University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 2 – Solutions

Due 12:30pm Monday, October 15, 2018

This homework begins with two simple problems to warm up your brain.

1. **Brain-warmers.**

   (a) Check from the 'bridge' equation $F = -T \log Z$ that $F = E - TS$ where E is the average energy and $S = -\partial_T F$ is the entropy.

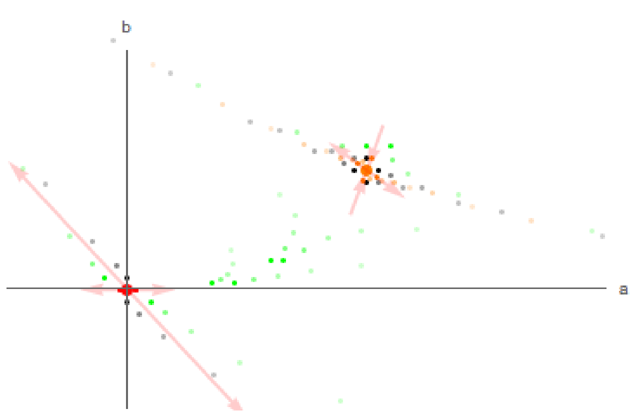
   (b) What is the relationship between the spin-spin correlation function $\langle s_i s_j \rangle$ and the probability that s_i and s_j are pointing in the same direction? (Hint: $(1 + s_i s_j)/2$ is a projector.)

2. **Biased unrestricted random walk.** Find the RMS size $R(M) \equiv \sqrt{\langle |\vec{R}_M|^2 \rangle_M}$ of the biased unrestricted random walk (each step is drawn from the distribution $p(\vec{r}) \propto e^{-\frac{|\vec{r}-\vec{r}_0|^2}{2\sigma^2}}$) of M steps. Show that when $M \gg 1$, this is $R(M) \to M|\vec{r}_0|$.

3. **Behavior near fixed-points.** Find the fixed points of the following map

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \mapsto \mathcal{R}(a,b) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - b + a^2/2 \\ (2-a)b \end{pmatrix} .$$

   Linearize the map about each of the fixed points and compute the scaling dimensions assuming this map is an RG transformation with zoom factor $\lambda = 2$. Draw a phase portrait indicating the fixed points and some of the flows into and out of them. Be careful of the signs.



Plotted here are trajectories of points under repeated application of the map $\mathcal{R}$ (later iterates are more transparent). Also plotted are the eigenvectors of the scaling matrix

$$R \equiv \begin{pmatrix} \partial_a \mathcal{R}_a & \partial_b \mathcal{R}_a \\ \partial_a \mathcal{R}_b & \partial_b \mathcal{R}_b \end{pmatrix}$$

at the two fixed points, which can be seen to follow the nearby trajectories. I find three fixed points: $p_0 = (0,0)$ and $p_1 = (1, 1/2)$ and $p_\infty = (\infty, \infty)$. The origin has scaling matrix

$$R_0 = R|_{(a,b)=(0,0)} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

with eigenvalues $1, 2$, which correspond to marginal and irrelevant perturbations respectively. In terms of scaling dimensions (with zoom factor $\lambda = 2$), this is $y = 0, 1$. The other fixed point has scaling matrix

$$R_1 = \begin{pmatrix} 2 & -1/2 \\ -1 & 1 \end{pmatrix}$$

with much uglier eigenvalues .63 and 2.4. The first eigenvector is irrelevant and the second is relevant. With zoom factor $\lambda = 2$, the associated scaling dimensions are

$$y = \log_2 \rho = -0.657, 1.24$$

The fact that fixed point at $(0,0)$ has a marginal perturbation in the $\pm a$ direction is manifested by the piling up of nearby points on the a-axis. This perturbation is *marginally irrelevant*, meaning that a point displaced from $(0,0)$ by $(-\epsilon, 0)$ does eventually return to $(0,0)$.

4. **Other fixed points of random walk.** [This problem is optional. Thanks to Brian Vermilyea for suggesting this example.] Show that the Lorentzian distribution $p_\sigma(\vec{r}) = \frac{\sigma/\pi}{\vec{r}^2+\sigma^2}$ is a fixed point of the coarse-graining transformation that takes $\vec{r} \to \vec{r}' = \sum_{i=1}^n \vec{r}_i$, *i.e.*

$$P(\vec{r}'') = p_\sigma(\vec{r})$$

for some appropriate rescaling $r'' = n^a r'$.

Let's study the problem in one dimension for simplicity. First find the fourier transform:

$$g(k) = \int_{-\infty}^{\infty} dr\; e^{\mathbf{i}kr} \frac{\sigma/\pi}{r^2+\sigma^2} = e^{-\sigma|k|}$$

where the integral can be done by residues, casing on the sign of k. Then

$$P(r_n) = \int \prod_{i=1}^{n} dr_i p_\sigma(r_i)\delta\left(r_n - \sum r_i\right) \tag{1}$$

$$= \int \prod_{i=1}^{n} \left(dr_i đk_i e^{-\mathbf{i}k_i r_i} g(k_i)\right) \int đk e^{-\mathbf{i}r_n k + \mathbf{i}\sum_i r_i k} \tag{2}$$

$$= \int đk e^{-\mathbf{i}r_n k} \int \prod_i đk_i g(k_i) \int \prod_j \left(\underbrace{dr_i e^{\mathbf{i}r_i(k-k_i)}}_{=2\pi\delta(k-k_i)}\right) \tag{3}$$

$$= \int đk g(k)^n e^{-\mathbf{i}r_n k} \tag{4}$$

$$= \int đk e^{-n\sigma|k|} e^{-\mathbf{i}r_n k} \tag{5}$$

$$= p_{n\sigma}(r_n), \tag{6}$$

the same distribution with $\sigma \to n\sigma$. So under the rescaling $r_n = nr''$, the distribution comes back to itself.

5. **Real-space RG for the SAW.** Consider again the problem of a self-avoiding walk on the square lattice. Construct an RG scheme with zoom factor $\lambda = 3$ (so that nine sites of the fine lattice are represented by one site of the coarse lattice). Find the RG map $K'(K)$, find its fixed points and estimate the critical exponent at the nontrivial fixed point. Is it closer to the numerical result than the $\lambda = 2$ schemes discussed in lecture and by Creswick?

   Actually the exponent is farther away. Sorry.

6. **Ising model in 1d by transfer matrix.**

   Consider a closed (periodic) chain of N classical spins $s_i = \pm 1$ ($s_{N+1} = S_1$) with Hamiltonian

   $$H = -J\sum_i s_i s_{i+1} - h\sum_i s_i + \text{const}, \quad s_{N+1} = s_1$$

   The partition function is $Z(\beta J, \beta h) = \sum_{\{s\}} e^{-\beta H}$; let's measure J, h in units of temperature, *i.e.* set $\beta = 1$.

   (a) Show that the partition function can be written as

   $$Z = \text{tr}_2 T^N$$

   where T is the 2×2 matrix

   $$T = \begin{pmatrix} e^{J+h} & e^{-J} \\ e^{-J} & e^{J-h} \end{pmatrix}$$

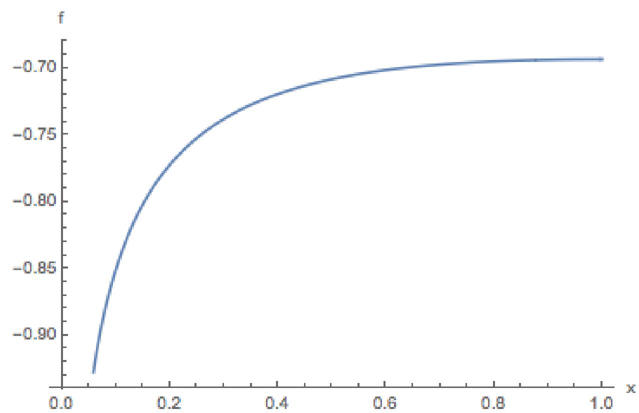
(called the *transfer matrix*) and $\text{tr}_2 M = M_{11} + M_{22}$ denotes trace in this two-dimensional space. Express Z in terms of the eigenvalues of T and find the free energy density $f = -\frac{T}{N}\log Z$ in the thermodynamic ($N \to \infty$) limit. Plot the free energy for $h = 0$ as a function of $x = e^{-4J}$ for $0 \leq x \leq 1$.

The real symmetric matrix $T = O\Lambda O^T$ can be diagonalized and

$$Z = \text{tr}T^N = \text{tr}\Lambda^N = \lambda_+^N + \lambda_-^N.$$

Assuming $\lambda_+ > \lambda_-$, we have

$$f = -\frac{1}{N}T\ln Z \overset{N\to\infty}{\to} -T\ln\lambda_+.$$



Notice it approaches $\ln 2$ at high temperature.

(b) Find an expression for the correlation function

$$G(m) \equiv \langle s_i s_{m+i}\rangle - \langle s_i\rangle\langle s_{i+m}\rangle$$

using the transfer matrix. Show that as $N \to \infty$,

$$G(m) \sim e^{-m/\xi}$$

where $\xi = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)}$ where $\lambda_1 > \lambda_2$ are eigenvalues of T. Note that $\xi \to \infty$ when $\lambda_1 \to \lambda_2$. For what values of β, h, J does this happen?

$$\langle s_i s_{i+r}\rangle = \frac{1}{Z}\mathrm{tr}\left(T^i\sigma^z T^r\sigma^z T^{N-i-r}\right) \tag{7}$$

$$= \frac{1}{Z}\mathrm{tr}\Lambda^{N-r}\underbrace{O^T\sigma^z O}_{\equiv S}\Lambda^r O^T\sigma^z O \tag{8}$$

$$= \frac{1}{Z}\sum_\ell \langle\ell|\lambda_\ell^{N-r}S\sum_m |m\rangle\langle m|S\lambda^r|\ell\rangle \tag{9}$$

$$= \frac{1}{Z}\sum_\ell \lambda_\ell^{N-r}\lambda_m^r\langle\ell|S|m\rangle\langle m|S|\ell\rangle \tag{10}$$

$$= \frac{\lambda_+^N S_{++}^2 + \lambda_-^N S_{--}^2 + \lambda_+^r\lambda_-^{N-r}S_{+-}S_{-+} + \lambda_-^r\lambda_+^{N-r}S_{+-}S_{-+}}{\lambda_+^N + \lambda_-^N} \tag{11}$$

$$\overset{N\to\infty}{\to} S_{++}^2 + \frac{\lambda_-^r}{\lambda_+^r}S_{+-}^2 \tag{12}$$

Here $\mathbb{1} = \sum_\ell |\ell\rangle\langle\ell|$ is a resolution of the identity in the T eigenbasis. Similarly,

$$\langle s_i\rangle = \frac{1}{Z}\mathrm{tr}\left(T^N\sigma^z\right) = \frac{\sum_\ell S_{\ell\ell}\lambda_\ell}{Z} \overset{N\to\infty}{\to} S_{++},$$

so the connected correlator is

$$G(r) = \langle s_i s_{i+r}\rangle - \langle s_i\rangle\langle s_r\rangle \overset{N\to\infty}{\to} \frac{\lambda_-^r}{\lambda_+^r}S_{+-}^2 = e^{-\frac{r}{\xi}}.$$

This gives $\xi = \ln\frac{\lambda_+}{\lambda_-}$. For $h = 0$ $\langle s\rangle = \mathrm{tr}T^N\sigma^z = 0$, so the disconnected bit vanishes.

The condition for infinite correlation length $\lambda_+ = \lambda_-$ implies

$$0 = \cosh^2\beta h + e^{-4\beta J} - 1 = \sinh^2\beta h + e^{-4\beta J}$$

but both terms are positive, and so must vanish separately. So the only solution is $h = 0, \beta J = \infty$.