

## Physics 217 Fall 2018 Assignment 4

Due 12:30pm Monday, October 29, 2018

### 1. Brain-warmer.

Check that

$$-\operatorname{arctanh}(x) = \frac{1}{2} \log \frac{1-x}{1+x},$$

and that this is  $\partial_x s(x)$  with

$$s(x) \equiv -\frac{1+x}{2} \log \frac{1+x}{2} - \frac{1-x}{2} \log \frac{1-x}{2}.$$

### 2. A case where mean field theory is right.

[from Goldenfeld, Ex 2-2] Consider the *infinite-range* Ising model, where the coupling constant is  $J_{ij} = J$  for all  $i, j$ , with no notion of who is whose neighbor. That is:

$$-H(s) = h \sum_i s_i + \frac{J_0}{2} \sum_{ij} s_i s_j.$$

(a) Explain why it's a good idea to let  $J_0 = J/N$  where  $N$  is the number of sites.

(b) Prove that

$$e^{\frac{a}{2N}x^2} = \int_{-\infty}^{\infty} dy \sqrt{\frac{Na}{2\pi}} e^{-\frac{Na}{2}y^2 + axy}, \quad \operatorname{Re} a > 0.$$

(c) Use this to show that

$$Z_\Lambda = \int_{-\infty}^{\infty} dy \sqrt{\frac{N\beta J}{2\pi}} e^{-N\beta L}$$

where

$$L(h, J, \beta, y) = \frac{J}{2} y^2 - T \log (2 \cosh (\beta(h + Jy))).$$

When can this expression be non-analytic in  $\beta$ ?

(d) In the thermodynamic limit ( $N \rightarrow \infty$ ), this integral can be evaluated exactly by the method of steepest descent. Show that

$$Z = Z(\beta, h, J) \simeq \sum_i \sqrt{J(y_i)} e^{-\beta N L(h, J, \beta, y_i)}$$

for some  $J(y_i)$ . Find the equation satisfied by the saddle point values  $y_i$ . Convince yourself that the  $\sqrt{J}$  prefactor can be neglected in the thermodynamic limit.

What is the probability of the system being in the state specified by  $y_i$ ? Use this to conclude that the magnetization is given by

$$m \equiv \lim_{N \rightarrow \infty} \frac{1}{\beta N} \partial_h \ln Z = y_0$$

where  $y_0$  is the position of the global minimum of  $L$ .

- (e) Now consider the case  $h = 0$ . Show by graphical methods that there is a phase transition and find the transition temperature  $T_c$ . What's the story with all the solutions, for  $T$  both below and above  $T_c$ ?
- (f) Calculate the (isothermal) susceptibility

$$\chi = \partial_h m.$$

For  $h = 0$ , show that  $\chi$  diverges both above and below  $T_c$ , and find the leading and next-to-leading behavior of  $\chi$  as a function of the reduced temperature  $t \equiv \frac{T - T_c}{T_c}$ .

3. **Tensor network renormalization.** [optional bonus problem. This problem is not due next week. Also I just wrote it so please ask if something is unclear.]

Consider the nearest-neighbor Ising model on the triangular lattice.

- (a) Show that the partition function may be written as the contraction of a *tensor network*:

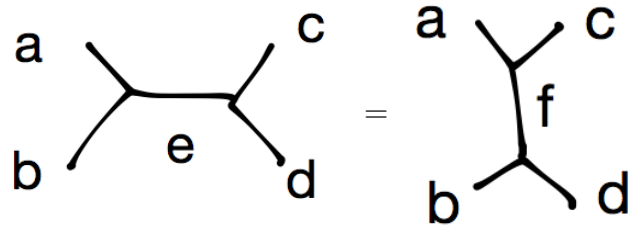
$$Z = \text{tr} e^{-\beta H} = \text{tr} T T T T T \cdots \equiv \sum_{ijklmno \cdots} T_{ijk} T_{klm} T_{mno} \cdots$$

where the tensors  $T_{ijk}$  are 3-index objects (tensors) which depend on the couplings, and which are associated with sites of the dual honeycomb lattice. They have one index for each of the incident edges of the honeycomb lattice. Find a set of  $T_{ijk}$ ,  $ijk \cdots = 0, 1$  which makes this equation true, for  $h = 0$ .

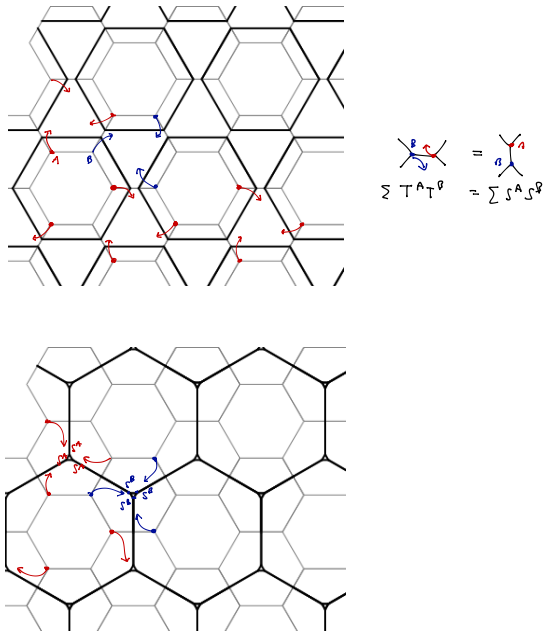
- (b) [slightly harder] Find a set of  $T$ 's which works for nonzero  $h$ .
- (c) [slightly harder still] Once we've written  $Z$  in this form, we can do a coarse-graining procedure in two steps. First consider a pair of neighboring honeycomb lattice sites, associated with two tensors  $\sum_e T_{abe} T_{ecd}$ . Regard this object as a  $D^2 \times D^2$  matrix with block indices  $ac$  and  $bd$ . By doing a singular-value decomposition of this matrix, rewrite the product as:

$$\sum_e T_{abe} T_{ecd} \equiv \sum_f S_{acf} S_{fbd}.$$

In diagrams, this looks like:



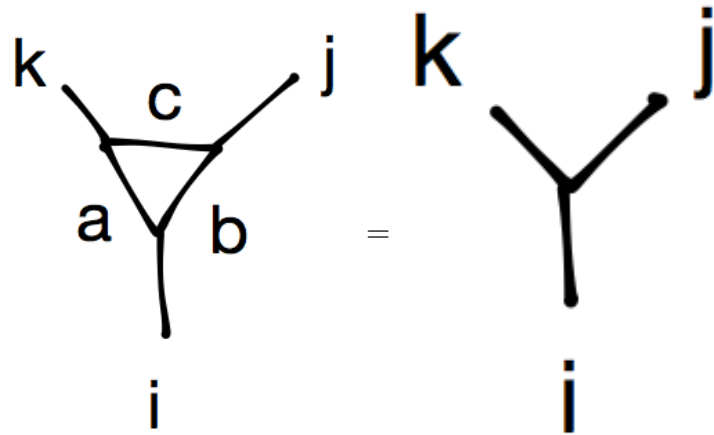
Doing this for a suitable collection of links (as in the figure),



we are left with triangles of  $S$ s. The second step of the coarse-graining scheme is to define a new  $T$  by

$$\sum_{a,b,c} S_{kac} S_{jcb} S_{iab} = T'_{kij}$$

or in pictures by:



This gives back an Ising model on the triangular lattice with a larger lattice spacing.

Implement this RG scheme numerically. Notice that the approximation comes in when we throw away singular values in step 1 (if we do not, the range of indices of the tensors (called the *bond dimension*) must grow with the number of steps). Compute the magnetization as a function of temperature.