University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 217 Fall 2018
# Assignment 2

**Due 12:30pm Monday, October 15, 2018**

This homework begins with two simple problems to warm up your brain.

1. **Brain-warmers.**

   (a) Check from the 'bridge' equation $F = -T \log Z$ that $F = E - TS$ where $E$ is the average energy and $S = -\partial_T F$ is the entropy.

   (b) What is the relationship between the spin-spin correlation function $\langle s_i s_j \rangle$ and the probability that $s_i$ and $s_j$ are pointing in the same direction? (Hint: $(1 + s_i s_j)/2$ is a projector.)

2. **Biased unrestricted random walk.** Find the RMS size $R(M) \equiv \sqrt{\langle |\vec{R}_M|^2 \rangle_M}$ of the biased unrestricted random walk (each step is drawn from the distribution $p(\vec{r}) \propto e^{-\frac{|\vec{r}-\vec{r}_0|^2}{2\sigma^2}}$) of $M$ steps. Show that when $M \gg 1$, this is $R(M) \to M|\vec{r}_0|$.

3. **Behavior near fixed-points.** Find the fixed points of the following map

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \mapsto \mathcal{R}(a,b) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - b + a^2/2 \\ (2-a)b \end{pmatrix} .$$

   Linearize the map about each of the fixed points and compute the scaling dimensions assuming this map is an RG transformation with zoom factor $\lambda = 2$. Draw a phase portrait indicating the fixed points and some of the flows into and out of them. Be careful of the signs.

4. **Other fixed points of random walk.** [This problem is optional. Thanks to Brian Vermilyea for suggesting this example.] Show that the Lorentzian distribution $p_\sigma(\vec{r}) = \frac{\sigma/\pi}{\vec{r}^2+\sigma^2}$ is a fixed point of the coarse-graining transformation that takes $\vec{r} \to \vec{r}' = \sum_{i=1}^n \vec{r}_i$, *i.e.*

$$P(\vec{r}'') = p_\sigma(\vec{r})$$

   for some appropriate rescaling $r'' = n^a r'$.

5. **Real-space RG for the SAW.** Consider again the problem of a self-avoiding walk on the square lattice. Construct an RG scheme with zoom factor $\lambda = 3$ (so that nine

sites of the fine lattice are represented by one site of the coarse lattice). Find the RG map $K'(K)$, find its fixed points and estimate the critical exponent at the nontrivial fixed point. Is it closer to the numerical result than the $\lambda = 2$ schemes discussed in lecture and by Creswick?

6. **Ising model in 1d by transfer matrix.**

   Consider a closed (periodic) chain of $N$ classical spins $s_i = \pm 1$ ($s_{N+1} = S_1$) with Hamiltonian

   $$H = -J \sum_i s_i s_{i+1} - h \sum_i s_i + \text{const}, \quad s_{N+1} = s_1$$

   The partition function is $Z(\beta J, \beta h) = \sum_{\{s\}} e^{-\beta H}$; let's measure $J, h$ in units of temperature, *i.e.* set $\beta = 1$.

   (a) Show that the partition function can be written as

   $$Z = \text{tr}_2 T^N$$

   where $T$ is the $2 \times 2$ matrix

   $$T = \begin{pmatrix} e^{J+h} & e^{-J} \\ e^{-J} & e^{J-h} \end{pmatrix}$$

   (called the *transfer matrix*) and $\text{tr}_2 M = M_{11} + M_{22}$ denotes trace in this two-dimensional space. Express $Z$ in terms of the eigenvalues of $T$ and find the free energy density $f = -\frac{T}{N} \log Z$ in the thermodynamic ($N \to \infty$) limit. Plot the free energy for $h = 0$ as a function of $x = e^{-4J}$ for $0 \leq x \leq 1$.

   (b) Find an expression for the correlation function

   $$G(m) \equiv \langle s_i s_{m+i} \rangle - \langle s_i \rangle \langle s_{i+m} \rangle$$

   using the transfer matrix. Show that as $N \to \infty$,

   $$G(m) \sim e^{-m/\xi}$$

   where $\xi = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)}$ where $\lambda_1 > \lambda_2$ are eigenvalues of $T$. Note that $\xi \to \infty$ when $\lambda_1 \to \lambda_2$. For what values of $\beta, h, J$ does this happen?