University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Fall 2018
Assignment 1 – Solutions

Due 12:30pm Monday, October 8, 2018

0. What kind of physics are you interested in working on?

1. **Finally, this is what you are supposed to be doing!** Make some fractals and compute their fractal dimensions. (This is something I enjoy, but when I do I feel guilty for not doing something more obviously productive. So here is an opportunity for you to do it without guilt – it's your homework.)

   Write a computer script (*e.g.* in Mathematica) that will draw them for you.

   Can you make a Sierpinski-like fractal whose fractal dimension is not of the form $\frac{\log m}{\log n}$ for $m, n \in \mathbb{Z}$? For the purposes of this problem, let's define 'Sierpinski-like' to mean that you could make a picture of it with a recursive function using something like Mathematica.

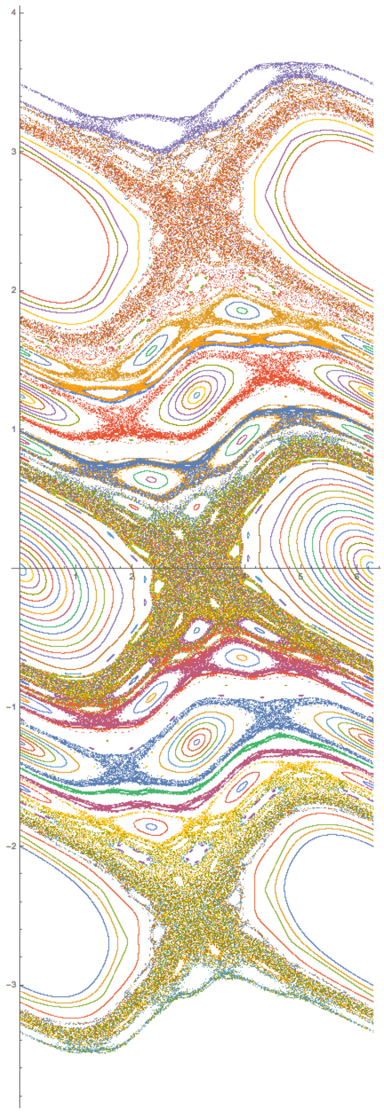
   Here is the Mathematica file where I made the fractals in the notes.

   Here is an infinite sierpinski triangle with javascript. Here is another javascript fractal (select a region to zoom). Computing its fractal dimension would have to be done numerically.

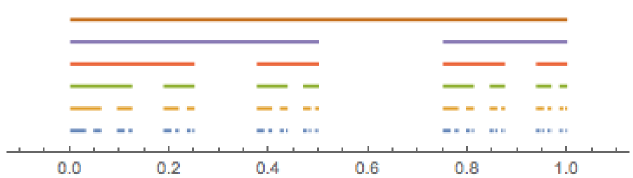
   One way to get a fractal dimension different from $\frac{\log m}{\log n}$ with $m, n \in \mathbb{Z}$ is to use the Cantor procedure, but remove the middle ath of each segment, where a is an irrational number.

   Another way might be to have the rule break the object up into sub-objects of different sizes. This will give something like $D = \frac{\log m + \log m'}{\log n}$. Oops, but $\log m + \log m' = \log mm'$ so if m, m' are integers this fails.

Maybe this is cheating, but here's a fractal I like. You can find out what it is here. I don't know what is its fractal dimension.



Here is an interesting example I learned from Aria Yom. Consider the limit of the process which keeps the third quarter of each interval:



Then

$$N(1) = 1, N(1/2) = 2, N(1/4) = 3, N(1/8) = 5, N(1/16) = 8...,$$

Fibonacci numbers. These satisfy

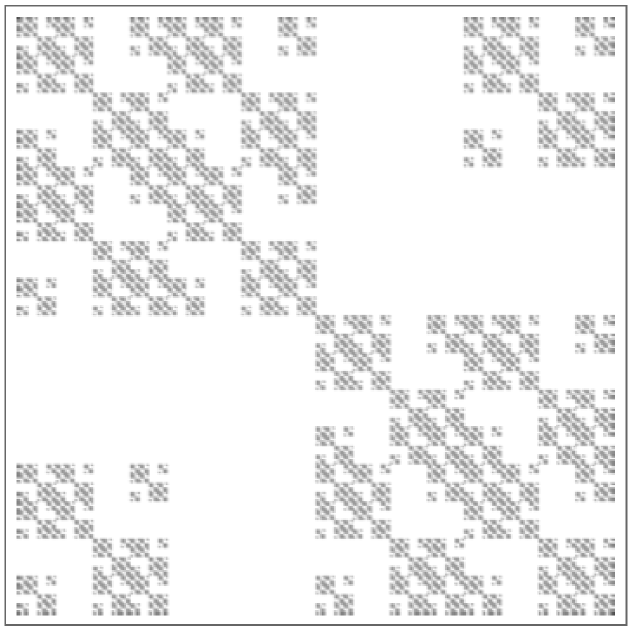
$$N(a) = N(a/2) + N(a/4)$$

which is solved by $N(a) \sim a^{-D}$ with

$$1 = 2^{-D} + 2^{-2D},$$

so that $2^{-D} = \frac{-1+\sqrt{5}}{2} = \phi - 1$ where ϕ is the golden ratio, and $D = -\log_2(\phi - 1)$.

A generalization of this in two dimensions looks like:



for which the numbers of squares of size a needed to cover the set satisfy

$$N(a) = 2N(a/2) + 2N(a/4)$$

whence $N(a) \sim a^{-D}$ with $1 = 2 \cdot 2^{-D} + 2 \cdot 2^{-2D}$, or $D = -\log(\sqrt{3} - 1)/2$.

2. **Failed Cantor set.** Suppose that in the defining procedure for the Cantor set, you remove the right third of the interval instead of the middle one. What is the fractal dimension of the resulting set? Where does the scaling argument given in class fail?

   The nth iterate has just one connected component which shrinks as $n \to \infty$. The dimension is just zero.

3. **Random walk in one dimension.** What is the fractal dimension of a one-dimensional random walk? Explain where the argument from lecture (that leads to $D = 2$) breaks down.

   Since it's a subset of $\mathbb{R}^1$, the dimension is bounded above by $D \leq 1$. Since it's made from line segments, it's bounded below by $d_T = 1 \leq D$. So $D = 1$. In one dimension, the steps of the walk necessarily overlap each other half the time. This allows the set to be covered with many fewer balls than in higher dimensions.

4. **The Temple's Cube.** *Lieutenant Curtis reported back to Captain Smith on their visit to the seat of the government of the Hrunkla. The king sat upon a giant cube which was broken down into smaller cubes all of the same size (consisting of four layers of cubes each one four by four). There were 16 white cubes...16 black cubes...16 green cubes....and 16 red cubes.... It was asserted that each vertical direction and each horizontal direction [and each third direction] through the cube had exactly one small cube of each color. Smith doubted this, but Curtis showed him how it might be done. How?* This interesting story (slightly edited for brevity) was from my son Isaac's homework from when he was in 4th grade. (For some reason, his current 7th grade homework is not nearly as interesting.) Here are my questions for you:

   (a) Suppose that there are 2^n colors, and $\frac{(2^n)^3}{2^n} = 2^{2n}$ blocks of each color. Find a self-similar solution of this generalization of the problem, that is, a giant $2^n \times 2^n \times 2^n$ cube with one block of each color in every row, column and whatever the third thing is called. Construct the solution hierarchically: use a solution for one value of n to construct a solution for next larger value where those cubes are subdivided.

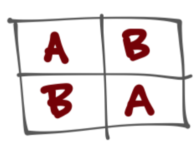
   Here is the RG solution I had in mind:

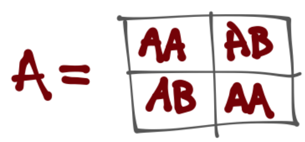
   • first the case with $n = 1$. label the two colors $a = 0$ and $a = 1$. place the color a at the lattice sites with coordinates (x, y, z) with $x + y + z = a \mod 2$.

   • for $n = 2$, label the four colors in binary $\{00, 01, 10, 11\} \equiv A_4$. make a lattice with half the lattice spacing, so that the sites have coordinates (x, y, z) with each of x, y, z chosen from the set A_4.

   • continue.

   Here is a pictorial explanation of the first two steps, for the 2d case:





The remaining parts of the problem are optional.

   (b) Find a condition on the solution with only 4 colors which picks out the $n = 2$ case of the self-similar solution from the other possible solutions.

   (c) Given a solution of the problem above, what operations map it to another solution? What is this group?

   (d) Tell me about other generalizations of this problem. The version with squares instead of cubes is a good warmup.

   (e) [super-bonus part] Find a set of energetic constraints on the configurations of the cubes (*e.g.* assign numbers to the colors) which favors the configurations described above.

(f) [super^2-bonus part] Find a quantum many-body physics application of these insights. For example, the condition that the colors be different could be a consequence of Fermi statistics.

5. **Open-ended question.** Suppose you know a function $\rho(\vec{r})$ which is nonzero on the support of a self-similar object and 0 elsewhere. Can you relate the fractal dimension of the object to properties of the indicator function $\rho(\vec{r})$?

   Comment/hint: There is an annoying question of what the value of ρ should be on the set $\mathcal{O}$. The case where $\rho(r \in \mathcal{O}) = 1$ is called an *indicator function*. More interesting probably is the *density* $\rho_{\mathcal{O}}(r) = \sum_{x\in\mathcal{O}} \delta(r - x)$, which has finite moments, generated by

$$\mathcal{A}(k) \equiv \langle\langle e^{\mathbf{i}\vec{k}\cdot\vec{r}}\rangle\rangle_{\mathcal{O}} \equiv \int dr \rho_{\mathcal{O}}(r) e^{\mathbf{i}\vec{k}\cdot\vec{r}},$$

   whose square is the structure factor $S(k) = |\mathcal{A}(k)|^2$.

   I wish the statement of this problem had been: suppose you measure the structure factor $S(k)$ of a fractal, *e.g.* by scattering light off of it. How do you extract the fractal dimension?

   The indicator function for a set of dimension less than 1 has measure zero, so all of its moments are zero. This is a fake problem – there is always a short-distance cutoff. Let us consider a Cantor-like fractal at some finite number of steps of the iteration procedure, so it is made up of finite but small intervals.

   If each of m copies of the fractal at positions r_i reproduce the whole fractal with zoom factor λ, then the indicator function satisfies

$$\rho(r) = \sum_{i=1}^{m} \rho(\lambda(r - r_i)).$$

   (For example, for the Sierpinski triangle $\lambda = 2$ and r_i are the corners of the surviving sub-triangles.) The fourier transform (generator of moments) then satisfies

$$\begin{aligned}
\mathcal{A}(k) &= \int d^d r e^{-\mathbf{i}kr} \rho(r) && (1)\\
&= \int d^d r e^{-\mathbf{i}kr} \sum_{i=1}^{m} \rho(\lambda(r - r_i)) && (2)\\
&= \sum_i \lambda^{-d} \int d^d r' e^{-\mathbf{i}(k/\lambda)r' - \mathbf{i}kr_i} \rho(r') \qquad (r' = \lambda(r - r_i)) && (3)\\
&= \lambda^{-d} \mathcal{A}(k/\lambda) \sum_i e^{-\mathbf{i}kr_i}. && (4)
\end{aligned}$$

   For long wavelength ($k \to 0$), we have $\sum_{i=1}^{m} e^{-\mathbf{i}kr_i} \sim m + \mathcal{O}(k^2)$, and the small-$k$ $\mathcal{A}(k)$ therefore satisfies

$$\mathcal{A}(k) = m\lambda^{-d}\mathcal{A}(k/\lambda).$$

This says

$$\mathcal{A}(k) \overset{k\to 0}{\sim} k^{\alpha}, \quad \alpha = \frac{\log m}{\log \lambda} - d = D - d$$

where D is the Hausdorff (ball-counting) dimension.

An interesting answer suggested by Feng Chen and Ethan Villarama suggests that we look at a subregion of the fractal, say a cube R^d of side length R and consider: $\int_{R^d} \rho(r) d^d r$ If we compare this number to a scaled-up region, then the ratio

$$\frac{\int_{\lambda R^d} \rho(r) d^d r}{\int_{R^d} \rho(r) d^d r}$$

will depend on λ by a power-law relation which depends on D. (I'm actually a little confused at the moment about what the power is.)

Another answer, for a fractal embedded in $\mathbb{R}^d$ with indicator function ρ, seems to be to consider the behavior of

$$F_{\rho,p}(h) \equiv \int d^d x |\rho(x+h) - \rho(x)|^p$$

as the offset $h \to 0$. According to Abry et al this should be a power law

$$F_{\rho,p}(h) \overset{h\to 0}{\sim} h^{\eta_\rho(p)}$$

and the power is

$$\eta_\rho(p) = d - D$$

where D is the fractal dimension, independent of p.