University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 1

Due 12:30pm Monday, October 8, 2018

0. What kind of physics are you interested in working on?

1. **Finally, this is what you are supposed to be doing!** Make some fractals and compute their fractal dimensions. (This is something I enjoy, but when I do I feel guilty for not doing something more obviously productive. So here is an opportunity for you to do it without guilt – it's your homework.)

   Write a computer script (*e.g.* in Mathematica) that will draw them for you.

   Can you make a Sierpinski-like fractal whose fractal dimension is not of the form $\frac{\log m}{\log n}$ for $m, n \in \mathbb{Z}$? For the purposes of this problem, let's define 'Sierpinski-like' to mean that you could make a picture of it with a recursive function using something like Mathematica.

2. **Failed Cantor set.** Suppose that in the defining procedure for the Cantor set, you remove the right third of the interval instead of the middle one. What is the fractal dimension of the resulting set? Where does the scaling argument given in class fail?

3. **Random walk in one dimension.** What is the fractal dimension of a one-dimensional random walk? Explain where the argument from lecture (that leads to $D = 2$) breaks down.

4. **The Temple's Cube.** *Lieutenant Curtis reported back to Captain Smith on their visit to the seat of the government of the Hrunkla. The king sat upon a giant cube which was broken down into smaller cubes all of the same size (consisting of four layers of cubes each one four by four). There were 16 white cubes...16 black cubes...16 green cubes....and 16 red cubes.... It was asserted that each vertical direction and each horizontal direction [and each third direction] through the cube had exactly one small cube of each color. Smith doubted this, but Curtis showed him how it might be done. How?* This interesting story (slightly edited for brevity) was from my son Isaac's homework from when he was in 4th grade. (For some reason, his current 7th grade homework is not nearly as interesting.) Here are my questions for you:

   (a) Suppose that there are 2^n colors, and $\frac{(2^n)^3}{2^n} = 2^{2n}$ blocks of each color. Find a self-similar solution of this generalization of the problem, that is, a giant $2^n \times 2^n \times 2^n$

cube with one block of each color in every row, column and whatever the third thing is called. Construct the solution hierarchically: use a solution for one value of n to construct a solution for next larger value where those cubes are subdivided.

The remaining parts of the problem are optional.

(b) Find a condition on the solution with only 4 colors which picks out the $n = 2$ case of the self-similar solution from the other possible solutions.

(c) Given a solution of the problem above, what operations map it to another solution? What is this group?

(d) Tell me about other generalizations of this problem. The version with squares instead of cubes is a good warmup.

(e) [super-bonus part] Find a set of energetic constraints on the configurations of the cubes (*e.g.* assign numbers to the colors) which favors the configurations described above.

(f) [super^2-bonus part] Find a quantum many-body physics application of these insights. For example, the condition that the colors be different could be a consequence of Fermi statistics.

5. **Open-ended question.** Suppose you know a function $\rho(\vec{r})$ which is nonzero on the support of a self-similar object and 0 elsewhere. Can you relate the fractal dimension of the object to properties of the indicator function $\rho(\vec{r})$?

   Comment/hint: There is an annoying question of what the value of ρ should be on the set $\mathcal{O}$. The case where $\rho(r \in \mathcal{O}) = 1$ is called an *indicator function*. More interesting probably is the *density* $\rho_{\mathcal{O}}(r) = \sum_{x \in \mathcal{O}} \delta(r - x)$, which has finite moments, generated by

$$\mathcal{A}(k) \equiv \langle\langle e^{\mathbf{i}\vec{k}\cdot\vec{r}} \rangle\rangle_{\mathcal{O}} \equiv \int dr \rho_{\mathcal{O}}(r) e^{\mathbf{i}\vec{k}\cdot\vec{r}},$$

   whose square is the structure factor $S(k) = |\mathcal{A}(k)|^2$.