do a coarse-graining procedure in two steps. First consider a pair of neighboring honeycomb lattice sites, associated with two tensors $\sum_e T_{abe}T_{ecd}$. Regard this object as a $D^2 \times D^2$ matrix with block indices ac and bd. By doing a singular-value decomposition of this matrix, rewrite the product as:

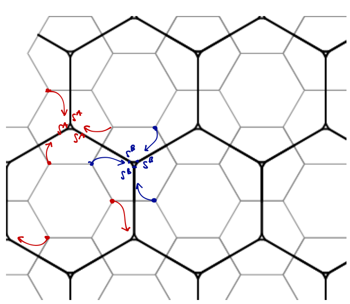
$$\sum_e T_{abe}T_{ecd} \equiv \sum_f S_{acf}S_{fbd}.$$

In diagrams, this looks like:



Doing this for a suitable collection of links (as in the figure),

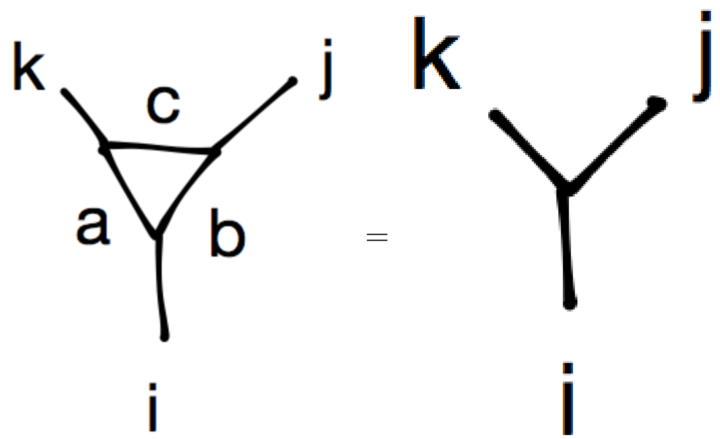




we are left with triangles of Ss. The second step of the coarse-graining scheme is to define a new T by

$$\sum_{a,b,c} S_{kac}S_{jcb}S_{iab} = T'_{kij}$$

or in pictures by:

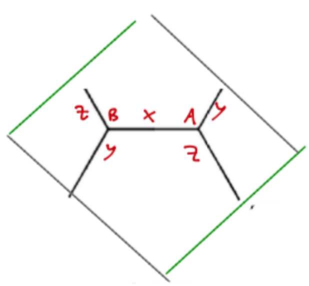


This gives back an Ising model on the triangular lattice with a larger lattice spacing.

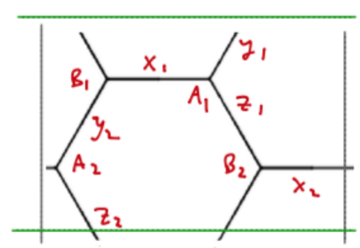
Implement this RG scheme numerically. Notice that the approximation comes in when we throw away singular values in step 1 (if we do not, the range of indices of the tensors (called the *bond dimension*) must grow with the number of steps). Compute the magnetization as a function of temperature.

In order to evaluate physical quantities, we start the procedure with some initial values of $x \equiv e^{-2\beta J}, y \equiv e^{-\beta h/6}$ and repeat until the tensors stop changing. Then we can evaluate Z as if we've reduced the lattice to a single unit cell or two, like

this

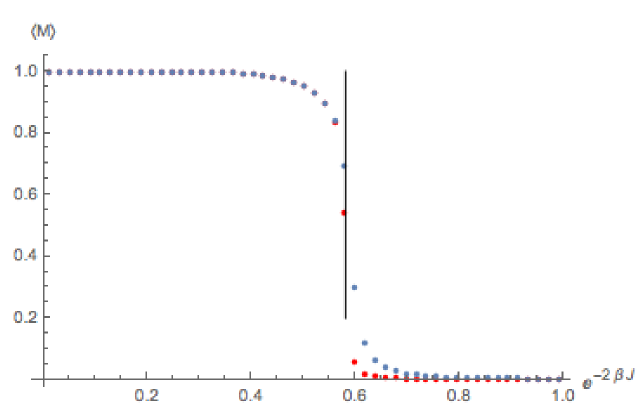


or this



.

For the magnetization, I find the following curve:



The blue dots are from max bond dimension 30 and took about 20 minutes to run on my old desktop. The vertical line is at $e^{-2\beta J} = .58$. By eyeball, this gives a reasonable value for $T_c \simeq \frac{-2}{\ln(.58)} = 3.7$ (the correct value (by duality) is $T_c^\triangle = 3.64$). The red dots are from bond dimension 15 takes a minute and gives a similar estimate for T_c. (Thanks to Shenglong Xu for some crucial help with the implementation.) Here's my notebook.

One shortcoming of this method is that it is not clear (at least to me) what is the analog of finite-size scaling, by which we could hope to make a more precise determination of the transition. It is tempting to do finite-bond-dimension scaling, *i.e.* superpose the magnetization curves for various bond dimension and see where they cross. Will this give the right T_c?

Since the paper by Levin and Nave, this subject has developed quite a bit. The state of the art, last I heard, was an improvement called HOTRG.