University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 3 – Solutions

Due 12:30pm Monday, October 22, 2018

1. **Brain-warmer.** Show that the relation $e^{2J'} = \cosh 2J$ can be rewritten as $v' = v^2$ in terms of $v \equiv \tanh J$.

2. **Decimation of 1d Ising model in a field.**

   Consider again a closed (periodic) chain of N classical spins $s_i = \pm 1$ with Hamiltonian

   $$H = -J \sum_i s_i s_{i+1} - h \sum_i s_i + \text{const}, \quad s_{N+1} = s_1$$

   The partition function is $Z(\beta J, \beta h) = \sum_{\{s\}} e^{-\beta H}$; let's measure J, h in units of temperature, *i.e.* set $\beta = 1$.

   Suppose that N is even.

   (a) Decimate the even sites:

   $$\sum_{s_{\text{even}}} e^{-H(s)} \equiv \Delta e^{H_{\text{eff}}(s_{\text{odd}})}.$$

   More explicitly, identify the terms in $H(s)$ that depend on any one even site, $H_2(s)$ and define its contribution to H_{eff} by

   $$\sum_{s_2} e^{-H_2(s)} \equiv \Delta e^{-\Delta H_{\text{eff}}(s_1,s_3)}$$

   Rewriting $H_{\text{eff}}(s_{\text{odd}}) = -J' \sum ss - h' \sum s - \text{const}$ in the usual form, find J', h' and the constant in terms of the microscopic parameters J, h.

   I find

   $$J' = \frac{1}{4\beta} \log \left(\frac{\cosh \beta(2J+h) \cosh \beta(2J-h)}{\cosh^2 \beta h} \right)$$
   $$h' = h + \frac{1}{2\beta} \log \left(\frac{\cosh 2\beta(2J+h)}{\cosh 2\beta(2J-h)} \right)$$
   $$\Delta = 2 \cosh \beta h \sqrt{\cosh \beta(2J+h) \cosh \beta(2J-h)}.$$

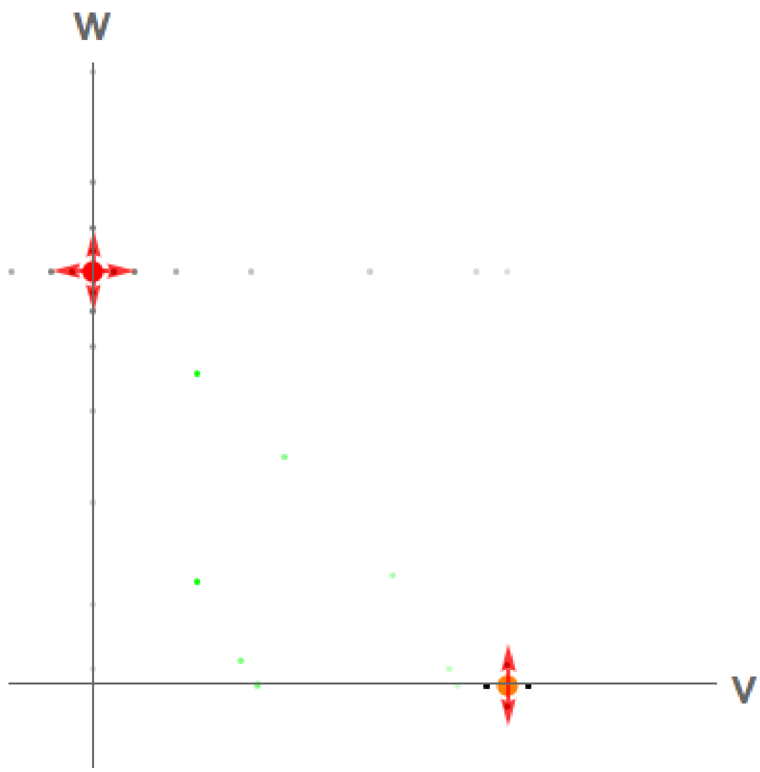
(b) Let $w \equiv \tanh \beta J, v \equiv \tanh \beta h$. Plot some RG trajectories in the v, w plane.

I'm sorry for switching the definitions of v, w relative to lecture.

In terms of v, w the map can (after some unpleasant work, sorry) be simplified to

$$v' = \frac{v(1+w)^2}{1+2v^2w+w^2}, \quad w' = \frac{w^2-1+x}{-w^2+1+x}$$

with $x^2 \equiv (1+w^2)^2 - 4v^2w^2, x > 0$.



(c) Find all the fixed points and compute the exponents near each of the fixed points.

The fixed points are $J^\star = \infty, h^\star = 0$ (*i.e.* $w = 1, v = 0$): the ferromagnetic, strong coupling fixed point, and $J^\star = 0$, any h (*i.e.* $w = 0$, any $v \in [-1, 1]$), the paramagnetic, high-temperature fixed point(s). Near the paramagnetic fixed point, we can linearize a bit to:

$$v' = v, w' = w^2/2.$$

The two eigenvalues of the 2×2 matrix

$$R = \frac{\partial(v', w')}{\partial(v, w)}$$

are

$$y_v = \frac{1}{\log 2} \log \partial_v v'|_{v_\star=0} = -\infty \qquad \textit{very} \text{ irrelevant.}$$

$$y_h = \frac{1}{\log 2} \log \partial_w w'|_{v_\star = 0} = 0 \quad \text{marginal.} \tag{1}$$

(we can read off the eigenvalues from the diagonal elements because one of the off-diagonal elements vanishes). Near the ferromagnetic fixed point, $w' = 2w, t' = 2t$, where $t \equiv 1 - v$. Here we have $y_v = y_t = 1$, two relevant perturbations.

3. **High temperature expansion for Ising model.**

In lecture, we rewrote the partition function of the nearest-neighbor Ising model (on any graph) as a sum over closed loops. Without a magnetic field, the loops were weighted by their length, just like in our discussion of SAWs. If we turn on a magnetic field, how does it change the form of the sum?

Actually, we rewrote the Ising partition sum as a sum over closed loops in two distinct ways: In the high-temperature representation, the loops represented contractions of the spins required to get a nonzero contribution in the spin sums $\sum_{s_i} s_i^k = \delta_{k,\text{even}}$. In the presence of a magnetic field, we can expand $e^{-\beta H}$ as

$$\prod_{\langle ij \rangle} (1 + v s_i s_j) \prod_i (1 + w s_i).$$

Now there is another way to get a factor of s_i – it just costs a factor of w, and allows the curves to end. The weight is now $v^{\text{length(C)}} w^2$, a factor of w for each endpoint. Actually, we may have sites covered by an odd number of links, but let's regard that as one curve ending somewhere in the interior of other curves.

In the low-temperature representation, the loops were domain walls, separating the regions of up spins from the down spins. In this representation, turning on a magnetic field means that a configuration is weighted by the *area* enclosed by the domain walls: the zeeman energy is $h(N_\uparrow - N_\downarrow) = h(N - 2N_\downarrow)$, where N is the total number of sites, and $N_\downarrow$ is the area on the side of the domain walls representing the down spins.