University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 4 – Solutions

Due 12:30pm Monday, October 29, 2018

1. **Brain-warmer.**

   Check that
   $$-\text{arctanh}(x) = \frac{1}{2}\log\frac{1-x}{1+x},$$
   and that this is $\partial_x s(x)$ with
   $$s(x) \equiv -\frac{1+x}{2}\log\frac{1+x}{2} - \frac{1-x}{2}\log\frac{1-x}{2}.$$

2. **A case where mean field theory is right.**

   [from Goldenfeld, Ex 2-2] Consider the *infinite-range* Ising model, where the coupling constant is $J_{ij} = J$ for all i, j, with no notion of who is whose neighbor. That is:
   $$-H(s) = h\sum_i s_i + \frac{J_0}{2}\sum_{ij} s_i s_j.$$

   (a) Explain why it's a good idea to let $J_0 = J/N$ where N is the number of sites.

   We want to normalize the coupling so that $F \propto N$.

   (b) Prove that
   $$e^{\frac{a}{2N}x^2} = \int_{-\infty}^{\infty} dy\sqrt{\frac{Na}{2\pi}}e^{-\frac{Na}{2}y^2+axy}, \quad \text{Re}\,a > 0.$$

   (c) Use this to show that
   $$Z_\Lambda = \int_{-\infty}^{\infty} dy\sqrt{\frac{N\beta J}{2\pi}}e^{-N\beta L}$$
   where
   $$L(h, J, \beta, y) = \frac{J}{2}y^2 - T\log\left(2\cosh\left(\beta(h + Jy)\right)\right).$$
   When can this expression be non-analytic in β?

   First, write $\sum_{ij} s_i s_j = \left(\sum_i s_i\right)^2 - N \equiv x - N$. Then
   $$e^{-\beta H} = e^{\frac{J\beta}{2N}x^2 - N + \beta hx}e^{-\beta J/2} = e^{-\beta J/2}\int_{-\infty}^{\infty} dy\; e^{-\frac{NJ\beta}{2}y^2 + \beta Jxy + \beta hx}.$$

Now do the sum over s_i:

$$Z_\Lambda = \sum_s e^{-\beta H} = e^{-\beta J/2} \int_{-\infty}^{\infty} dy\ e^{-\frac{NJ\beta}{2}y^2} \prod_{i=1}^{N} \underbrace{\sum_{s_i} e^{\beta(Jy+h)s_i}}_{=2\cosh\beta(Jy+h)}$$

(d) In the thermodynamic limit ($N \to \infty$), this integral can be evaluated exactly by the method of steepest descent. Show that

$$Z = Z(\beta, h, J) \simeq \sum_i \sqrt{J(y_i)} e^{-\beta N L(h,J,\beta,y_i)}$$

for some $J(y_i)$. Find the equation satisfied by the saddle point values y_i. Convince yourself that the $\sqrt{J}$ prefactor can be neglected in the thermodynamic limit.

The saddle point equation is the usual mean field equation. The prefactor adds a term to the action which is down by a power of N.

What is the probability of the system being in the state specified by y_i? Use this to conclude that the magnetization is given by

$$m \equiv \lim_{N\to\infty} \frac{1}{\beta N} \partial_h \ln Z = y_0$$

where y_0 is the position of the global minimum of L.

$P(y_i) \propto e^{-\beta N L(y_i)}$, so probability of the contributions with larger action is suppressed by $e^{-\beta N \Delta L}$.

(e) Now consider the case $h = 0$. Show by graphical methods that there is a phase transition and find the transition temperature T_c. What's the story with all the solutions, for T both below and above T_c?

This is just as in the notes on mean field theory.

(f) Calculate the (isothermal) susceptibility

$$\chi = \partial_h m.$$

For $h = 0$, show that χ diverges both above and below T_c, and find the leading and next-to-leading behavior of χ as a function of the reduced temperature $t \equiv \frac{T-T_c}{T_c}$.

Near the transition, m is small, so we can Taylor expand in $m = y_0$: $\tanh x = x - x^3/3 + \mathcal{O}(x^5)$. So

$$\chi = \partial_h m = \partial_h \left(\tanh(\beta(Jm+h))\right) = \frac{\beta(J\chi+1)}{\cosh^2(\beta(Jm+h))} \tag{1}$$

$$= \beta(J\chi+1) - \beta^3(J\chi+1)(Jm+h)^2 + ... \tag{2}$$

Now we can set $h = 0$ and solve for χ:

$$\chi = \frac{\beta(1 - \beta^2(Jm)^2 + ...}{1 - \beta J + \beta^3 J^3 m^2 + ...}$$

We can eliminate m by the mean-field equation $m = \tanh \beta J m = \beta J m - (\beta J m)^3/3$, which says

$$0 = m(1 - \beta J + m^2 \beta^3 J^3/3). \tag{3}$$

Above T_c the solution of (3) is $m = 0$ and we have

$$\chi|_{t\to 0^+} = \frac{\beta}{1-\beta J} = \frac{1}{T_c t}.$$

Below T_c, the solution is

$$m^2 = 3\frac{\beta J - 1}{\beta^3 J^3}$$

(note that this is positive for $T < T_c$!) so

$$\chi|_{t\to 0^-} \approx \frac{\beta(1 - \frac{3(\beta J - 1)}{\beta J})}{1 - \beta J + 3(\beta J - 1)} = \frac{\beta}{2(\beta J - 1)} - \frac{3}{2J} = \frac{1}{2|t|T_c} - \frac{3}{2T_c}.$$

Note that χ is always positive; this is a necessary condition for stability of the equilibrium!

3. **Tensor network renormalization.** [optional bonus problem. This problem is not due next week. Also I just wrote it so please ask if something is unclear.]

This problem is continued to next week's problem set.

Consider the nearest-neighbor Ising model on the triangular lattice.

(a) Show that the partition function may be written as the contraction of a *tensor network*:

$$Z = \mathrm{tr} e^{-\beta H} = \mathrm{tr} TTTTT \cdots \equiv \sum_{ijklmno\cdots} T_{ijk} T_{klm} T_{mno} \cdots$$

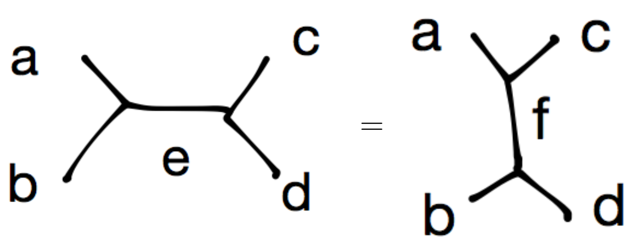
where the tensors T_{ijk} are 3-index objects (tensors) which depend on the couplings, and which are associated with sites of the dual honeycomb lattice. They have one index for each of the incident edges of the honeycomb lattice. Find a set of T_{ijk}, $ijk\cdots = 0, 1$ which makes this equation true, for $h = 0$.

(b) [slightly harder] Find a set of Ts which works for nonzero h.

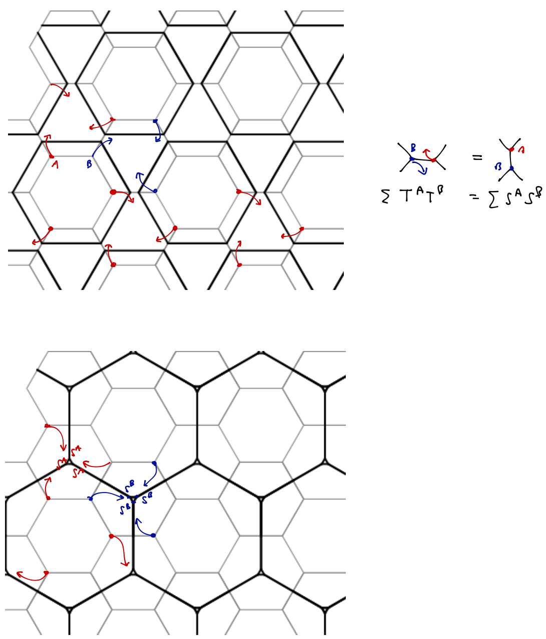
(c) [slightly harder still] Once we've written Z in this form, we can do a coarse-graining procedure in two steps. First consider a pair of neighboring honeycomb lattice sites, associated with two tensors $\sum_e T_{abe}T_{ecd}$. Regard this object as a $D^2 \times D^2$ matrix with block indices ac and bd. By doing a singular-value decomposition of this matrix, rewrite the product as:

$$\sum_e T_{abe}T_{ecd} \equiv \sum_f S_{acf}S_{fbd}.$$

In diagrams, this looks like:



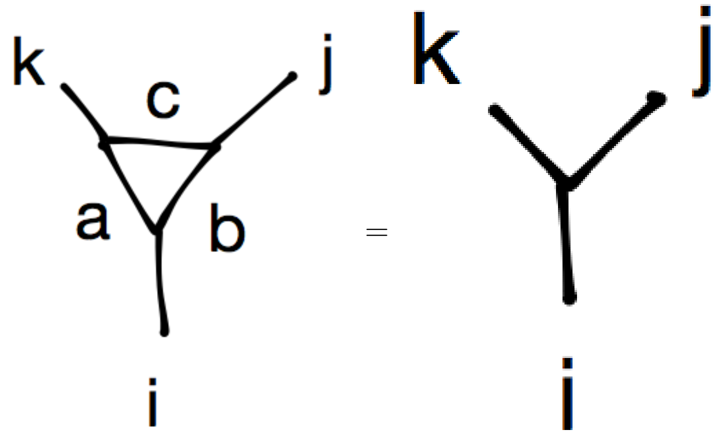
Doing this for a suitable collection of links (as in the figure),



we are left with triangles of Ss. The second step of the coarse-graining scheme is to define a new T by

$$\sum_{a,b,c} S_{kac}S_{jcb}S_{iab} = T'_{kij}$$

or in pictures by:



This gives back an Ising model on the triangular lattice with a larger lattice spacing.

Implement this RG scheme numerically. Notice that the approximation comes in when we throw away singular values in step 1 (if we do not, the range of indices of the tensors (called the *bond dimension*) must grow with the number of steps). Compute the magnetization as a function of temperature.