

Physics 217: The Renormalization Group

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0.1 Introductory comments

The ‘renormalization group’ (RG) is a poor name for the central concept in many-body physics. It is a framework for addressing the question: what is the relationship between microscopic laws and macroscopic observations?

Or, closer to home, it allows us to answer questions such as: Why don’t you need to understand nuclear physics to make your tea in the morning?¹

Briefly, the RG is the realization that systems of many degrees of freedom (especially when they have local interactions) should be understood *hierarchically*, *i.e.* scale by scale.

There is a lot more to say to contextualize the RG, which, as you can see from the previous question, is really a piece of metaphysics, that is, it is a framework for how to do physics. But since it is such a broad and far-reaching concept, in order to avoid being vague and useless, it will be better to start with some concrete and simple ideas, before discussing of some of its many consequences.

A word about prerequisites: The official prerequisite for this course is graduate statistical mechanics. I think you would be able to get by with a good undergrad class.

The historical origins of the RG (at least its name) are tied up with high-energy particle physics and quantum field theory. That stuff involves quantum mechanics in a serious way. Much of the content of this course can be understood without quantum mechanics; the fluctuations could all be thermal. At various points along the way I will point out the connections with quantum field theory.

So this is mostly a course in statistical field theory (\equiv statistical mechanics of many degrees of freedom). But there are many other applications of the RG which don’t quite fit in this category which I also hope to discuss.

Also, I think our discussion will all be non-relativistic, $v \ll c$.

Initial Tentative Plan:

1. Scaling and self-similarity
2. RG treatment of random walks
3. Ising models
4. Critical phenomena (a great victory of the RG). $4 - \epsilon$ expansions

¹This framing of the question I heard from Savas Dimopoulos.

5. RG treatment of iterative maps and the period-doubling approach to chaos
6. RG treatment of percolation and lattice animals
7. RG understanding of the method of matched asymptotic expansions
8. RG treatment of stochastic PDEs

As the title indicates, this is a very rough guess for what we'll do. An early target will be a renormalization-group understanding of the central limit theorem.

Sources for these notes (anticipated):

Introduction to Renormalization Group Methods in Physics, by R. J. Creswick, H. A. Farach, C. P. Poole.

Lectures on Phase Transitions and the Renormalization Group, by Nigel Goldenfeld.

Statistical Physics of Fields, by Mehran Kardar.

Introduction to Statistical Field Theory, by Eduard Brézin.

Renormalization Group Methods, a guide for beginners, by W. D. McComb.

Scaling and Renormalization in Statistical Physics, by John Cardy.

Modern Theory of Critical Phenomena, by S.-K. Ma (UCSD).

Statistical Field Theory, by David Tong.

0.2 Conventions

The convention that repeated indices are summed is always in effect unless otherwise indicated.

A useful generalization of the shorthand $\hbar \equiv \frac{\hbar}{2\pi}$ is

$$\mathfrak{d}k \equiv \frac{dk}{2\pi}.$$

I will also write $\delta(q) \equiv (2\pi)^d \delta^d(q)$.

I will try to be consistent about writing fourier transforms as

$$\int \frac{d^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k) \equiv \int \mathfrak{d}^d k e^{ikx} \tilde{f}(k) \equiv f(x).$$

RHS \equiv right-hand side.

LHS \equiv left-hand side.

BHS \equiv both-hand side.

IBP \equiv integration by parts.

$+\mathcal{O}(x^n)$ \equiv plus terms which go like x^n (and higher powers) when x is small.

I write log for base two and ln for base e .

$A \stackrel{!}{=} B$ means we demand $A = B$.

$A \equiv B$ is a definition.

I reserve the right to add to this page as the notes evolve.

Please send me email if you find typos or errors or violations of the rules above.

1 Scaling and self-similarity

[This discussion largely follows the early chapters of the book by Creswick et al.] First some somewhat-vague definitions to get us started. An object is *self-similar* if its parts, when magnified by a suitable scale factor λ , look like the whole. (Here is an example.) Something is *scale-invariant* if this is true for every λ . (Self-similarity is sometimes called ‘discrete scale invariance’.) An important generalization is the notion of *statistical self-similarity* – something which is sampled from a distribution which is self-similar.

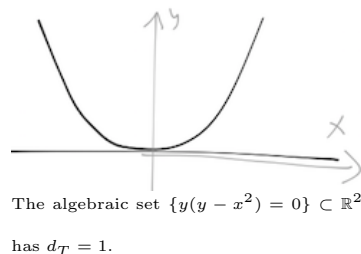
The point in life of the renormalization group is to provide a way of thinking about (and ideally relating quantitatively) what’s going on at different scales of magnification. So something which is self-similar or scale-invariant is a simple special case for the RG.

As we’ll see, a symptom of scale invariance is a power law.

1.1 Fractal dimension

The word ‘dimension’ is used in many ways in this business. Let’s consider a set of points in d -dimensional Euclidean space, \mathbb{R}^d . In the previous sentence ‘dimension’ is the minimum number of coordinates needed to specify the location of a point (this is usually called ‘Euclidean dimension’). It’s an integer.

A subset of \mathbb{R}^d specified by some algebraic equations on the coordinates (we can call this an algebraic set) generically has a Euclidean dimension which is an integer (though it may not be the same integer for every point). That is, locally around almost every solution of the equations, the object will look like a piece of \mathbb{R}^{d_T} for some $d_T \leq d$ (sometimes this notion is called ‘topological dimension’).



Here is a different, RG-inflected definition of the dimension of an object $O \subset \mathbb{R}^d$, called fractal dimension or Hausdorff dimension: cover the object O with d -balls of diameter a ,

$$B_{r_0}(a) \equiv \{\vec{r} \in \mathbb{R}^d \text{ such that } |\vec{r} - \vec{r}_0| \leq a/2\}. \quad (1.1)$$

Let

$N(a) \equiv$ the minimum number of such balls required to cover O ,

minimizing over the locations of their centers. Do this for various values of a . Then, if this function is a power law,

$$N(a) \sim a^{-D} \quad (1.2)$$

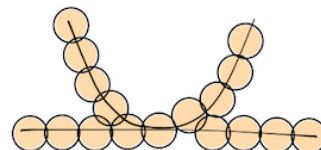
then D is the fractal dimension of O . Even if $N(a)$ is not a power law, we can define $D \equiv -\log_a N(a)$.

A few observations:

- Notice that D may itself depend on the range of ball-sizes a we consider, that is, the same scaling relation may not hold for all a . Often (always) there is a short-distance (“UV”) cutoff on the regime where the scaling relation (1.2) holds – if our object is the coastline of France, it is maybe not so useful to consider femtometer-sized balls. Also, there is often a long-distance (“IR”) cutoff – in the same example, Earth-sized balls will not give an interesting power law (it just gives $N(r_{\oplus}) = 1$).

For objects defined by algebraic equations, $D = d_T$. For

- example, in the example at right, the required number of balls of size a goes like $1/a$.



- O is a set of points in \mathbb{R}^d , the dimension of the objects composing O (maybe points themselves, maybe line segments...) have some topological dimension d_T , and

$$d_T \leq D \leq d.$$

Where does the right bound come from? By placing d -balls centered at the sites of a cubic lattice (with a diameter a proportional to the lattice spacing) we can cover a whole region of \mathbb{R}^d with a number that goes like a^{-d} .

It behooves us to give some examples with $D \notin \mathbb{Z}$. Such things are called *fractals* and are often defined by a recursive process.

1. A Cantor set in $d = 1$ can be defined beginning with a line segment of length a_0 . Divide it in thirds and remove the middle segment. Repeat for each sub-segment. At the end of this process, we get a bunch of points ($d_T = 0$) in $d = 1$. According to our definition (1.1), 1-ball of diameter a is an interval of length a . After n steps of the above procedure, we end up with $2^n \equiv N$ line segments of length $a_n = a_0 3^{-n}$, $n = 1, 2, 3, \dots$ (that’s what I’m drawing). Since we only remove stuff, we can cover the whole thing with these, so we have a lower bound of $N(a_n) = 2^n$, and you can’t do better. Eliminate n : $n = -\frac{\log \frac{a_n}{a_0}}{\log 3}$ (think of this as $n(a) = -\frac{\log a/a_0}{\log 3}$), so

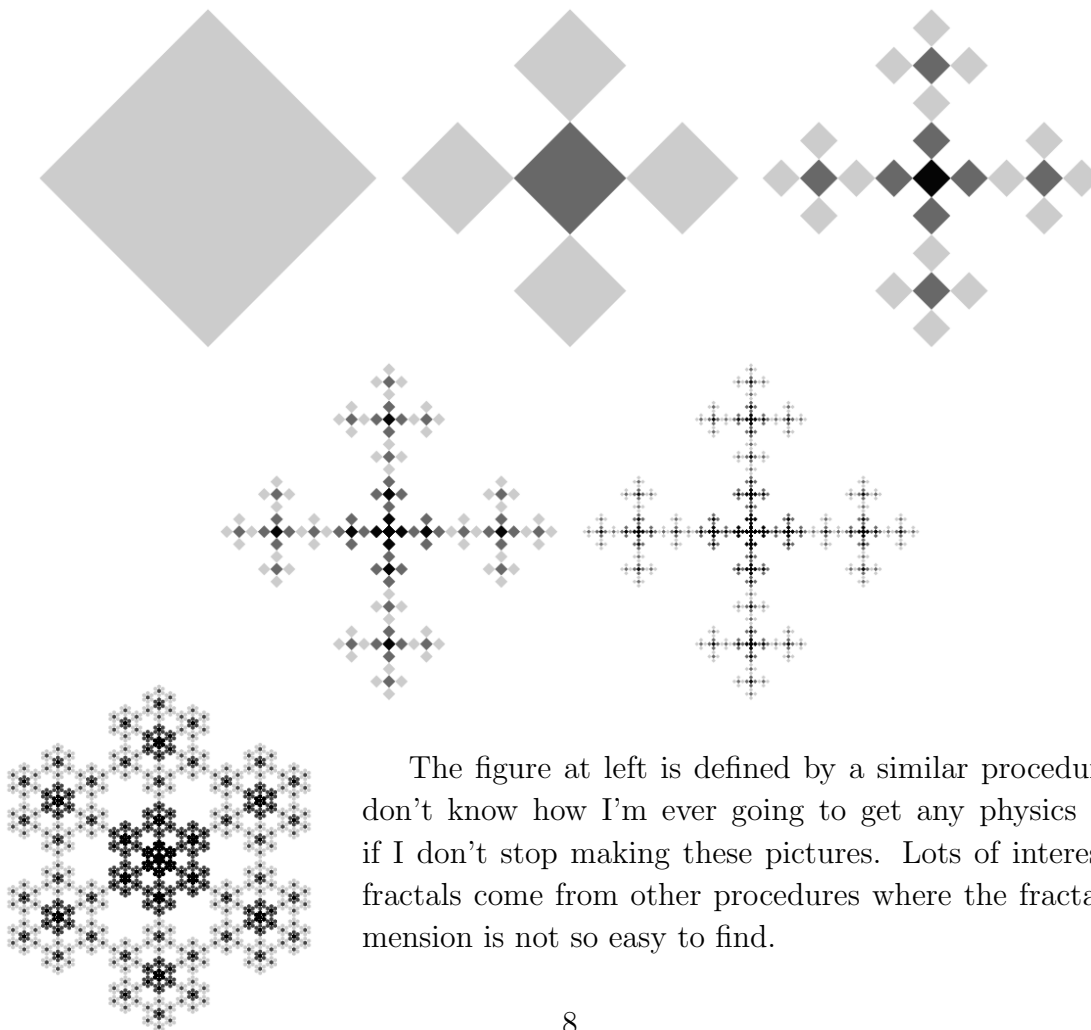
$$N(a) = 2^{n(a)} = 2^{-\frac{\log \frac{a}{a_0}}{\log 3}} = \left(\frac{a}{a_0}\right)^{-\frac{\log 2}{\log 3}}$$

which gives fractal dimension

$$D = \frac{\log 2}{\log 3} \simeq .63 \in (0, 1) . \tag{1.3}$$

Notice that this object is self-similar with scale factor $\lambda = 3$: the two remaining thirds are identical to the original up to a rescaling of the length by a factor of three. This fact can be used to infer the power-law, since it means $N(a) = 2N(3a)$. So if $N(a) \sim a^{-D}$, we must have $a^{-D} = 2(3a)^{-D} \implies 1 = 2 \cdot 3^{-D}$ which is (1.3).

- Here's an example in $d = 2$. Take a square with side length a_0 . Now divide it into nine squares by dividing each side by three. Remove every other sub-square, leaving the corners. Repeat. This procedure gives the accompanying figures. The resulting figure is again self-similar with $\lambda = 3$ and has $N(a) = 5N(3a)$ – we need only five times as many balls of diameter a to cover the region as balls of diameter $3a$. Therefore, if there is a scaling relation $N(a) \sim a^{-D}$, we need $D = \frac{\log 5}{\log 3} \simeq 1.46$. Note that this is sensibly between 1 and 2.



The figure at left is defined by a similar procedure. I don't know how I'm ever going to get any physics done if I don't stop making these pictures. Lots of interesting fractals come from other procedures where the fractal dimension is not so easy to find.

1.2 Fractal dimension of a random walk

So far we've been discussing fractals defined by an artificial procedure. Consider a *random walk* in Euclidean space of dimension $d \geq 2$. Starting from some origin, we take discrete steps, where the displacement \vec{r} of each step is chosen (*e.g.*) independently from some distribution $p(\vec{r})$. For example, we could take our steps to have fixed length a_0 , and uniformly distributed direction. For now, we assume this distribution and that the walk is otherwise unrestricted.

Of interest is the net displacement after M steps

$$\vec{R}_M \equiv \sum_{i=1}^M \vec{r}_i.$$

This is a random variable with average

$$\langle \vec{R}_M \rangle_M \equiv \int d^d r_1 \cdots d^d r_M p(\vec{r}_1) \cdots p(\vec{r}_M) \sum_{i=1}^M \vec{r}_i$$

which vanishes by our assumption of rotation invariance of the individual distributions. On the other hand, $\langle \vec{r}_i \cdot \vec{r}_j \rangle \propto \delta_{ij}$, so the square-length of the net displacement has

$$\langle |\vec{R}_M|^2 \rangle_M = \sum_i^M \sum_j^M \langle \vec{r}_i \cdot \vec{r}_j \rangle_M = \sum_{j=1}^M \langle |r_j|^2 \rangle = M a_0^2.$$

The RMS displacement $R(M) \equiv \sqrt{\langle |\vec{R}_M|^2 \rangle_M} = \sqrt{M} a_0$ goes like the square root of the number of steps, a probably-familiar result on which we are going to get some new perspective now.

What is the fractal dimension of a random walk?

A walk of M steps can be regarded as M/n subwalks of n steps (choose M so that these are integers). By the above result, the RMS displacement of the subwalks is $r(n) = \sqrt{n} a_0$; choose M big enough so that this is a good approximation. This suggests that we may think of a random walk (RW) of M steps of length a_0 as a RW of M/n steps each of length (approximately) $a_1 \equiv \sqrt{n} a_0$. Notice that this 'coarse-grained' step size is not actually the same for each subwalk. (We are relying on the central limit theorem here to say that the distribution of subwalk sizes is well-peaked around the central value. We'll give an RG proof of that result next.)

This perspective allows us to estimate the fractal dimension of an unrestricted RW. Let $N(a)$ be as above the number of balls of diameter a needed to cover a walk (probably) of M microscopic steps of size a_0 . When the ball-size is about the same as the stepsize, we need one ball for each step (this is overcounting but should give the right scaling), we'll have

$$N(a) \sim M, \quad \text{for } a \sim a_0.$$

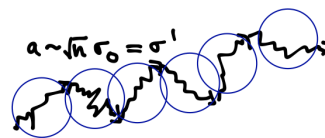
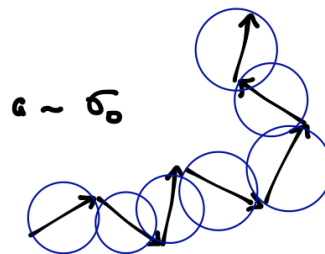
For the sub-steps, the same relation says we should cover each subwalk step (length $\sqrt{n}a_0$, of which there are M/n) with a ball, so

$$N(\sqrt{n}a_0) \sim \frac{M}{n}.$$

Combining the previous two displayed equations (eliminate M) gives

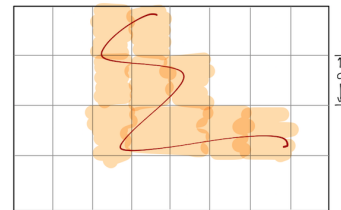
$$M = N(a_0) = nN(\sqrt{n}a_0) \implies N(a) \sim a^{-2}$$

which says that the fractal dimension of the (unrestricted, in $d \geq 2$) random walk is $D = 2$.



A few points regarding the notion of fractal dimension.

1. The Hausdorff dimension we've defined is not the only candidate for such a scale-dependent and possibly-fractional notion of dimension. In fact there are *many* others, and they are not all equivalent. Two that are notable are the *box-counting* dimension, where one covers the whole \mathbb{R}^d with a grid of boxes of side length a and counts the number $N(a)$ of boxes containing an element of the set as a function of a ; if $N(a) \sim a^{-D_{bc}}$ then this defines the box-counting dimension D_{bc} . This one is easier to implement numerically since it doesn't involve a minimization procedure.



Another one is the *correlation dimension*, which is related to a problem on the homework.

2. As a practical physicist, why should you care about this result? Here's one kind of answer: suppose you have in your hands some object which is locally one-dimensional, but squiggles around in a seemingly random way. It is governed by some microscopic dynamics which are mysterious to you, and you would like

to know if you can model it as an unrestricted random walk. One diagnostic you might do is to measure its fractal dimension; if it's not $D = 2$ then for sure something else is going on in there. (If it *is* $D = 2$ something else still might be going on.)

[End of Lecture 1]

3. For some statistically self-similar sets, a single fractal dimension does not capture the full glory of their fractaliciousness, and it is useful to introduce a whole spectrum of fractal dimensions. Such a thing is called *multifractal*.

I hope to say more about both of the previous points later on in the course.

1.3 RG treatment of random walk

Now we'll study the random walk a bit more precisely, and use it to introduce the RG machinery. To be specific, suppose that each microscopic step is sampled from the (Gaussian) distribution

$$p(\vec{r}) = \mathcal{N} e^{-\frac{|\vec{r}|^2}{2\sigma_0^2}}, \quad \mathcal{N} = (2\pi\sigma_0^2)^{-d/2}.$$

As before, the detailed form of the single-step distribution will be unimportant for the questions of interest to us – the technical term is ‘irrelevant’; this will be an outcome of the RG analysis. In this case, we have $\langle \vec{r} \rangle = 0$, $\langle \vec{r} \cdot \vec{r} \rangle = \sigma_0^2$.

Let $\vec{r}' \equiv \sum_{i=1}^n \vec{r}_i$. Think of this as a ‘coarse-grained step’ – imagine that the single steps (of RMS size σ_0) are too small to see, but for n big enough, n of them can get somewhere. The distribution for the coarse-grained step is:

$$\begin{aligned}
 P(\vec{r}') &= \int d^d r_1 \cdots d^d r_n p(\vec{r}_1) \cdots p(\vec{r}_n) \delta \left(\vec{r}' - \underbrace{\sum_{i=1}^n \vec{r}_i}_{= \int d^d k e^{i\vec{k} \cdot (\vec{r}' - \sum_i \vec{r}_i)} \right) \\
 &\quad \text{(Do } n \cdot d \text{ Gaussian integrals.)} \\
 &\quad \text{Note that } d^d k \equiv \frac{d^d k}{(2\pi)^d} \text{.)} \\
 &= \int d^d k \exp \left(-n |\vec{k}|^2 \frac{\sigma_0^2}{2} - i\vec{k} \cdot \vec{r}' \right) \\
 &\quad \text{(One more Gaussian integral.)} \\
 &= \mathcal{N}' \exp \left(-\frac{|\vec{r}'|^2}{2n\sigma_0^2} \right), \quad \mathcal{N}' \equiv (2\pi n\sigma_0^2)^{-d/2}. \tag{1.4}
 \end{aligned}$$

This is the same form of the distribution, with the replacement $\sigma_0 \rightarrow \sigma' \equiv \sqrt{n}\sigma_0$. We can make it *actually* the same distribution if we rescale our units (the second half of the RG transformation): rescale $r' \equiv \sqrt{n}r''$, where the zoom factor is chosen to keep the width of the distribution the same after the coarse-graining step. Remembering that distributions transform under change of variables by

$$P(\vec{r}')d^d r' = P(\vec{r}'')d^d r''$$

we have

$$P(\vec{r}'') = \frac{1}{(2\pi\sigma_0)^{d/2}} e^{-\frac{|\vec{r}''|^2}{2\sigma_0^2}}$$

– the same distribution as we had for single step. Therefore, a random walk is (probably) a fractal – it is self-similar on average.

The two steps above – (1) coarse graining and (2) rescaling – constitute a *renormalization group transformation* (more on the general notion next). The ‘coupling constant’ σ_0 transforms under this transformation, in this case as

$$\sigma_0 \mapsto \sigma_{\text{renormalized}} = \sigma_0,$$

i.e. it maps to itself; such a parameter is called *marginal* and is a special case.

Consider the RMS distance covered by a walk in M steps,

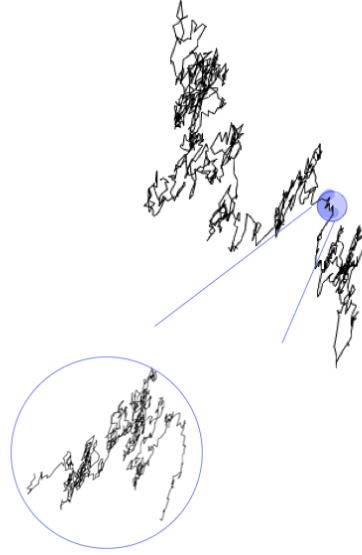
$$R(M)^2 \equiv \left\langle \left| \sum_{i=1}^M \vec{r}_i \right|^2 \right\rangle_M .$$

It depends on M and the typical step size, which is σ (since $\sigma^2 = \int d^d r |\vec{r}|^2 p(\vec{r})$). Dimensional analysis tells us that we must have $R(M) \propto \sigma$ and the statistical self-similarity we’ve just found suggests a power law dependence on M :

$$R(M) \sim \sigma M^\nu$$

which scaling relation defines the exponent ν . The coarse-grained walk (no rescaling) takes $M' = N/n$ steps. Demanding the same outcome for the RMS displacement in both the microscopic description and in the coarse-grained description says

$$\sigma M^\nu = \underbrace{\sigma'}_{\sigma'=\sqrt{n}\sigma} (M')^\nu = \sqrt{n}\sigma \left(\frac{M}{n}\right)^\nu = n^{\frac{1}{2}-\nu}\sigma M^\nu. \quad (1.5)$$



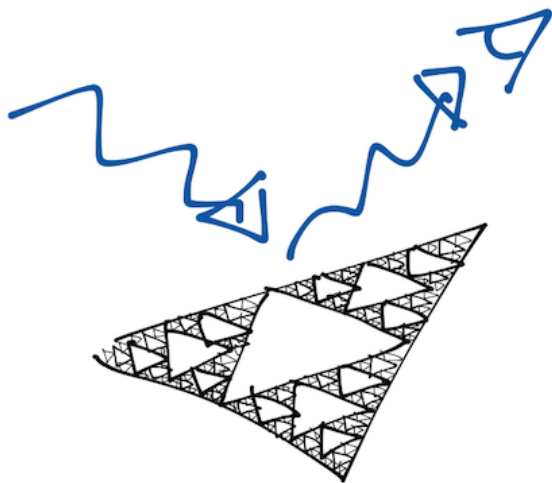
(In the context of quantum field theory, a relation with the same logical content is called a Callan-Symanzik equation.) In order for this to be true for all n , we must have

$$\nu = \frac{1}{2}.$$

Recalling that the fractal dimension $D = 2$ also came from $\sigma' = \sqrt{n}\sigma_0 = n^{1/D}\sigma_0$, we've shown that an unrestricted random walk in $d \geq 2$ has a relationship between the fractal dimension and the RMS displacement exponent: $\nu = 1/D$.

Measurability of the fractal dimension. I've spoken above about the fractal dimension of a random walk, for example of a random polymer configuration, as an 'observable'. How could you measure it?

Suppose the monomers making up the polymer scatter light (elastically). The fractal dimension can be extracted from the *structure factor* $S(k)$, as measured by the intensity of scattering of light off the object, as a function of the wavenumber k of the light. (This statement is related to the open-ended question on the first homework.)



1.4 Anatomy of an RG scheme

As we saw in (1.5), we are taking a passive point of view on the RG transformations: the observable physics (whatever it may be, $R(M)$ in the example above) is the same, and we are only changing our description of that physics.

An RG transformation has two steps:

1. **Coarse-graining or decimation:** The idea of this step is familiar from the central idea of how thermodynamics emerges from statistical mechanics: we should

average over the stuff we can't keep track of (microscopic configurations of the system), holding fixed the stuff we do keep track of (the thermodynamic variables like energy and volume). In the connection mentioned in the previous sentence, we do it all at once.

The key new idea of the RG is to do it a little bit at a time. That is: Integrate out or average over some set of short-distance/fast degrees of freedom, holding fixed a set of long-wavelength/slow degrees of freedom.

Notice that this step is not necessarily reversible: the value of a definite integral (or sum) does not uniquely determine the integrand (or summand). We lose information in this step. This means that a set of transformations defined this way is not in fact a group in the mathematical sense, since there is no inverse element (it is a *semigroup*). So much for that.

The idea is that we are squinting, so that the smallest distance Δx we can resolve gets a little bigger, say before the coarse-graining, we had a resolution $\Delta x = \epsilon$, and afterwards we only keep track of stuff that varies on distances larger than $\Delta x = \lambda\epsilon$ for some scale factor $\lambda > 1$.

2. **Rescaling:** Now we change units to map the coarse-grained system back onto the original one, so that $\lambda\epsilon \mapsto \epsilon$. We do this so that we can compare them.

Now we're going to think about the space on which this transformation is acting. Its coordinates are the parameters of the system, such as the parameters defining the probability distribution such as σ_0 for the random walk, or the couplings in the Hamiltonian if $p = e^{-\beta H}/Z$. Let's call the set of such parameters $\{h_j\}$, where j is an index which runs over as many parameters as we need to consider²³. These parameters get transformed according to

$$\{h_j\} \xrightarrow{\text{steps 1, 2}} \{h'_j \equiv \mathcal{R}_j(\{h\})\}.$$

This map is something we can do over and over, coarse-graining (zooming out) by a factor of λ each time, until we get to macroscopic sizes. The repeated application of

²For example, in the random walk case, other parameters we could include are b, c, \dots in

$$p(\vec{r}) = \exp - \left(\vec{b} \cdot \vec{r} + \frac{r^2}{2\sigma^2} + cr^4 + \dots \right).$$

³One of the many crucial contributions of Ken Wilson to this subject was (I think) allowing for the possibility of including arbitrarily many parameters. The terror you are feeling at this possibility of an infinite-dimensional space of coupling parameters will be allayed when we discover the correct way to organize them two pages from now.

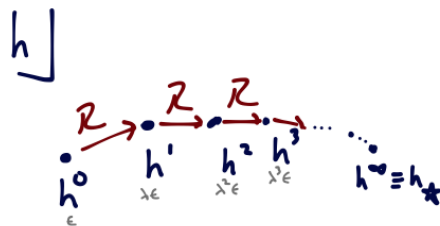
the map $h'_j \equiv \mathcal{R}_j(h)$ describes a dynamical system on the space of parameters. If we are interested in macroscopic physics, we care about what happens when we do it lots of times:

$$h \mapsto \mathcal{R}(h) \mapsto \mathcal{R}(\mathcal{R}(h)) \equiv \mathcal{R}^2(h) \mapsto \mathcal{R}^3(h) \rightarrow \dots$$

(When studying such a possibly-nonlinear dynamical system more generally, it is a good idea to ask first about the possible late-time behavior.)

What can happen? There are three possibilities:

1. We can reach a *fixed point*, $h_* = \mathcal{R}(h_*)$. (We'll include $h \rightarrow \infty$ in some direction in this case. That just means we chose a bad parametrization.)
2. We can go around in circles forever. This is called a *limit cycle*.
3. We can jump around chaotically forever.



The first case, where there is a fixed point, is the one about which we have a lot to say, and fortunately is what seems to happen usually.

A crucial point: the distribution described by such a fixed point of the RG is self-similar, by the definition of the RG transformation. (If this is true when our zooming size $\lambda \rightarrow 1$, then it is actually scale-invariant.)

1.5 Scaling behavior near a fixed point

Now, suppose we have found a fixed point h_* of our RG transformation, which is a candidate for describing the macroscopic behavior of our system. It is then a good idea to look at the behavior in the neighborhood of the fixed point (this is also a good piece of advice for general dynamical systems): linearize about the fixed point. We will see that this analysis immediately spits out the phenomenology of scaling behavior near a critical point. If that is not a familiar notion, don't worry, we'll come back to it.

First, define the set of points which flows to the fixed point to be the

$$\text{critical surface of } h_* \equiv \{h \mid \lim_{n \rightarrow \infty} \mathcal{R}^n(h) = h_*\} \equiv \mathcal{S}(h_*)$$

– this is the basin of attraction of the fixed point in question.

Linearizing about the fixed point, let $h_j \equiv h_j^* + \delta_j$, where $|\delta| \ll 1$ will be our small parameter. This maps under the RG step according to

$$h_j \equiv h_j^* + \delta_j \mapsto h'_j = \mathcal{R}_j(h^* + \delta) \stackrel{\text{Taylor}}{=} \underbrace{\mathcal{R}_j(h^*)}_{=h_j^*} + \delta_k \underbrace{\frac{\partial h'_j}{\partial h_k}}_{\equiv R_{jk}}|_{h^*} + \mathcal{O}(\delta^2)$$

where in the last step we assumed that the RG map \mathcal{R} is analytic in the neighborhood of the fixed point, *i.e.* that it has a Taylor expansion. How could it not be? We got it by doing some finite sums of analytic functions. By $+\mathcal{O}(\delta^2)$ I mean plus terms that go like δ^2 and higher powers of delta which are small and we will ignore them. If we ignore them, then the map on the deviation from the fixed point δ is a linear map:

$$\delta_j \mapsto \delta'_j = R_{jk} \delta_k.$$

We know what to do with a linear map: find its eigenvalues and eigenvectors:

$$R_{jk} \phi_k^{(n)} = \rho_n \phi_j^{(n)}. \quad (1.6)$$

Notice that nothing we've said guarantees that R_{jk} is a symmetric matrix, so its right and left eigenvectors need not be the same (the eigenvalues are), so we'll also need

$$\tilde{\phi}_j^{(n)} R_{jk} = \rho_n \tilde{\phi}_k^{(n)}.$$

Together, these are orthonormal

$$\sum_j \tilde{\phi}_j^{(n)} \phi_j^{(n')} = \delta_{n,n'} \quad (1.7)$$

and complete

$$\sum_n \tilde{\phi}_j^{(n)} \phi_k^{(n)} = \delta_{jk}.$$

About the eigenvalues, notice the following. We've defined the RG transformation $\mathcal{R} \equiv \mathcal{R}_\lambda$ to accomplish a coarse-graining by a scale factor λ . We can imagine defining such a transformation for any λ , and these operations form a semigroup under composition

$$\mathcal{R}_\lambda \mathcal{R}_{\lambda'} = \mathcal{R}_{\lambda\lambda'}.$$

This is useful because it says that the eigenvalues of the linearized operators

$$R_\lambda \phi^{(n)} = \rho_n(\lambda) \phi^{(n)}$$

must satisfy the same multiplication law⁴

$$\rho_n(\lambda)\rho_n(\lambda') = \rho_n(\lambda\lambda'). \quad (1.8)$$

But a function which satisfies this rule must have the form⁵

$$\rho_n(\lambda) = \lambda^{y_n} \quad (1.9)$$

for some y_n independent of λ .

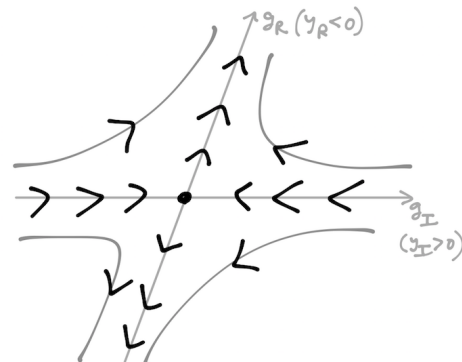
The eigenvectors of R give a preferred coordinate basis near the fixed point:

$$\delta_j = \sum_n g_n \phi_j^{(n)}, \quad g_n \stackrel{(1.7)}{=} \sum_k \tilde{\phi}_k^{(n)} \delta_k,$$

which we will use from now on. y_n is called the *scaling dimension* of the coupling g_n .

Now we can see the crucial RG dichotomy which tames the infinitely many couplings: If $|\rho_n| < 1$ ($y_n < 0$) then as we act with \mathcal{R} many times to get to long wavelengths, then $g_n \rightarrow 0$. Such a coupling is called *irrelevant*: it goes away upon repeated RG transformations and its effects on macroscopic physics can be ignored. Notice that since the perturbation is getting smaller, the approximation $|\delta| \ll 1$ becomes better and better in this case.

In contrast, if $|\rho_n| > 1$ ($y_n > 0$) then as we act with \mathcal{R} many times to get to long wavelengths, then g_n grows. Such



(Notice that the eigenvectors need not be orthogonal.)

⁴Why do R_λ for different λ have the same eigenvectors?

It really follows from the semigroup property. The eigenvectors are physical things: an eigenvector determines some operator O with the following property: if I add O to the fixed-point hamiltonian, $H_\star + gO$, an RG transformation does not generate any other operators, i.e. it gives $H = H_\star + \alpha gO$ for some α .

On the other hand, the choice of by how much to zoom out (λ) is an arbitrary one. Doing the RG step by λ twice should give the same result as doing it once by 2λ . So in particular either one should give the same set of special directions.

⁵ The function $y(\lambda) \equiv \log \rho_n(\lambda)$ then satisfies $y(\lambda) + y(\lambda') = y(\lambda\lambda')$. First this implies $y(1) = 0$. If we consider $\lambda' = 1 + \epsilon$, we have

$$\begin{aligned} y(\lambda) + y(1 + \epsilon) &= y(\lambda + \lambda\epsilon) \\ y(\lambda) + y(1) + \epsilon y'(1) &= y(\lambda) + \lambda\epsilon y'(\lambda) + \mathcal{O}(\epsilon^2) \end{aligned}$$

which says that y satisfies the differential equation $y'(\lambda) = \frac{y'(1)}{\lambda}$ which is solved by

$$y(\lambda) = y'(1) \ln \lambda.$$

I'm not sure if the statement (1.9) follows if we only know (1.8) for discrete values of λ . Does it?

a parameter is called *relevant*, and represents an instability of the fixed point: our linearization breaks down after repeated applications of \mathcal{R} and we leave the neighborhood of the fixed point.

The case of a coupling with $y_n = 0$ which doesn't change is called *marginal*.

In these terms, the critical surface (actually its tangent space near the fixed point) is determined by

$$\mathcal{S}(h_*) = \{g_n = 0 \text{ if } y_n > 0\}.$$

In particular, the *codimension* of the critical surface in the space of couplings is the number of relevant perturbations of the fixed point.

[\[End of Lecture 2\]](#)

2 Random walks

Next we generalize our ensemble of random walks to illustrate some features of the RG that were missing from our simple pure Gaussian example above.

2.1 Biased gaussian walk

First, we can see an example of a relevant operator if we study a *biased* walk, with

$$p(\vec{r}) = (2\pi\sigma^2)^{-d/2} \exp\left(-\frac{|\vec{r} - \vec{r}_0|^2}{2\sigma^2}\right). \quad (2.1)$$

Again define the distribution for the coarse-grained step to be

$$\begin{aligned} P(\vec{r}') &= \int \prod_{i=1}^n (d^d r_i p(\vec{r}_i)) \delta\left(\vec{r}' - \sum_i \vec{r}_i\right) \\ &\stackrel{\text{(more Gaussian integrals)}}{=} (2\pi n\sigma^2)^{-d/2} \exp\left(-\frac{|\vec{r}' - n\vec{r}_0|^2}{2n\sigma^2}\right). \end{aligned} \quad (2.2)$$

So, after the coarse-graining step, we have

$$\begin{cases} \sigma' &= \sqrt{n}\sigma \\ \vec{r}'_0 &= n\vec{r}_0 \end{cases}.$$

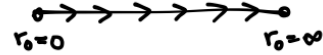
After the rescaling step, to keep the width of the distribution fixed, we have

$$\begin{cases} \sigma^{(R)} &= \sigma \\ \vec{r}_0^{(R)} &= \sqrt{n}\vec{r}_0 \end{cases}.$$

So R is diagonal already. This says that the bias of the walk is a relevant operator of dimension $y_0 = \frac{1}{2} > 0$.

We have here an explicit example of an RG map \mathcal{R} . Let's study its fixed points. There's one at $(\sigma, \vec{r}_0 = 0)$ (for any σ , so actually it is a family of fixed points parametrized by the marginal coupling σ) which is the unbiased walk we studied earlier. This fixed point is unstable because if we turn on a little r_0 it will grow indefinitely.

And there's another fixed point at $(\sigma, \vec{r}_0 = \infty)$. This is where we end up if we perturb the unbiased fixed point. The distribution (2.1) says (by direct calculation) that



$$R(M) = \sqrt{\langle |\vec{R}_M|^2 \rangle_M} = \sqrt{M^2 |\vec{r}_0|^2 + M\sigma^2} \xrightarrow{M \gg 1} M |\vec{r}_0|.$$

This means that for large a , we'll need $N(a) \sim 1/a$ spheres of diameter a to cover the walk – it will be one dimensional.

This means that a system defined by some microscopic distribution of the form (2.1) with some value of \vec{r}_0 and σ will look like a Brownian walk of the type described above, with fractal dimension $D = 2$, if you look at it closely, with a resolution $\delta x \ll \sigma$. But from a distance (resolution worse than $\delta x \gg \sigma$), it will look like a one-dimensional path ($D = 1$) in the \vec{r}_0 direction. For example, the number of balls defining the fractal dimension behaves as

$$N(a) \sim \begin{cases} a^{-2}, & a \ll \sigma \\ a^{-1}, & a \gg \sigma \end{cases}.$$

2.2 Universality class of the (unrestricted) random walk

Now let the distribution from which a single step is sampled be *any* rotation-invariant distribution $p(\vec{r}) = p(|\vec{r}|)$ with finite moments. For example, the fixed-step-length distribution $p(\vec{r}) = \frac{1}{4\pi a^2} \delta(|\vec{r}| - a)$ is a good one to keep in mind. (This is still not the most general walk, since we're still assuming the steps are independent. More on that next.) The distribution for the coarse-grained step is

$$\begin{aligned} P(\vec{r}') &= \int \prod_{i=1}^n d^d r_i p(\vec{r}_i) \delta\left(\vec{r}' - \sum_i \vec{r}_i\right) \\ &= \int d^d k e^{-i\vec{k}\cdot\vec{r}'} \left(\langle e^{i\vec{k}\cdot\vec{r}} \rangle\right)^n. \end{aligned} \quad (2.3)$$

The quantity

$$\langle e^{i\vec{k}\cdot\vec{r}} \rangle = \int d^d r p(\vec{r}) e^{i\vec{k}\cdot\vec{r}} \equiv g(k)$$

is called the characteristic function of the distribution $p(\vec{r})$, and is a generating function for its moments:

$$\langle r^m \rangle = (-\mathbf{i}\partial_k)^m g(k)|_{k=0}.$$

The Taylor expansion in k of its logarithm is the cumulant expansion:

$$\log g(k) = \sum_m \frac{(\mathbf{i}k)^m}{m!} C_m, \quad C_m = (-\mathbf{i}\partial_k)^m \log g|_{k=0}.$$

The important point for us is the expansion:

$$\langle e^{\mathbf{i}\vec{k}\cdot\vec{r}} \rangle = \exp \left(\mathbf{i}\vec{k} \cdot \langle \vec{r} \rangle - \frac{1}{2} \sum_{\mu,\nu} k_\mu k_\nu \underbrace{(\langle r_\mu r_\nu \rangle - \langle r_\mu \rangle \langle r_\nu \rangle)}_{=\sigma_0^2 \delta_{\mu\nu}} + \frac{(\mathbf{i}k)^3}{3!} C_3 + \mathcal{O}(k^4) \right).$$

If we don't truncate the sum in $\sum_m \frac{(\mathbf{i}k)^m}{m!} C_m$, then the $\{C_m\}$ are just another set of coordinates on the space of couplings for the walk. Why should we treat the integration variable k in (2.3)

$$P(\vec{r}') = \int d^d k e^{-\mathbf{i}\vec{k}\cdot\vec{r}'} e^{-\frac{n}{2}\sigma_0|\vec{k}|^2 + \mathcal{O}(nk^3)}$$

as small? Because the integrand is suppressed by the Gaussian factor. If the Gaussian bit dominates, then the integrand has support at $k \lesssim \frac{1}{\sqrt{n\sigma_0^2}}$, at which the m th term in the cumulant expansion contributes to the exponent in (2.3) as

$$nk^m C_m \sim n^{1-\frac{m}{2}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{if } m > 2,$$

where the important thing for getting zero is just that C_m is finite and independent of n and k . This is the statement that the couplings C_m for $m > 2$ are irrelevant. Then we can do the remaining Gaussian integral (ignoring the small corrections which are suppressed by $e^{-n^{1-\frac{m}{2}} C_m}$)

$$P(\vec{r}') \stackrel{n \gg 1}{\cong} (2\pi n\sigma_0^2)^{d/2} e^{-\frac{1}{2} \frac{|\vec{r}' - n\langle \vec{r} \rangle|^2}{n\sigma_0^2}}.$$

What's this? This is the Gaussian we used at the beginning, with $r_0 = n \langle \vec{r} \rangle$.

This result, that the distribution for a sum of many random variables independently distributed according to some distribution with finite moments, is usually called the Central Limit Theorem or the Law of Large Numbers. (For more on the derivation I recommend the discussion in Kardar volume 1.)

In the framework of the RG it is an example of *universality*: all such probability distributions are in the basin of attraction of the gaussian random walk – they are said to be in the same *universality class* meaning that they have the same long-wavelength

physics. In particular, their RMS displacement goes like $R_M \sim M^{1/2}$ for large number of steps M , and (for $d \geq 2$) their fractal dimension is $D = 2$.

Notice that we did not prove that the Gaussian fixed point is the only one: we had to assume that we were in its neighborhood in order to use the $k \sim n^{-1/2}$ scaling – this scaling is a property of the neighborhood of the fixed point, just like the exponents y we got by linearizing about the general fixed point in §1.5.

We could try to find other fixed points in the space of d -dimensional walk distributions. For example, we could have chosen the scaling to fix the coefficient C_m for any m . In that case we would find that the $m - 1$ perturbations $C_{l < m}$ are relevant and all the $C_{l > m}$ are irrelevant. The special case where we fix C_1 (i.e. choose $k \sim 1/n$) gives the same fixed-point we reached for the biased walk. The fixed points with $C_{m > 2}$ fixed would have more than one relevant operator (we will learn to call this ‘multicritical’), which means reaching them requires tuning several parameters. For better or worse, these fixed point distributions with $m > 2$ don’t seem to exist as probability distributions, because they would have to have zero variance⁶.

Also, the assumption in the statement of the CLT also has an RG analog: if the initial distribution does not have finite moments, then our expansion in terms of cumulants is no good. An example is a Lorentzian distribution, $p(r) = \frac{\sigma/\pi}{r^2 + \sigma^2}$. In fact in a certain sense the Lorentzian is a fixed point (if we set $n = 2$ where n is the parameter in the coarse-graining transformation as above).

(We will see another fixed point next when we include interactions between the steps of the walk.)

One lesson which does generalize, however, is that most of the possible perturbations of the fixed point are irrelevant, and there is only a small number of relevant or marginal perturbations.

2.3 Self-avoiding walks have their own universality class

[Still from Creswick! I like this book. According to Amazon, Dover has put out a second edition.] Suppose that the random 1d objects we are studying are actually *polymers* – long chain molecules made of ‘monomers’ which cannot be in the same place, *i.e.* they have some short-ranged repulsion from each other. We can model this as lattice paths without self-intersection, or *self-avoiding walks* (SAWs). Does this microscopic modification of our ensemble change the long-wavelength physics?

It certainly changes our ability to do all the sums. If our polymer has n monomers,

⁶Thanks to Tarun Grover for pointing this out to me. Maybe they do exist as simple analogs of ‘[complex fixed points](#),’ where we drop some positivity assumptions

we'd like to know about the numbers

$$M_n(\vec{R}) \equiv \# \text{ of SAWs with } n \text{ steps connecting } \vec{0} \text{ to } \vec{R}. \quad (2.4)$$

Then we could figure out the RMS displacement from head-to-tail of the n step polymer (actually we are not distinguishing between head and tail):

$$R(n)^2 \equiv \langle |\vec{R}|^2 \rangle_n = \frac{\sum_{\vec{R}} M_n(\vec{R}) |\vec{R}|^2}{M_n}.$$

The denominator here is $M_n \equiv \sum_{\vec{R}} M_n(\vec{R})$. As with the unrestricted random walk, we might expect to have (we will) a scaling relation

$$R(n) \sim n^\nu \quad (2.5)$$

with some characteristic exponent ν .

Enumerating M_n is not so easy. For the square lattice, $M_1 = 4$, $M_2 = 4 \cdot 3$ since there are three choices for the second step, $M_3 = 4 \cdot 3 \cdot 3$, but after that we can make loops (for some of the previous choices) and it gets ugly and grows rapidly. A generating function which packages this information is the grand-canonical-type object

$$G(K, \vec{R}) \equiv \sum_n K^n M_n(\vec{R}) \quad (2.6)$$

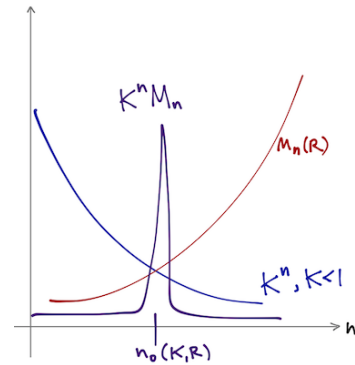
where K is a *fugacity* whose size determines the relative contributions of walks of different lengths to $G(K)$.

Let

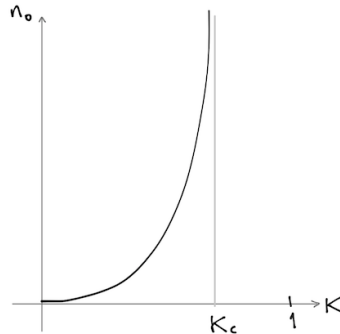
$$\xi^2(K) \equiv \sum_R \frac{\sum_n |\vec{R}|^2 K^n M_n(\vec{R})}{G(K)} \quad (2.7)$$

be the square of the RMS displacement at fixed K , the typical size of a SAW.

In (2.6) $G(K) \equiv \sum_R G(K, R)$. In this ensemble, for $K < 1$, the falloff of K^n with n fights against the growth of M_n to produce a sharp peak at some $n_0(K)$.



There is a value of K where this peak step-length diverges, since it is finite for $K \rightarrow 0$ and infinite for $K \geq 1$.



Preview: if M_n grows exponentially, with some power-law prefactor,

$$M_n \sim K_c^{-n} n^{\gamma-1}$$

then $n_0(K)$ occurs at the maximum of

$$K^n M_n = e^{n \ln(K/K_c) + (\gamma-1) \ln n}$$

$$\implies n_0(K) = -\frac{\gamma-1}{\ln(K/K_c)} \stackrel{K \rightarrow K_c}{\sim} (\gamma-1) \left(\frac{K_c - K}{K_c} \right)^{-1}$$

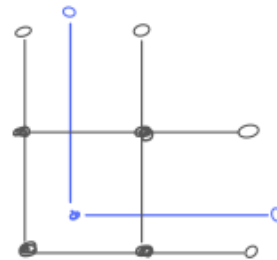
which diverges at (the aptly named) K_c , and the typical walk size goes like

$$\xi(K) \stackrel{K \rightarrow K_c}{\sim} R(n_0(K)) \sim n_0(K)^\nu \sim \left(\frac{K_c - K}{K_c} \right)^{-\nu}$$

So from this grand-canonical point of view, the reason there are scaling relations and power laws is the existence of this critical point where the length of the typical walk diverges. End of preview.

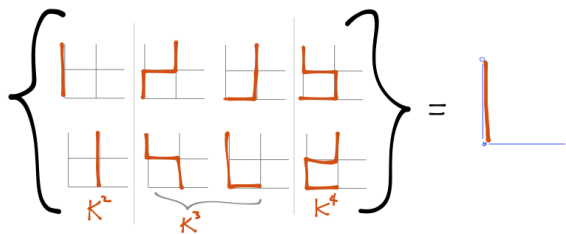
Let's implement an RG for a 2d SAW on the square lattice. What is the space of couplings we should consider? Previously, our only coupling was the bond-fugacity K , that is, a walk was weighted by K^n with n the number of bonds covered by the walk. We could also consider multiplying this weight by $K_2^{n'}$ where n' is the number of next-nearest neighbor bonds covered, or K_3^a where a is the area of the region between the SAW and some reference curve. Any property of the SAW you can quantify can appear in the weight, if you want. Call the weight $W(K)$, where K now represents some collection of such parameters. When pressed, I'll just consider the one fugacity K for the number of steps.

Here's the coarse-graining we'll do: take SAWs Γ on the fine-grained lattice Λ with weight $W_\Gamma(K)$. We will use these to build SAWs Γ' on a coarser lattice Λ' , with some relative zoom factor λ . For example, if λ is an integer, we could take Λ' to be a square lattice with lattice spacing λa where a is the lattice spacing of Λ . ($\lambda = 2$ in the figure at right.)



The weights are related by

$$W_{\Gamma'}(K') = \sum_{\Gamma \in \Gamma'} W_{\Gamma}(K)$$



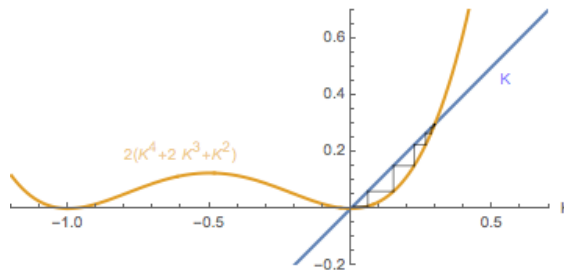
where we regard Γ' as defining an equivalence class of walks on the finer lattice. Here is an example of a possible rule for determining the inclusion $\Gamma \in \Gamma'$, for $\lambda = 2$. It is very non-unique. Depicted is a unit cell of Γ' (blue) and the overlapping unit cells of Γ (black). A SAW which enters the cell at the lower left must leave through the right or the top. The model has a $\pi/2$ rotation symmetry so we can consider just the latter.

Since on the coarser lattice, each of these represents, just a single step, $W_{\Gamma'}(K') = K'$. The result is

$$K' = 2(K^4 + 2K^3 + K^2). \quad (2.8)$$

Let me emphasize that the details of this real-space RG procedure are not to be taken too seriously, and other similarly-legitimate schemes produce somewhat different polynomials on the RHS here.

At right is a visualization of the map (2.8). Fixed points are (by definition) intersections between the curve (2.8) and the line $K' = K$. The map (2.8) has three fixed points: [\[End of Lecture 3\]](#)



1. $K = 0$, which is an ensemble dominated by very short walks, and in particular finite-length ones.
2. $K = \infty$, which is dominated by crazy lattice-filling walks. Maybe interesting.
3. $K = K_c \simeq 0.297$. This third one is where we go from finite walks at K slightly below K_c to infinite walks at $K > K_c$.

The jagged line between $K' = K$ and the curve defined by (2.8) depicts the repeated action of the map with an initial condition near (but slightly below) the fixed point at $K = K_c$. As you can see from the jagged line, the fixed point K_c is unstable – the perturbation parametrized by $K - K_c$ is relevant. Its dimension determines the exponent ν defined in (2.5) as follows.

Because we are zooming out by a factor of λ , the typical size will rescale as

$$\xi(K) = \lambda \xi'(K').$$

Near the critical point,

$$\xi(K) \stackrel{K \rightarrow K_c}{\sim} |K - K_c|^{-\nu} \stackrel{!}{=} \underbrace{\lambda}_{=2} \underbrace{\xi'(K')}_{=\xi(K')} = 2 \underbrace{|K'(K) - K_c|}_{=\frac{\partial K'}{\partial K}|_{K_c}(K-K_c)}^{-\nu}$$

Therefore

$$|K - K_c|^{-\nu} = \lambda \left(\left| \frac{\partial K'}{\partial K} \right|_{K_c} \right)^{-\nu} |K - K_c|^{-\nu}$$

from which we conclude

$$\nu = \frac{\ln \lambda}{\ln \left| \frac{\partial K'}{\partial K} \right|_{K_c}} = 0.771.$$

Numerical simulations give $K_c = 0.379$ and $\nu = 0.74$.

Where are we making an approximation in the above? For example, some configurations on the fine lattice have no counterpart on the coarse lattice (an example is a walk which enters the cell and leaves again the same way). We are hoping that these don't make an important contribution to the sum. The real-space RG can be systematically improved by increasing the zoom factor λ (clearly if we coarse-grain the whole lattice at once, we'll get the exact answer).

The important conclusion, however, is pretty robust: the $d = 2$ SAW has a different exponent than the unrestricted walk:

$$\nu_{SAW} > \nu_{RW} = 1/2.$$

This makes sense, since it means that $R_{\text{RMS}}(\text{SAW}) > R_{\text{RMS}}(\text{unrestricted})$ for many steps – the SAW takes up more space (for a fixed number of steps) since it can't backtrack. The fractal dimension is therefore smaller $D_{\text{SAW}} = \frac{1}{\nu} \simeq 1.3 < 2$.

Different exponents for the same observable near the critical point means different universality class.

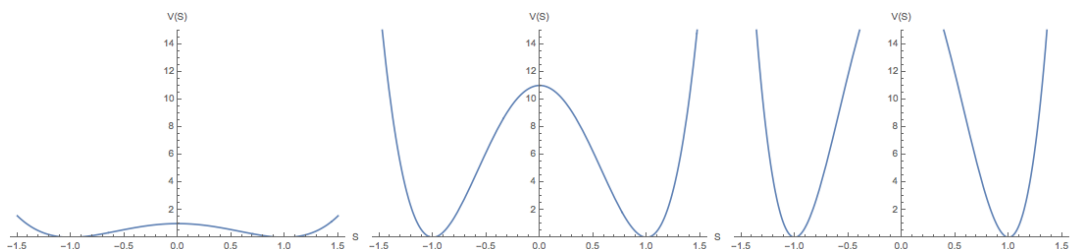
Teaser: This ensemble of self-avoiding walks is the $n \rightarrow 0$ limit of the $O(n)$ model! More specifically, the critical point in temperature of the latter model maps to the large-walk limit: $T - T_c \sim M^{-1}$. This realization will allow us to apply the same technology we will use for the Ising model (which we could call the $O(1)$ model) and its $O(n)$ generalizations to this class of models.

3 Ising models

Words about the role of models, solvable and otherwise, and universality:

Fixed points of the RG are valuable. Each one describes a possible long-wavelength behavior, and each one has its own basin of attraction. That basin of attraction includes lots of models which are in some sense ‘different’: they differ in microscopic details of values of couplings, and sometimes even more dramatically. Two important examples: (1) a lattice model and a continuum model can both flow to the same fixed point. The idea is that if the correlation length is much longer than the lattice spacing, the lattice variable looks like a continuous field, and we can interpolate between the lattice points. And at a fixed point scale invariance requires that the correlation length be infinity (or zero).

(2) a model with two states per site (like an Ising magnet, the subject of this section) and a model with infinitely many states at each site can flow to the same fixed point. Here’s a picture of how that might come about. Suppose we have at each site a variable called S which lives on the real line, and it is governed by the potential energy function $V(S) = g(S^2 - 1)^2$. (So for example the Boltzmann distribution is $e^{-\beta V(S)}$.) The parameter g might be relevant, in the sense that $g \rightarrow \infty$ at long wavelengths. This process of making g larger is depicted in the following figures (left to right $g = 1, 11, 21$):



As you can see, it becomes more and more energetically favorable to restrict S to just the two values $S = \pm 1$ as g grows.

I’ve just made a big deal about universality and the worship of fixed points of the RG. Part of the reason for the big deal is that universality greatly increases the power of simple models: if you can understand the physics of some simple (even ridiculously over-idealized) model and show that it’s in the same universality class as a system of interest, then you win.

[Goldenfeld §2.5, Creswick §5, lots of other places] The Ising model is an important common ground of many fields of science. At each site $i \in \Lambda$ (Λ may be a chain, or the square lattice, or an arbitrary graph, and $i = 1 \dots |\Lambda| \equiv N(\Lambda) = N$ is the number of sites), we have a binary variable $s_i = \pm 1$ called a spin, whose two states are sometimes called up and down. There are 2^N configurations altogether. (Although I

will sometimes call these ‘states’ I emphasize that we are doing classical physics.)

The name ‘Ising model’ connotes the following family of energy functionals (also known as Hamiltonians):

$$-H(s) = \sum_{i \in \Lambda} h_i s_i + \sum_{ij} J_{ij} s_i s_j + \sum_{ijk} K_{ijk} s_i s_j s_k + \dots \quad (3.1)$$

where this sum could go on forever with terms involving more and more spins at once. (The RG will generically generate all such terms, with coefficients that we can hope do not cause too much trouble.) With this definition, the model may describe magnetic dipoles in a solid, a lattice gas (where $s_i = \pm 1$ correspond to presence or absence of a particle at i), constrained satisfaction problems, neural networks, ... anything with bits distributed over space. This list also could go on forever^{7,8}.

Equilibrium statistical mechanics. Why might we care about $H(s)$? We can use it to study the equilibrium thermodynamics of the system, at some temperature $T \equiv 1/\beta$. Let’s spend a few moments reminding ourselves about the machinery of equilibrium statistical mechanics. The key ‘bridge’ equation between the microscopic world (stat mech) and the macroscopic world (thermo) in thermal equilibrium is

$$e^{-\beta F} = \sum_s e^{-\beta H(s)} \equiv Z.$$

Here in our theorists’ paradise, we measure temperature in units of energy, $k_B = 1$. Notice that in classical equilibrium stat mech, the temperature is redundant with the overall scaling of the Hamiltonian, only the combinations $\beta h, \beta J, \dots$ appear in the partition function, so a system with twice the temperature and twice the couplings will have the same physics. The sum here is over the 2^N configurations of the spins:

$$\sum_s \equiv \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \sum_{s_3=\pm 1} \dots \sum_{s_N=\pm 1} \equiv \prod_{i=1}^N \sum_{s_i=\pm 1} \equiv \text{tr}$$

and we will sometimes write tr for ‘trace’. I emphasize that we are doing classical physics here.

Why do we care about the free energy F ? For one thing, it encodes the thermodynamics of the system: the average energy is

$$E \equiv \langle H \rangle \equiv \frac{1}{Z} \text{tr} H e^{-\beta H} = -\partial_\beta \log Z,$$

⁷Here is an example I learned of recently of how an Ising model is used for data clustering.

⁸Sometimes the word ‘Ising’ is used to indicate the presence of the \mathbb{Z}_2 symmetry under $s \rightarrow -s$ which is present when only even terms appear in H ($h = 0, K = 0$).

the entropy is

$$S = -\partial_T F,$$

the heat capacity is

$$C_V = \partial_T E = \frac{1}{T^2} (\langle H^2 \rangle - \langle H \rangle^2),$$

a dimensionless measure of the number of degrees of freedom. Notice the familiar thermodynamic identity $F = E - TS$ follows by calculus.

More ambitiously, if we knew how F depended on all the coupling parameters $\{h_i, J_{ij}, K_{ijk} \dots\}$ in (3.1), we would know all of the correlation functions of the spins, for example

$$\partial_{h_i} F = -T \partial_{h_i} \log Z = -T \frac{1}{Z} \text{tr} \frac{s_i}{T} e^{-\beta H} = -\langle s_i \rangle.$$

And similarly,

$$\partial_{h_i} \partial_{h_j} F = (\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle) T^{-1} \equiv G_{ij} T^{-1}.$$

It is a generating function for these (connected) correlation functions.

Clean and local Ising models. Two important specializations of (3.1) are quite important in physics (not always in the other applications of the Ising model). We will (usually) restrict to the important special case with the following two assumptions.

1. the couplings (J_{ij} and friends) are *local* in the sense that the coupling between two sites goes away ($J_{ij} \rightarrow 0$) if the sites are far apart ($|r_i - r_j| \rightarrow \infty$).

A reason to care about the two point function in the case where there is a notion of locality, then, is that it allows to define a *correlation length*, ξ :

$$G_{ij} \stackrel{r \gg a}{\sim} e^{-r_{ij}/\xi}$$

– here a is the range of the interactions, or the lattice spacing, and $r_{ij} \equiv |r_i - r_j|$ is the distance between the locations of spins i and j . The correlation length will depend on the parameters in H and on the temperature, and it measures the distance beyond which the spin orientations are uncorrelated. More formally, $\xi^{-1} \equiv -\lim_{r \rightarrow \infty} \partial_r \ln G_{i, i+r}$ (but of course the ∞ here has to stay within the box containing the system in question).

2. the couplings are translation invariant: $J_{ij} = Jf(|r_i - r_j|)$ for some function of the distance $f(r)$. (If one thinks of variations of J_{ij} with i, j as coming from some kind of microscopic disorder, one refers to this case as *clean*.) We will often consider the case where $f(r)$ only has support when $r = a$ is one lattice spacing. (Notice that $s^2 = 1$ means that we can ignore the case when $r = 0$.)

These two assumptions are independent, but we will usually make both. So: on any graph (of N sites), the nearest-neighbor ‘clean’ Ising model has energy functional

$$-H = h \sum_i s_i + J \sum_{\langle ij \rangle} s_i s_j$$

where $\langle ij \rangle$ means the set of nearest neighbors i, j .

An important observable (especially in this case) is the *magnetization*

$$M \equiv \frac{1}{N} \sum_{i=1}^N \langle s_i \rangle = -\frac{1}{N} \partial_h F.$$

Also of interest is the spin susceptibility:

$$\chi \equiv \frac{1}{N} \partial_h M = \frac{1}{T} \frac{1}{N^2} \sum_{ij} (\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle) = \frac{1}{N^2 T} \sum_{ij} G_{ij}.$$

When $J > 0$, the energy of a configuration is lower if neighboring spins point the same way; in this ‘ferromagnetic’ case everybody can be happy (and $M \neq 0$). In the antiferromagnetic case $J < 0$, neighbors want to disagree. All spins can agree to disagree if the graph has no loops. Any loop with an odd number of sites, like a triangle, leads to a *frustration* of the antiferromagnetic interaction, which requires compromise and leads to drama.

Lack of drama for bipartite lattices. A bipartite lattice is one which can be divided into two distinct sublattices A, B each of which only neighbors sites of the other lattice. That is $\langle ij \rangle$ contains only pairs, one from A and one from B . For example, hypercubic lattices are all bipartite: let the A lattice be those sites (x, y, \dots) whose (integer) coordinates add up to an even number $x + y + \dots \in 2\mathbb{Z}$. The honeycomb lattice is also bipartite. The triangular lattice is not.⁹

[End of Lecture 4]

A consequence of bipartiteness is that any loop traverses an *even* number of sites, since it must alternate between the two sublattices. Hence there is no frustration for a (nearest-neighbor!) Ising antiferromagnet on a bipartite lattice. In fact, a stronger statement is true. Since

$$H_{h=0,J}(s^A, s^B) = -J \sum_{\langle ij \rangle} s_i^A s_j^B$$

⁹Notice, by the way, that bipartite does not require that A and B be isomorphic or even that they have the same number of sites. For example, if we simply removed a (periodic) subset of sites (and all the associated links) from the A sublattice of a lattice, we would still have a bipartite lattice. You can find an example by googling ‘Lieb lattice’. Beware confusion in the literature on this point.

if we flip the spins of one sublattice, we also reverse J :

$$H_{h=0,J}(s^A, -s^B) = +J \sum_{\langle ij \rangle} s_i^A s_j^B = H_{h=0,-J}(s^A, s^B).$$

But for any function ϕ

$$\sum_{\{s\}} \phi(\{s\}) = \sum_{\{s\}} \phi(\{-s\}) \quad (3.2)$$

by renaming summation variables. Therefore on a bipartite lattice

$$Z(h=0, J) = \sum_{\{s\}} e^{-\beta H_{h=0,J}(s^A, s^B)} = \sum_{\{s\}} e^{-\beta H_{h=0,-J}(s^A, s^B)} = Z(h=0, -J).$$

So on a bipartite lattice, a ferromagnet and an antiferromagnet have the same thermodynamics.

The nearest-neighbor restriction is essential here. Even for a one-dimensional chain, we can make frustration by adding antiferromagnetic interactions for $s_i s_{i+1}$ and for $s_i s_{i+2}$.

Symmetry and spontaneous symmetry breaking. Notice that the general Ising Hamiltonian (3.1) enjoys the following property

$$H_{h,J,K\dots}(s) = H_{-h,J,-K\dots}(-s)$$

– flipping all the spins and flipping the coefficients of odd powers of the spins preserves the energy. In particular, if $h=0, K=0$, all odd powers do not appear, and flipping the spins is a symmetry of the Hamiltonian. What consequence does this have for thermodynamics?

$$Z(-h, J, -K, T) = \sum_{\{s\}} e^{-\beta H_{-h,J,-K}(s)} = \sum_{\{s\}} e^{-\beta H_{h,J,K}(-s)} \stackrel{(3.2)}{=} Z(h, J, K, T).$$

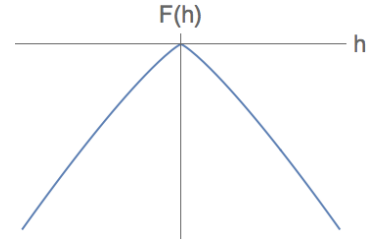
And therefore the free energy in particular satisfies $F(-h, J, -K, T) = F(h, J, K, T)$. Let's set $K=0$ from now on. This operation $s_i \rightarrow -s_i$ is a \mathbb{Z}_2 transformation in the sense that doing it twice is the same as doing nothing. It is a symmetry when $h=0$. (Only when $h=0$ does the transformation map the ensemble to itself.)

Question: does this mean that when $h=0$ we must have zero magnetization,

$$M = \frac{1}{N} \sum_i \langle s_i \rangle \propto \partial_h F \stackrel{?}{=} 0 \quad ?$$

A nonzero value of M (without an applied field h) is called *long-range order*, because it means that distant spins must conspire to point in the same direction.

Answer: It would if $F(h)$ had to be a smooth, differentiable function. In order for $\langle s \rangle_{h=0}$ to be nonzero, $F(h)$ must have a different derivative coming from positive and negative h , as in the figure at right. This phenomenon is called *spontaneous symmetry breaking* because the symmetry reverses the sign of the magnetization $M \rightarrow -M$.



But this phenomenon, of $\partial_h F|_{h=0^+} \neq \partial_h F|_{h=0^-}$ requires the function $F(h)$ to be non-analytic in h at $h = 0$. This is to be contrasted with the behavior for a finite system ($N < \infty$) where

$$Z(h) = \sum_{\{s\}} e^{-\beta H(s)} = e^{-\beta h N m_1} c_1 + e^{-\beta h N m_2} c_2 + \dots + e^{-\beta h N m_n} c_n$$

where $n = 2^N$ is the number of configurations, c_a are positive coefficients (functions of βJ) independent of h , and m_a is the magnetization in configuration a ($m_a = \sum_i s_i(a)/N$). The important point is just that there is a finite number of terms in this sum, and Z is therefore a polynomial in $e^{-\beta h}$, so $F(h) = -T \log Z(h)$ is only singular when $Z = 0$ or $Z = \infty$, which doesn't happen for finite values of β, h .

In conclusion: spontaneous symmetry breaking requires the thermodynamic limit $N \rightarrow \infty$.

3.1 Decimation RG for 1d nearest-neighbor Ising model

Let's step back from these grand vistas and apply the RG for the Ising model in one dimension. Consider a chain of sites $i = 1 \dots N$, arranged in a line with spacing a , and with an even number of sites, $N \in 2\mathbb{Z}$. And for definiteness, if you must, take periodic boundary conditions $s_{N+1} = s_1$. Turn off the magnetic field, so

$$H = -J \sum_{i=1}^N s_i s_{i+1} .$$

We'll speak about the ferromagnetic case, $J > 0$ (though the same results apply to $J < 0$ since the chain is bipartite). The partition function

$$Z = \text{tr} e^{-\beta H} = Z(\beta J)$$

is calculable exactly in many ways, each of which instructive. Since the partition function only depends on the combination βJ , let us set $\beta = 1$.

In the spirit of the RG, let us proceed by a hierarchical route, by decimating the even sites:

$$\sum_{\{s\}_{i,\text{even}}} e^{-H(s)} = e^{-H_{\text{eff}}(s_{\text{odd}})}$$

On the right hand side, we have defined the *effective* hamiltonian for the spins at the odd sites. The odd sites are separated by distance $a' = 2a$ and there are half as many of them. We can use this as the first half of an RG implementation (the second half is rescaling). We've zoomed by a factor of $\lambda = a'/a = 2$.

In this 1d case we can actually do these sums:

$$\sum_{s_2=\pm 1} e^{+Js_2(s_1+s_3)} = 2 \cosh(J(s_1 + s_3)) \equiv \Delta e^{J's_1s_3}$$

where in the last step we defined Δ, J' , constants independent of the configuration of the remaining, not-yet-decimated spins. Then the result for the whole trace over even spins is

$$e^{-H_{\text{eff}}(s')} = \Delta^{N/2} e^{J' \sum_i s_{2i+1} s_{2(i+1)+1}}, \quad \implies \quad H_{\text{eff}}(s') = \sum_{i, \text{ odd}} J' s_i s_{i+2} + \frac{N}{2} \log \Delta.$$

The Δ business just adds a constant to the (free) energy, which divides out of the partition function and we don't care about it here.

We can figure out what the new parameters are by checking cases, of which only two classes are distinct:

$$\begin{aligned} \text{if } s_1 = s_3: \quad 2 \cosh 2J = \Delta e^{J'} & \xrightarrow{\text{product}} \Delta^2 = 4 \cosh 2J \\ \text{if } s_1 = -s_3: \quad 2 = \Delta e^{-J'} & \xrightarrow{\text{ratio}} e^{2J'} = \cosh 2J. \end{aligned} \quad (3.3)$$

The solution can be usefully written as

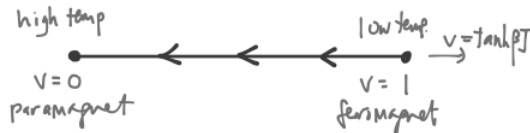
$$v' = v^2, \quad \Delta = 2\sqrt{\cosh 2J} \quad (3.4)$$

where $v \equiv \tanh J \in [0, 1]$ (using hyperbolic trig identities). The map (3.4) is another explicit example of an RG map on the parameters. In this case, unlike the previous SAW example, it happens to be exact.

The RG preserves symmetries. Why is the effective hamiltonian of the same form as the original one? The couplings like the magnetic field multiplying odd numbers of spins vanish by the Ising spin-flip symmetry of the original model. (More precisely: because of the locality of H , we can determine H_{eff} by decimating only a finite number of spins. This rules out generation of nonzero h' by some version of spontaneous symmetry breaking. This requires locality of the interactions.) This line of thinking leads us to expect that the effective hamiltonian should generally have the same symmetries as the original one.

The 4-spin interaction vanishes because in 1d, each site has only two neighbors with whom it interacts, each of which has only one other neighbor. So that was a bit of an accident.

This map has two fixed points: One is $v^* = 0$, which is $\beta J = 0$, meaning infinite temperature, or no interactions; this one is ‘boring’ from the point of view of the



study of many-body physics and collective phenomena, since the spins don’t care about each other at all. The other fixed point is $v_* = 1$, which is $\beta J = \infty$, meaning zero temperature or infinite interaction strength. This is a ferromagnetic fixed point where it is very urgent for the spins to agree with each other. The fact that there is no fixed point at a finite temperature means that there is no critical behavior in the 1d nearest-neighbor Ising model; only at $T = 0$ do the spins align with each other.

More explicitly, how does the correlation length behave? In zooming out by a factor of λ , it changes by

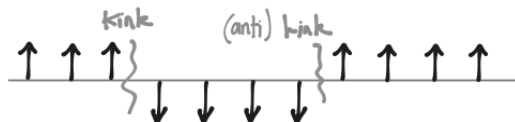
$$\xi(v) = \lambda \xi(v') = 2 \xi(v^2) \quad \implies \quad \xi = -\frac{K}{\log v} \xrightarrow{T \rightarrow 0} \frac{K}{2} e^{2J/T} \quad (3.5)$$

(where K is a constant not determined by this argument) which is finite for $T > 0$.¹⁰

Why did it happen that there is no critical point at $T > 0$? A point of view which illuminates the distinction between 1d and $d > 1$ (and is due to Peierls and now permeates theoretical condensed matter physics) is to think about the statistical mechanics of *defects* in the ordered configuration.

Consider a favored configuration at low-temperature, where all spins point the same way. Small deviations from this configuration require reversing some of the spins and will cost energy $2J$ above the aligned configuration for each dissatisfied bond. In 1d, a single dissatisfied bond separates two happy regions, and is called a *kink* or *domain wall*. Notice that the energy is independent of the size of each happy region (which is called a *domain*). n domains of reversed spins cost energy $4Jn$, since each domain has two boundary links.

In 1d, each region of spins that we reverse has two boundaries, a kink and an antikink.



At $T = 0$, the state minimizes the energy and there is no reason to have any kinks. But at $T > 0$, we care about (*i.e.* the macroscopic equilibrium configuration minimizes) the free energy $F = E - TS$, and the fact that there are *many* kink configurations matters.

¹⁰A log is a special case of a power law: Taylor expand v^ν in ν about 0.

¹¹Preview: near less weird fixed points, the correlation length will diverge like a power law $\xi(T) \sim (T - T_c)^{-\nu}$ instead of this weird function.

How many are there? If there are n segments of $s = -1$ in a sea of $s = +1$ then we must decide where to place $2n$ endpoints. The number of ways to do this is:

$$\Omega(n) \simeq \binom{N}{2n} = \frac{N!}{(2n)!(N-2n)!} \stackrel{1 \ll n \ll N}{\sim} e^{N \log N - 2n \log 2n - (N-2n) \log(N-2n)}$$

where in the last step we used Stirling's formula. So the free energy for $2n$ kinks is

$$F_n = n \cdot 4J - T \log \Omega(n) \simeq 4Jn - T (N \log N - 2n \log 2n - (N-2n) \log(N-2n)) .$$

In equilibrium, the free energy is minimized with respect to any variational parameters¹² such as n , which happens when

$$-\frac{4J}{T} = 2 \log 2n - 2 \log(N-2n) \quad \implies \quad \frac{n}{N} = \frac{1}{2} \frac{e^{-2J/T}}{1 + e^{-2J/T}} \stackrel{T \lesssim 2J}{\sim} \frac{1}{2} e^{-2J/T} .$$

As a check, the correlation length is approximately the size of the domains, which is approximately the inverse kink density:

$$\xi \sim \left(\frac{n}{N}\right)^{-1} \sim 2e^{2J/T}$$

which again agrees with our RG result (3.5).

The codewords for this phenomenon are: the destruction of long-range order by the proliferation of topological defects. (They are topological, for example, in the sense that a kink must follow an antikink, and the number of kinks on circle must equal the number of antikinks.)

In the special case of 1d, we can be completely explicit and verify the result for the correlation length by calculating the correlation function.

First of all,

$$G(r) = \langle s_{i+r} s_i \rangle \tag{3.6}$$

(let's keep the disconnected bits in there for now) is independent of i because of translation invariance. The Boltzmann factor can be written as

$$e^{\beta J \sum_{\langle ij \rangle} s_i s_j} = \prod_{\langle ij \rangle} e^{\beta J s_i s_j}$$

Since $s = \pm 1$, we are multiplying

$$e^{\beta J s_i s_j} = \cosh \beta J + s_i s_j \sinh \beta J = \cosh \beta J (1 + v s_i s_j)$$

¹²Actually, we'll prove this statement in section 4.

where $v \equiv \tanh \beta J$ (as above). Think about expanding this product over links into a sum. Each term in the sum gets either a 1 or a $vs_i s_j$ from each link. Any term in the sum can be visualized by coloring the links which contribute a $vs_i s_j$.

When we multiply this out, the dependence on any one of the spins s_i can be only two things: 1 if the term has an even number of factors of s_i , or s_i if it has an odd number. Here's the Ising model integration table:

$$\sum_{s_i} 1 = 1 + 1 = 2, \quad \sum_{s_i} s_i = 1 - 1 = 0. \quad (3.7)$$

In the last two paragraphs, we haven't used the restriction to 1d at all. (This will be useful in §3.2.) Consider a single spin s_2 of an infinite 1d chain; if it is not one of the two sites i or $i + r$ in (3.6) the factors which matter to it are¹³:

$$\sum_{s_2} (1 + vs_1 s_2)(1 + vs_2 s_3) \stackrel{\text{FOIL!}}{=} \sum_{s_2} (1 + vs_2(s_1 + s_3) + v^2 s_1 s_3) = 2(1 + v^2 s_1 s_3).$$

This is just of the same form as if we had a direct link between 1 and 3 with weight v^2 (up to the overall prefactor). Therefore, doing this repeatedly (r times) for the sites in between i and $i + r$,

$$G(r) = \frac{\text{tr} \mathbf{s}_i 2^r (1 + v^r \mathbf{s}_i \mathbf{s}_{i+r}) \mathbf{s}_r}{\text{tr} 2^r (1 + v^r \mathbf{s}_i \mathbf{s}_{i+r})} = v^r$$

(The red factors are the ones that survive the trace.) Therefore

$$\xi^{-1} = -\partial_r \log G(r) = -\log v$$

which agrees with (3.5) with $K = 1$.

The thing that's special about 1d here is that only a single term in the expansion of the product survives the sum. This is because there is only one *path* between the two sites i and $i + r$. If we had taken N finite and periodic boundary conditions, there would be another path (around the back of the circle) and hence another term in the answer

$$G(r) = v^r + v^{N-r} \stackrel{N \gg r}{\sim} v^r.$$

In $d > 1$ there are many paths and the answer is more interesting, as we'll see below.

[End of Lecture 5]

¹³Note that this expression gives a very simple derivation of (3.4).

3.2 High-temperature expansion

Return now to the moment at (3.7), right before we restricted our discussion to one dimension. We had written the partition function of the nearest-neighbor Ising model (on any graph) as a product over links

$$Z = \cosh^{N_\Lambda}(\beta J) \sum_s \prod_{\langle ij \rangle} (1 + v s_i s_j) \quad (3.8)$$

and argued that expanding this binomial gives a sum over *paths* in the graph. More explicitly, we think of the two terms in each link factor in (3.8) as a sum over another dynamical variable, $n_{\langle ij \rangle} = 0, 1$:

$$1 + v s_i s_j = \sum_{n_{ij}=0,1} (v s_i s_j)^{n_{ij}}.$$

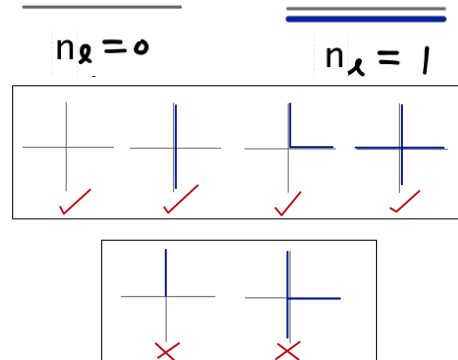
So we can write the Ising partition function as

$$Z = \cosh^{N_\Lambda}(\beta J) \prod_\ell \sum_{\{n_\ell=0,1\}} \sum_{\{s\}} \prod_{\langle ij \rangle} (v s_i s_j)^{n_{ij}}.$$

Now we can do the sums over the spins using our ‘integration table’ above (3.7). For each spin, the sum is

$$\sum_{s_i=\pm 1} s_i^{\sum_{\langle ij \rangle} n_{ij}} = \delta \left(\sum_{\langle ij \rangle} n_{ij} \equiv 0 \pmod{2} \right)$$

Here we’ve defined the notation ‘ $\langle ij \rangle$ ’ to mean ‘neighbors j of a fixed site i ’. That is: the sum is only only nonzero if an even number of the links ending at site i have $n_{ij} = 1$. If we represent $n_l = 1$ by coloring the link l , the configurations which survive this constraint are made of closed loops.



$$Z = \cosh^{N_\Lambda}(\beta J) \prod_{\langle ij \rangle} (1 + v s_i s_j) = \cosh^{N_\Lambda}(\beta J) \sum_C v^{\sum_l n_l(C)} \quad (3.9)$$

where we are summing over configurations of binary numbers $n_l = 0, 1$ on the links that are *closed* in the sense that

$$\sum_{\langle ij \rangle} n_{\langle ij \rangle} \equiv_2 0 \quad \forall i.$$

That is: we sum over lattice curves which have an even number of links going into each site. The contribution of a curve C (which is not necessarily connected) is weighted by $v^{\text{length}(C)}$.

This rewriting of the Ising partition sum will be useful below.

3.3 RG evaluation of physical quantities

Behavior of the correlation length under RG. We've defined the correlation length using the spin-spin correlator $G(r)$, in terms of its rate of falloff for large r . Let us use this to examine its behavior under the RG more directly. To do this, denote more explicitly

$$G_H(r) \equiv \frac{\text{tr} s_i s_{i+r} e^{-H}}{\text{tr} e^{-H}}.$$

Now suppose that i and $i+r$ are both odd sites (so that they survive our decimation); in that case we can still do all the decimation as in the partition function :

$$G_H(r) \equiv \frac{\text{tr}_{e,o} s_i s_{i+r} e^{-H(s_e, s_o)}}{\text{tr} e^{-H(s_e, s_o)}} = \frac{\text{tr}_o s_i s_{i+r} \text{tr}_e e^{-H(s_e, s_o)}}{\text{tr}_o \text{tr}_e e^{-H(s_e, s_o)}}.$$

I emphasize that the argument of G_H is measured in units of the lattice spacing, *i.e.* the number of lattice sites between the spins. But recall that $e^{-H'(s_o)} \propto \text{tr}_e e^{-H(s_e, s_o)}$ defines the effective Hamiltonian for the remaining odd sites, so this is precisely

$$G_{H'}(r/2) \equiv \frac{\text{tr}_o s_i s_{i+r/2} e^{-H'(s_o)}}{\text{tr} e^{-H'(s_o)}},$$

where now there are only half as many sites in between the spins in the new coarser lattice. Under this RG, we are zooming out by a factor of 2. Altogether, $G_{H'}(r/2) = G_H(r)$. Combining this with the definition of ξ , we have

$$\xi_{H'} = \frac{1}{2} \xi_H \tag{3.10}$$

(as we said earlier).

The notation ξ_H is to emphasize that the correlation length is completely determined by the Hamiltonian (I am assuming thermal equilibrium here). At a fixed point, the Hamiltonian does not change under the RG, so the correlation length can't either. This can be consistent with (3.10) in one of two ways

$$\xi_\star = 0 \quad \text{or} \quad \xi_\star = \infty.$$

The first case means that spins at different sites do not care about each other, as at $T = \infty$. I've already disparaged this case as boring. The second case of a divergent correlation length characterizes critical behavior and we define it to be *interesting*.

[Cardy §3.4, Domany, RG notes, chapter 1] **Free energy density.** Next I want to show how to calculate the free energy from an ‘RG trajectory sum’. It is a reason to care about the constants in the effective hamiltonian, as in a' in $H'(s') = \sum J' s' s' + a' N'$. In the example above, we found $a' = \frac{1}{N} \log \Delta$, where Δ was some function of the microscopic J .

Let the free energy density (free energy per site) be

$$f \equiv -\frac{T}{N} \log Z_N(K).$$

Here I am denoting by K the collection of all couplings, and labelling the partition function Z_N by the number of sites. More explicitly,

$$Z_N \equiv \sum_{\{s\}} e^{-\beta H(s)} \equiv \sum_{\{s\}} e^{-\beta(NC + \tilde{H}(s))} \equiv e^{-N\beta C} \tilde{Z}_N.$$

Here \tilde{H} is a Hamiltonian modified by subtracting a constant so that

$$\sum_{\{s\}} \tilde{H}(s) = 0$$

so that it has no constant piece (for quantum mechanical folks: it is like a ‘normal-ordered’ Hamiltonian). And $\tilde{Z}_N \equiv \sum_s e^{-\beta \tilde{H}(s)}$, and naturally we’ll denote

$$\tilde{f} \equiv -\frac{T}{N} \log \tilde{Z}_N.$$

This last expression is a little less innocent than it seems: I am anticipating here that the free energy is extensive – has a leading piece at large N that grows like N , $F \stackrel{N \gg 1}{\approx} Nf + \mathcal{O}(N^0)$ – so that \tilde{f} is independent of N in the thermodynamic limit. (We’ll give an RG-based proof of this statement in §5.) Then $f(K) = C + \tilde{f}(K)$.

Now some RG content: the partition function is invariant under the RG:

$$\begin{aligned} Z_N &= e^{-\frac{NC}{T}} \tilde{Z}_N = e^{-\frac{NC}{T}} e^{-\frac{N'a'}{T}} \tilde{Z}_{N/b}(K') \\ &= e^{-\beta(NC + N'a' + N^{(2)}a^{(2)})} \tilde{Z}_{N/b^2}(K^{(2)}) \\ &= e^{-\beta(NC + N'a' + \dots + N^{(n)}a^{(n)})} \tilde{Z}_{N/b^n}(K^{(n)}). \end{aligned} \quad (3.11)$$

Here we’ve defined $N^{(n)}$ to be the number of sites decimated at step n , and N/b^n is the number of sites remaining. For the example above, these are the same, and $b = 2$: $N^{(n)} = N/2^n$. As above $K^{(n)} = \mathcal{R}^n(K)$ is the image of the couplings under n -times repeated RG transformation. (Notice that if we were in d dimensions, we would have $b = \lambda^d$, where λ is the linear zoom factor, and the number of sites decimated would not

equal the number remaining even for $\lambda = 2$.) Taking logs of the BHS of the previous equation

$$f(K) = C + \sum_{k=1}^n \frac{N^{(k)}}{N} a^{(k)} + \frac{N}{b^n} \tilde{f}(K^{(n)}). \quad (3.12)$$

If we iterate the RG transformation enough times, and $\tilde{f}^{(n)}$ is finite, its contribution is suppressed by $b^{-n} \rightarrow 0$.

Magnetization. The magnetization can be calculated by taking derivatives of the previous result:

$$M \propto \partial_h f = \langle s_i \rangle$$

but here is some cleverness. By translation invariance the BHS is independent of i . Therefore, we can choose i to be a site that survives all the decimation. Then

$$\langle s_i \rangle_H = \frac{\sum_s s_i e^{-H}}{\sum_s e^{-H}} = \frac{\sum_{s_0} s_i \sum_{s_e} e^{-H(s_0, s_e)}}{\sum_{s_0} \underbrace{\sum_{s_e} e^{-H(s_0, s_e)}}_{=e^{-H'(s_0)}}} = \frac{\sum_{s_0} s_i e^{-H'(s_0)}}{\sum_{s_0} e^{-H'(s_0)}} = \langle s_i \rangle_{H'}.$$

We have just shown that the magnetization is an RG invariant. This result required that we are using a *decimation* scheme, where the spins surviving the RG are a subset of the initial spins. I will come back to alternatives soon, and we will see why we need them. This means we can compute the magnetization for a macroscopic system just by following the flow to the end:

$$\langle s_i \rangle = \frac{\sum_{s_i=\pm 1} s_i e^{-H^\infty(s_i)}}{\sum_{s_i} e^{-H^\infty(s_i)}}$$

but $H^\infty(s_i) = a^\infty + h^\infty s_i$ (these are the only two possible terms) and h^∞ is the fixed-point value of the Zeeman field. So

$$\langle s_i \rangle = \frac{\sum_{s_i=\pm 1} s_i e^{-h^\infty s_i}}{\sum_{s_i} e^{-h^\infty s_i}} = \frac{-e^{+h^\infty} + e^{-h^\infty}}{e^{+h^\infty} + e^{-h^\infty}} = -\tanh h^\infty.$$

I emphasize again that this works only for decimation schemes.

3.4 Need for other schemes

Let's think about decimation of the Ising model on the square lattice. Again it is bipartite, and we can do the sum of each spin on one of the sublattices fixing the spins on the other, one at a time:

$$\sum_{s_x} e^{J s_x (s_1 + s_2 + s_3 + s_4)} \equiv \psi(s_1 + s_2 + s_3 + s_4).$$

The argument of the function ψ defined by this equation only takes the values $0, \pm 2, \pm 4$. We've set the Zeeman field $h = 0$, so it is even $\psi(-x) = \psi(x)$, and there are only three values of the argument we care about. For these values, it can be written as

$$\psi(s_1 + s_2 + s_3 + s_4) = e^{a' + J'(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1 + s_1 s_3 + s_2 s_4) + M' s_1 s_2 s_3 s_4}$$

with values of a', J', M' determined by J which you can figure out. The first term a' is just a constant. The first four terms multiplied by J' are nearest-neighbor interactions on the new (square) lattice with lattice spacing $\sqrt{2}a$ (rotated by $\pi/4$). This means $\lambda = \sqrt{2}$; the number of remaining spins is $N/2$, so $b = \lambda^{d-2} = 2$ as expected in two dimensions. The next two terms are next-nearest-neighbor exchange couplings (s_1 and s_3 are separated by $2a$) of the same size. Finally, M' multiplies a qualitatively-new 4-spin interaction, proportional to J^4 . Ick!

This isn't so bad if we think of the initial Hamiltonian as sitting in a special corner of the large and high-dimensional space of possible couplings, and the RG just moves us to a more generic point:

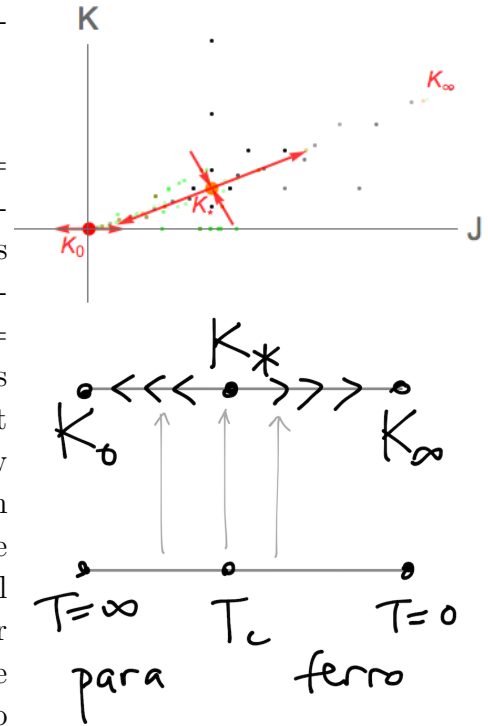
$$(J, 0, 0, \dots) \xrightarrow{\mathcal{R}} (J', K', M' \dots).$$

That's just a little ugly. But there's a reason why it's objectively bad: we can't repeat this RG step. After the first iteration, we generate couplings between spins of the same sublattice of the remaining square lattice. This means we can't just sum them independently anymore. We could do some uncontrolled truncation, or we can find a better scheme. There are 2d lattices for which a decimation scheme can work (*i.e.* can be iterated).

We can nevertheless persevere by truncating the generation of couplings. For example, if we keep terms only to order J^2 and order K , we do not generate any further couplings beyond J, K , and we find a closed set of RG recursion equations:

$$J' = K + 2J^2, \quad K' = J^2.$$

These equations have three fixed points: $(J, K) = (0, 0), (\infty, \infty)$ and $(1/3, 1/9)$. The nearby flow diagram is indicated at right. Fixing the couplings and varying T amounts to the replacement (J, K) to $(J/T, K/T)$. Increasing the temperature corresponds to scaling J, K down towards $K_0 = (0, 0)$, the infinite-temperature fixed point, where everyone is decoupled. This point and the zero-temperature fixed point (K_∞ , where all couplings are infinite) are separated by a new fixed point *with a single relevant perturbation*. Let's focus on just the relevant dimension (which is not orthogonal to the temperature direction), so we can draw a one-dimensional plot (after all, we are already ignoring infinitely many other irrelevant directions). We see that there is a critical value T_c below which we flow to K_∞ , and above which we flow to K_0 . A fixed point with a single relevant operator describes a critical point, a continuous phase transition between two phases.



3.5 Low-temperature expansion, and existence of phase transition in $d > 1$

Maybe you still don't believe me that there has to be a phase transition in the nearest-neighbor Ising model, even in $d = 2$. At arbitrarily high temperatures, there is definitely no spontaneous symmetry breaking, since each spin is just looking out for itself and there can be no collective behavior, and $\langle s \rangle = m = 0$. At $T = 0$, the spins all align (as they do in $d = 1$, too). Here is an argument (due to Peierls, still) that the ordered state survives to some finite temperature for $d \geq 2$.

A configuration of lowest energy, say all $s_i = +$, has energy $E_0 = -JN_l$, where N_l is the number of links of the graph (this is $2N$ for the square lattice since there are two links in each unit cell, one up and one right). The minimal excitation above the ordered configuration flips one spin and has energy $E_0 + 2zJ$ where z is the number of neighbors of the flipped spin. We can estimate the entropy of a dilute gas of n such flipped spins, with energy $E(n) \sim E_0 + 2Jzn$; the number of configurations is again approximately $\Omega(n) = \binom{N}{n}$, and so their free energy is

$$F \stackrel{\text{Stirling}}{\sim} nz2J - T(N \log N - (N - n) \log(N - n) - n \log n).$$

(Actually, the flipped spins have a short-ranged attraction because if they are adjacent they share a happy bond. We ignore this; think about why we can get away with it.) This is minimized by an equilibrium density of flipped spins

$$\frac{n_{\text{eq}}}{N} \simeq e^{-2zJ/T}.$$

All this so far is just like in the $1d$ argument, except we replaced 2 neighbors with z neighbors, and counted spin flips rather than domain walls.¹⁴

Here's the catch: The magnetization is not so strongly affected by a flipped spin as it is by a domain wall. It is only decreased from the maximum ($m = 1$) to

$$m = 1 - 2\frac{n_{\text{eq}}}{N} \simeq 1 - 2e^{-2zJ/T} \simeq 1 \quad \text{if } T \ll zJ.$$

So this means that at low (but nonzero) temperature, the magnetization survives. And therefore something interesting has to happen at some intermediate temperature.

[\[End of Lecture 6\]](#)

¹⁴Why did we count domain walls in $d = 1$? Because in $d = 1$, the energy of a row of k flipped spins in a row is the same for any k . The elementary dynamical object is really the kink itself in $d = 1$. This is the tip of an iceberg called 'fractionalization'.

3.6 A word from our sponsor

We've been spending a lot of time talking about Ising models. Let's take a break and talk about another role it plays in physics.

Lattice gas. Suppose our dynamical variables are the locations $r_1..r_N$ of a collection of point particles. The grand canonical partition function is

$$\Xi(\zeta) = \sum_N \frac{\zeta^N}{N!} \int d^d r_1 \dots d^d r_N e^{-\beta \sum_{i<j} V(r_i - r_j)} \quad (3.13)$$

where ζ is a fugacity for particle number, and $V(r)$ is an interparticle potential, which usually has a short-range repulsion and long-range attraction (most kinds of particles find each other vaguely attractive from far away...). The kinetic energy was $\sum_i \frac{p_i^2}{2m}$, but we did the p integrals already: $\int d^d p e^{-\beta \frac{p^2}{2m}} = (\pi m T)^{d/2}$.

These integrals in (3.13) are hard. If our interest is in critical behavior, we can zoom out, and take the particles to live at the sites of a lattice $r \in \Lambda$, so our dynamical variables are instead the number of particles at site r , $n(r)$. To implement the short-range repulsion, we take $n(r) = 0, 1$. Then we study

$$\Xi_\Lambda(\zeta) = \sum_{\{n(r)=0,1\}} \zeta^{\sum_r n(r)} e^{-\frac{1}{2}\beta \sum_{r,r'} J_{r,r'} n(r)n(r')}$$

where $J(r-r')$ implements the long-ranged part of the potential. If we change variables to $s(r) \equiv 2n(r) - 1 = \pm 1$, we have

$$H(s) = -\frac{1}{2}\beta \sum_{r,r'} J_{r,r'} s_r s_{r'} - \beta \sum_r h_r s_r + \text{const}$$

with $\beta h_r = \frac{1}{2} \log \zeta + \beta \sum_{r'} J_{r,r'}$. This is an Ising model. The ferromagnetic ordering transition is the liquid-gas transition! Recalling that this occurs at $h = 0$, we see that the $s \rightarrow -s$ symmetry of the Ising model (with $h = 0$) is a symmetry of the lattice gas only near the critical point – it is an ‘emergent symmetry’.

Another useful interpretation of the same model is as a ‘binary fluid’, where $n = 0, 1$ represent occupation by two kinds of fluid elements.

3.7 Duality

[Feynman, *Statistical Mechanics* §5.4, Parisi, §4.6]

Let's set $J = 1$ for a bit, since only the combination βJ appears in this discussion.

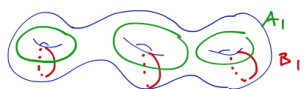
Now consider again the low-temperature expansion: previously we thought about flipping some dilute collection of isolated spins, each of which costs a (Boltzmann) factor of $e^{-2\beta Jz}$ in the partition function. More accurate is to think of this as a sum over *islands* of reversed spins. If we were speaking about a 3d lattice, they would be regions, with two-dimensional boundaries. Let's focus on the case of a 2d lattice, so that the boundaries of the islands are one-dimensional curves.

If we keep track of the boundaries between these regions, we have complete information about the spin configuration, up to an overall reversal. The weight for a given configuration C is $e^{-2\beta\ell(C)}$ where $\ell(C)$ is the total length of boundary in that configuration. We could include a field h in this description, that multiplies the weight by

$$e^{-\beta h(A_{\text{in}}(C) - A_{\text{out}}(C))}$$

where $A_{\text{in/out}}(C)$ is the number of sites inside/outside the chosen configuration of boundaries.

Can we represent the entire partition sum as a sum over these boundaries (which are called *domain walls*)? It is not just a sum over curves. Notice that the boundaries are always closed curves – it is a deep topological fact that a boundary has no boundary. Furthermore, the boundary curve is always *contractible* in the sense that by flipping back some of the spins in the region to agree with the rest, we can gradually shrink it away to nothing. Here's an example of some curves that are not contractible, on the



surface of a complicated pastry:

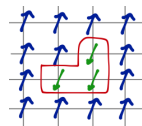
The curves A and B are not the boundary of any 2d region on the surface of the pastry. Let us restrict ourselves for now to lattices which do not contain such curves (they are called *simply-connected*).

It is useful to introduce at this point the *dual lattice*: for a 2d lattice Λ , this is a lattice $\hat{\Lambda}$ whose sites correspond to the plaquettes of Λ . A link of Λ separates two plaquettes of Λ ; it corresponds to a link of $\hat{\Lambda}$ connecting the two corresponding sites of



The domain walls of a spin configuration on the sites of Λ cover a set

of links of $\hat{\Lambda}$:



But our description of the low-temperature expansion on Λ as

$$Z_\Lambda(T) = 2 \sum_C e^{-2\beta\ell(C)} \quad (3.14)$$

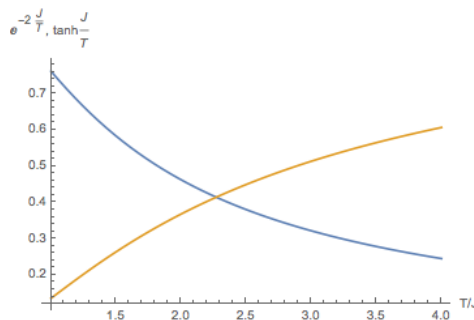
has exactly the same form as our high-temperature expansion (3.9) if we identify

$$e^{-2\beta} = \hat{v} \equiv \tanh \hat{\beta} .$$

This equation relates high temperature (small J) on Λ to low temperature (large J) on $\hat{\Lambda}$. It is due to Kramers and Wannier.

For the square lattice, the dual lattice is the square lattice again! This means that if there is only *one* critical point (remember: fixed points of the RG are rare and valuable), it must be a fixed point (not only of the RG but also) of the duality transformation on the couplings:

$$e^{-2\beta J} = \hat{v} \equiv \tanh \beta J .$$



The dual of the honeycomb lattice is the triangular lattice (and vice versa). To learn their critical temperature, we add one more maneuver, called *star-triangle transformation*: The honeycomb lattice is bipartite, and the two sublattices are triangular lattices. By decimating one of the two sublattices, we can relate

$$Z_{\square}^N(J) = \Delta^{N/2} Z_{\triangle}^{N/2}(K)$$

where Δ and K are determined from J by:

$$\sum_{s_x=\pm 1} e^{J s_x (s_1 + s_2 + s_3)} = \cosh^3 J [1 + \tanh^2 J \cdot (s_1 s_2 + s_2 s_3 + s_3 s_1)] \equiv \Delta e^{K(s_1 s_2 + s_2 s_3 + s_3 s_1)} .$$

Combining this with the duality relation we can relate the critical temperature of the Ising model on the triangular lattice to itself.

Here is a table of the critical values of $\frac{1}{\beta J}$ for various lattices. z is the coordination number, the number of neighbors of each site.

Λ	z	T_c/J
–	2	0
\square	3	1.52
\square	4	2.27
\triangle	6	3.64

The first entry is the 1d chain. You can see that the critical temperature rises with coordination number.

Notice that the disordered (high-temperature) phase is dual to the ordered (low-temperature) phase. That this is not a contradiction is related to the factor of 2 in front of the partition sum in (3.14): the description in terms of domain walls doesn't really know about the magnetization.

If you can't wait to learn more about the many generalizations of Kramers-Wannier duality, here are some references: [Kogut](#), [Savit](#).

There is more to be said about this sum over curves. They can be used to solve the 2d Ising model exactly. They are the worldlines of free fermions.

3.8 Block spins

Here we introduce a more general class of coarse-graining transformations, called *blocking*. The essential rule is that the partition function is an RG invariant:

$$Z = \sum_s e^{-H(s)} \stackrel{!}{=} \sum_{s'} e^{-H'(s')}. \quad (3.15)$$

Previously, in the decimation schemes, the coarse-grained variables $\{s'\} \subset \{s\}$ were a subset of the microscopic variables. This is a special case of the more general blocking rule

$$e^{-H'(s')} \equiv \sum_s \prod_{\text{blocks}, b} T(s'; s_{i \in b}) e^{-H(s)}$$

where T is a function which decides how the *block spin* s' depends on the spins s_i in the block. Decimation is the special case where we weight the opinion of one of the spins over all the others:

$$T_{\text{decimate}}(s'; s_{i \in b}) = \delta_{s', s_2}.$$

Another option is majority rule:

$$T(s'_b; s_{i \in b}) = \begin{cases} 1, & \text{if } s'_b \sum_{i \in b} s_i > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that for each block, $\sum_{s'=\pm 1} T(s'; s) = 1$ guarantees (3.15). Furthermore, it is useful if $T(s'; s) \geq 0$, so that everything is a probability. Also, it is best if T preserves the symmetries of the system.

4 Mean Field Theory

Mean field theory (MFT) is always simple and sometimes right, and it is all around us in physics departments, so we must understand well when to believe it. We will see that it goes bad near critical points, and the RG will come to our rescue. It is great for getting a big picture of the phase diagram.

We'll give three roads toward MFT, in order of decreasing squiggleness. For definiteness, consider the Ising model, on any graph Λ :

$$Z = \sum_s e^{-H(s)}, \quad H(s) = -\frac{1}{2} \sum_{i,j \in \Lambda} J_{ij} s_i s_j - h \sum_i s_i.$$

(I've put the $\frac{1}{2}$ to agree with our previous definition of J , because here the sum is over all i, j .) Mean field theory is an attempt to fulfill the urge everyone has to be able to do the sums over the spins one at a time. If only J were zero, we could do this, for example to compute the magnetization:

$$m = \langle s \rangle = \frac{\sum_{s=\pm 1} s e^{\beta h s}}{\sum_{s=\pm 1} e^{\beta h s}} = \tanh \beta h. \quad (4.1)$$

But $J \neq 0$ is much more interesting. So what to do?

Our first approach to MFT is via political science. Mean field theory is the physics realization of libertarian political philosophy¹⁵. This has two ingredients.

(1) No one cares about anyone else. What I mean by this is: put yourself in the position of one of the spins in the Ising model. How does it even know about its neighbors? Its role in the hamiltonian is

$$H(s_i) = s_i \left(-\frac{1}{2} \sum_{j \neq i} J_{ij} s_j - h \right).$$

From its point of view, this is just like some external magnetic field depending on what its neighbors are doing. What's s_j ? Well, it's probably equal to its average value, $\langle s_j \rangle = m$. So let's just forget everyone else, and assume they are average and incorporate them into an effective magnetic field:

$$h_{\text{eff}} \equiv \frac{1}{2} \sum_j J_{ij} m + h.$$

The second tenet is

(2) Everyone's the same (and I am just like everyone). That is: if there is only

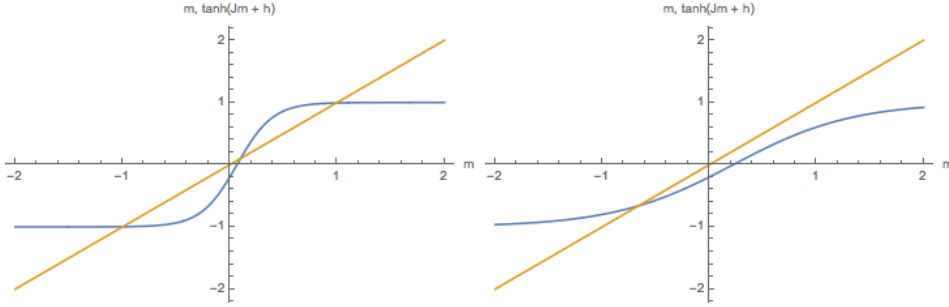
¹⁵Disclaimer: I actually don't know anything at all about political philosophy and made all this up during lecture.

one spin in the world, and this is the field it sees, then we can compute m using (4.1):

$$m \stackrel{(4.1)}{=} \tanh \beta h_{\text{eff}} = \tanh (zJm + h).$$

Here I defined $J \equiv \frac{1}{2} \sum_j J_{ij}$. This is an equation for m ! We can solve it!

At least graphically or numerically we can solve it. Here is m (yellow) and $\tanh(zJm + h)$ (blue) plotted versus m for two values of J (large and small compared to T , with some small h)



Here's our second approach to MFT. Basically, here we will be more explicit about what we're leaving out (but it is the same as the previous discussion). We rewrite the interaction term in the Ising hamiltonian as

$$\begin{aligned} s_i s_j &= (m + (s_i - m))(m + (s_j - m)) \equiv (m + \delta s_i)(m + \delta s_j) \\ &= m^2 + m(\delta s_i + \delta s_j) + \mathcal{O}(\delta s)^2 = -m^2 + m(s_i + s_j) + \mathcal{O}(\delta s)^2. \end{aligned} \quad (4.2)$$

We are going to treat the fluctuation about the mean δs as small. Then

$$\begin{aligned} -H &= \frac{1}{2} \sum_{ij} J_{ij} (m(s_i + s_j) - m^2) + h \sum_i s_i + \mathcal{O}(\delta s)^2 \\ &= -\frac{1}{2} N J m^2 + (zJm + h) \sum_i s_i + \mathcal{O}(\delta s)^2. \end{aligned} \quad (4.3)$$

N is the number of sites, and $J \equiv \sum_j J_{ij}$. The contribution Jm to the external field from the neighbors is sometimes called the 'molecular field'. What we are neglecting here (when we drop the $\mathcal{O}(\delta s)^2$ in a moment) is the correlations between the spins at different sites i, j . This is not small if $|r_i - r_j| < \xi$, by definition of the correlation length ξ . Brutally ignoring the correlations, then, we can do all the sums have

$$Z \simeq e^{-\frac{1}{2} N \beta J m^2} (2 \cosh \beta (zJm + h))^N \equiv Z_{\text{MFT}}$$

So in this approximation, the free energy density is

$$f_{\text{MFT}}(m) \equiv -\frac{T}{N} \log Z_{\text{MFT}} = \frac{1}{2} J m^2 - T \log \cosh \beta (zJm + h) + \text{const.}$$

I claim, and will prove next, that $f_{\text{MFT}}(m) \geq f$ is an upper bound on the correct free energy. This is true for every m , and so the best bound comes from minimizing over m . That condition gives back the equation for m (namely $m = \tanh \beta(zJm + h)$) that we got from self-consistency above. (And it will tell us what to do in the case of $J \gg T$ where there are three solutions.)

Our third approach is the *variational method*. [There is a good discussion of this in Parisi's book.] It will give our proof that $f_{\text{MFT}}(m)$ upper bounds f . The idea can be found from a Bayesian viewpoint on statistical mechanics. Let's put this in a box:

Bayesian viewpoint on statistical mechanics. Suppose we are given a (classical) physical system, defined by a configuration space (*e.g.* the values of a bunch of spins $\{s\}$) and a Hamiltonian ($H(s)$). Further suppose that the only thing we know about the state of the system is the average energy E . What probability distribution $P(s)$ should we use to make predictions? We don't want to add unjustified information. One way is to find the distribution P_\star which maximizes the (Shannon) entropy

$$S[P] \equiv - \langle \log P \rangle_P = - \sum_s P(s) \log P(s),$$

¹⁶ subject to the constraint that $E = \langle H \rangle_{P_\star} \equiv E[P_\star]$. The distribution should also be normalized $\sum_s P(s) = 1$. We can impose these conditions with lagrange multipliers:

$$\Phi[P] \equiv S[P] + b(E[P] - E) + a(\sum_s P(s) - 1) = - \sum_s P(s) \log(P(s)) + \sum_s (bH(s) + a)P(s) - bE - a$$

$$\begin{aligned} \frac{\delta \Phi[P]}{\delta P(s)} &= - \log P(s) - 1 + bH(s) + a \\ \implies P_\star(s) &= e^{bH(s) + a - 1} \end{aligned}$$

where a, b must be determined to satisfy the two constraints.

If instead of fixing the average energy, we want to fix the temperature $1/\beta$, what do we do? We should instead find the distribution $P_\star(s)$ which *minimizes* the free energy

$$F[P] = E[P] - S[P]/\beta$$

¹⁶A useful way to think about this quantity is the following. Given a distribution $P(s)$, the quantity $-\log P(s)$ is called the *surprise* of the configuration s – the bigger it is, the more surprised you should be if s actually obtains. So the Shannon entropy is simply the *average surprise* (or maybe the *expected surprise*). Clearly, all else being equal, we will make the best predictions using the distribution that minimizes the expected surprise. If you like this perspective on the world, the place to get more is E. T. Jaynes, *Probability Theory: The Logic of Science*.

as a functional of P . It is still normalized, so we need to use a lagrange multiplier again, and minimize

$$F_\lambda[P] \equiv F[P] + \lambda(\sum_s P(s) - 1)$$

which is extremized when

$$0 = H(s) + T + \lambda + T \log P(s)$$

from which we again recover the Boltzmann distribution, $P(s) = e^{-\beta H(s)}/Z$ (the multiplier λ is eliminated in favor of Z by normalizing).

This derivation is useful philosophically (for example, it evades all the vexing questions about ergodicity), and it also implies a variational bound on the free energy F . That is, if we pick some arbitrary *other* distribution $P_{\text{off-the-street}}(s)$, then we know that its free energy is bigger than the correct equilibrium free energy:

$$F[P_{\text{off-the-street}}] \geq F[e^{-\beta H}/Z].$$

[End of Lecture 7]

So: to recover mean field theory, we choose a distribution which we like because we know how to calculate its averages, that is, one which *factorizes*:

$$P_{\text{MFT}}(s) = \prod_i p_i(s_i) \tag{4.4}$$

where we can parametrize the individual factors as $p_i(s) = e^{-\beta h s_i}/z_i$ or as

$$p_i(s) = \frac{1 + m_i}{2} \delta_{s,1} + \frac{1 - m_i}{2} \delta_{s,-1}.$$

It is normalized since each factor is. For any distribution of the form (4.4) all the moments factorize:

$$\langle g_1(s_1) g_2(s_2) \rangle_P = \langle g_1(s_1) \rangle_{p_1} \langle g_2(s_2) \rangle_{p_2}$$

and each factor is

$$\langle g(s_1) \rangle_{p_1} = \frac{1 + m_i}{2} g(1) + \frac{1 - m_i}{2} g(-1).$$

And in particular,

$$\langle s_i \rangle = m_i, \quad \langle H \rangle = -\frac{1}{2} \sum_{ij} J_{ij} m_i m_j - \sum_i h m_i$$

and its entropy is

$$S[P] = -\langle \log P \rangle_P = \sum_i s(m_i), \quad s(x) \equiv -\frac{1+x}{2} \log \frac{1+x}{2} - \frac{1-x}{2} \log \frac{1-x}{2}.$$

Now we apply the variational bound. The free energy $F(m_i) = F[P_{\text{MFT}}^{(m_i)}] \geq F$ upper bounds the true free energy for any m , so we do best by minimizing it:

$$0 = \partial_{m_i} F = - \sum_j J_{ij} m_j - h_i + T \operatorname{arctanh} m_i$$

which gives back

$$m_i = \tanh \beta \left(\sum_j J_{ij} m_j + h_i \right)$$

the mean field equation.

This perspective on mean field theory has the further advantage that it is systematically improvable. For example, rather than writing a completely factorized distribution $p(s) = \prod_i p_i(s_i)$, we could instead consider, for example, a trial state of the form

$$p(s) = \prod_b p_b(s_{i \in b})$$

where b represent some blocks of sites. Such a state is more general than the MFT ansatz, and will have more variational parameters, and necessarily gives a better estimate of the correct free energy. Further thinking in this direction leads to *cluster mean field theory* and *belief propagation algorithms*.

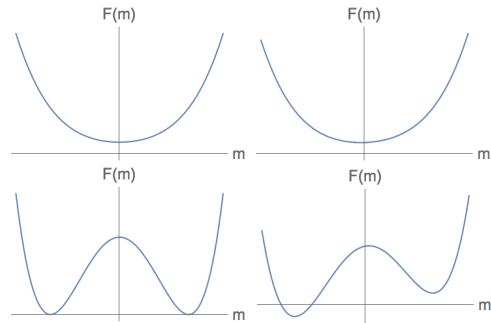
On the **form of the mean-field free energy**. The most important conclusion from the mean field theory is that (for $h = 0$) there are two phases distinguished by whether or not the \mathbb{Z}_2 symmetry is spontaneously broken – at high T , we have $m = 0$, and at low T $m \neq 0$. In between there is a phase transition¹⁷, where m suddenly grows from zero. If we set $h = 0$ and study small m , we can expand f_{MFT} in m and find

$$f_{\text{MFT}}(m) \simeq a + \frac{1}{2} B m^2 + c m^4 + \dots \quad (4.5)$$

where a, c are constants. The coefficient B is

$$B = (1 - \beta J) \equiv bt,$$

where $t \equiv \frac{T - T_c}{T_c}$ is the “reduced” temperature. If $c > 0$, this function looks like one of the figures at right, where the top left figure is for $T > T_c^{\text{MF}} = J$ and the bottom left



¹⁷In case I forgot to say so, a *phase transition* occurs when physical quantities are non-analytic in the parameters at some point in the parameter space – it means that Taylor expanding physics on one side of the phase transition gives the wrong answer (for something) on the other side.

figure $T <$ the critical temperature. If $c < 0$, then we have to keep more terms in the expansion to know what happens. (The right column is with $h < 0$.) So you can see that the minimum of f occurs at $m = 0$ for $T > T_c$ (disordered phase) and $m \neq 0$ for $T < T_c$ (ordered phase). This figure makes it clear that the third solution of the MF equations (at $m = 0$) that exists for $T < T_c$ is a *maximum* of the free energy – it is unstable.

4.1 Landau-Ginzburg theory

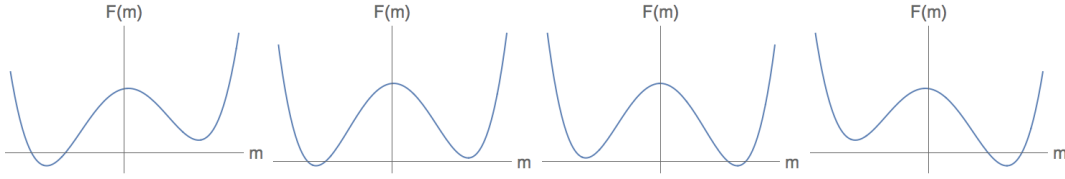
[Parisi §4.3, 5.2] Before drawing any further physical conclusions from the MFT free energy we just derived, let me say some words in defense of this form of the free energy (4.5). These are the words (the idea is due to Landau; this is a paraphrase):

What else could it be?

If the free energy is analytic near $m = 0$, it looks like this. So all that song and dance about justifying mean field theory is really irrelevant to the conclusions we draw about the phase transition from $m = 0$ (at $T > T_c$) to $m \neq 0$ (at $T < T_c$). The dependence of B on $T - T_c$ follows from (4.5) itself! With this assumption, $f_{\text{MFT}}(m)$ is the most general answer, consistent with the symmetry under $\begin{cases} m & \mapsto -m \\ h & \mapsto -h \end{cases}$ (at the same time).

So: the only real assumption leading to (4.5) is the analyticity of $f(m)$. Some points: (1) we will see immediately below that analytic $f(m)$ does *not* mean that the physics is analytic in external parameters – we *can* get critical behavior from this framework. (2) When we find out that MFT gives wrong predictions for critical exponents, we will have to find out how and why we get an $f(m)$ which is not analytic. (3) The fact that the coefficient of m^2 is proportional to the deviation from the critical temperature follows from our analysis of (4.5). The only input from the microscopic calculation (with all the approximations above) is how do the coefficients a, b, c, d depend on the microscopic couplings. Notice that the actual magnetization $m = N^{-1} \sum_{i=1}^N \langle s_i \rangle$ is an average of numbers each ± 1 , and therefore lies between these two numbers. The minimum of $f(m)$ will not satisfy this constraint for all values of $a, b, c, d...$ consistent with the input above: this is a “UV constraint on IR physics” of the kind that the string theorists dream about.

Types of phase transitions. A *first order phase transition* is one where the minimum of the free energy *jumps* from one value to another, distant value, like if the potential evolves as in this comic strip as a function of the parameter in question:



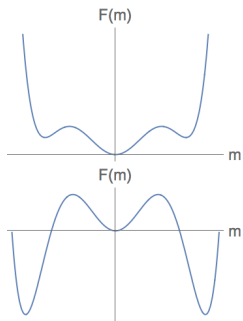
The two configurations need have nothing to do with each other, and there is no notion of universal properties of such a transition. The correlation length need not grow. This is what happens when we vary h from positive to negative, at nonzero $t < 0$. The correlation length stays fixed, but the minimum jumps from $-m_0$ to $+m_0$ as h goes through zero (as in the comic strip above).

The alternative is a *continuous phase transition* which is more interesting, because then, as we will see, there is a field theory which encodes a collection of universal phenomena at and near the critical point.

(Sometimes, one hears about ‘ n th-order’ phase transitions, where the n th derivative of the free energy is discontinuous for various $n \geq 2$, but I haven’t found the need to distinguish between these. Moreover, it is only in mean field theory that the free energy goes like *integer* powers of t (as in (4.6) below); more generally, taking enough derivatives of the free energy will give a *divergent* (not just discontinuous) behavior at the transition. So this more detailed ‘classification’ (due to Ehrenfest) is both incomplete and not useful.)

Notice that when we say that ‘a transition is continuous’ it can depend on what parameter we are varying: at $T < T_c$, as a function of the magnetic field, the transition from one minimum to the other of the Ising model is first order. (This is what’s illustrated in the comic above). But at $h = 0$, there is a continuous transition as T is varied through T_c .

Here are some simple examples of the power of the LG point of view: If we break the Ising symmetry the transition should generically be first order. This allows a cubic term in the potential, and it means that as we cool from high temperatures, one of the two minima at $m \neq 0$ will have $f(m) < f(0)$ before (at a higher temperature than the one where) $f''(0)$ becomes negative.



A continuous transition is, however, not an inevitable consequence of Ising symmetry: if $c < 0$, then we must consider the m^6 term. Depending on the signs, there is a regime where the minima at $m \neq 0$ descend before $f''(0)$ goes negative.

Usually (but not always) $T_c^{\text{MF}} > T_c$, since the fluctuations we are ignoring disfavor the ordered state. (Sometimes in fact $T_c \leq 0$.)

Mean field critical exponents. The very fact that there is a notion of T_c in MFT is worth remarking on. Lots of stuff is non-analytic at T_c !

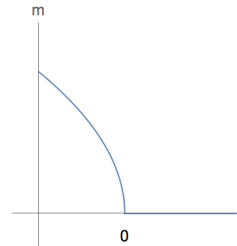
Near T_c , we can expand

$$f_{\text{MF}}(m) = a + btm^2 + cm^4 + \mu hm + \dots$$

where $T \equiv \frac{T-T_c}{T_c}$ is the non-dimensionalized deviation from the critical temperature. Notice that a, b, c, μ really do depend on T , but only weakly (*i.e.*, $a = a_0 + a_1t + \dots$).

When $h = 0$, the free energy is minimized when :

$$m = \begin{cases} 0, & t > 0 \\ \pm \sqrt{\frac{b}{2c}} \sqrt{-t}, & t < 0 \end{cases}$$



The magnetization critical exponent is called β and $\beta_{\text{MFT}} = \frac{1}{2}$.

When $h \neq 0$ (and small) and $t > 0$, we can ignore everything but $f \sim a + btm^2 + \mu hm$ (beware typos in Cardy's book here) to find

$$m \sim \frac{\mu h}{bt}.$$

This exponent in t (which determines $\chi = \partial_h m \sim t^{-\gamma}$) is called γ , and $\gamma_{\text{MFT}} = 1$.

Right at the transition, $t = 0$, we must keep the quartic term and we find

$$m \sim \left(\frac{\mu h}{c} \right)^{1/3}.$$

This exponent is called δ and $\delta_{\text{MFT}} = \frac{1}{3}$. (I'm mentioning this botany of greek letters because there are people for whom these letters are close friends.)

Finally, the free energy density evaluated at the minimum, at $h = 0$, is

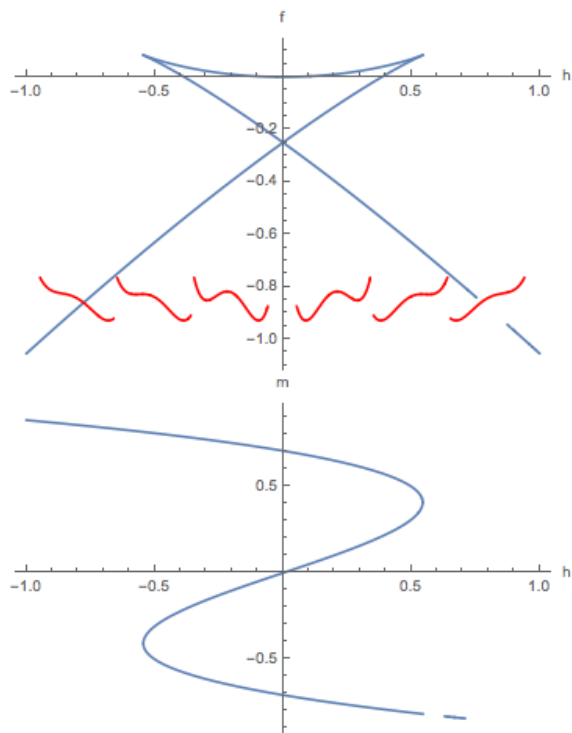
$$f(t) = \begin{cases} a, & t > 0 \\ a - \frac{(bt)^2}{2c}, & t < 0 \end{cases} \quad (4.6)$$

which means that $\partial_t^2 f$ jumps at the transition; this jump is actually an artifact of MFT.

Otherwise, the behavior in general predicted by MFT is good, but we'll see that the values of these exponents aren't always right (and why and when, and then we'll understand how to fix them). In particular, mean-field critical exponents are always rational numbers. In contrast, for the 3d Ising model, $\beta = 0.326419(3)$, which isn't looking very rational. This value comes from the [conformal bootstrap](#) program to solve and classify fixed points.

Notice that the critical exponents do not depend on the particular values of the parameters $a, b, c, \mu \dots$. This is one reason to hope that they can be understood, and that they are universal in the sense defined earlier.

It is worth thinking about what the extrema of this potential do as we vary the parameters. At right is a plot of the free energy evaluated at all of the critical points of $f(m)$ as h varies (the other couplings are fixed to $T < T_c$). (This picture is sometimes called a ‘swallowtail’.) Inset in red is the shape of the potential at the corresponding value of h . Plotted below is the corresponding magnetization. Notice that the number of (real) critical points goes from 1 to 3 as $|h|$ is decreased below some value; the two new extrema are pair-produced from the complex plane, that is, the new extrema come in pairs and have a *larger* free energy. Notice further that $\partial_h^2 f > 0$ along the top trajectory – this is the maximum near the origin. The other one is actually a local minimum – a *metastable state*, responsible for hysteresis phenomena at the first-order transition. More on the physics of this in §5.5.



LG Theory for other symmetries. Here is another illustration of the Power of Landau. We've been studying models with a \mathbb{Z}_2 symmetry acting by $\begin{cases} m & \mapsto -m \\ h & \mapsto -h \end{cases}$. Suppose instead of this, we made the replacement $\mathbb{Z}_2 \rightsquigarrow \text{O}(n)$ rotation symmetry acting on a generalization of the magnetization with n components, $m \rightsquigarrow m_a$, in that case the external field would be $h \rightsquigarrow h^a$, and the transformation rule would be

$$\begin{cases} m_a & \mapsto R_a^b m_b \\ h^a & \mapsto h^a R_a^b \end{cases}$$

(where R is an orthogonal matrix ($R^t R = 1$) so that $m_a h^a$ is invariant and can be added to the hamiltonian). Incidentally, the field m_a playing the role of the magnetization, the thing that orders at low temperatures, is called the *order parameter*.

What's the free energy in this case? If it's analytic in m , for small m it must be of the form¹⁸

$$f_{\text{LG}}(m_a) = a + bt \sum_a m_a^2 + c \left(\sum_a m_a^2 \right)^2 + \dots + \mu h^a m_a .$$

For $t < 0$, a minimum occurs at $m_a = (m_0, 0, 0\dots)_a$ with $m_0 = \sqrt{\frac{-bt}{2c}}$, as well as at all its images under an arbitrary $\text{O}(n)$ rotation. The $\text{O}(n)$ symmetry is broken in the sense that the free energy minima form a nontrivial orbit of the symmetry (and furthermore, the free energy at the minimum will be non-analytic in h near $h = 0$). This degeneracy has the following consequence. If we expand $m_a = (m_0, 0, 0\dots)_a + \delta m_a$ about the minimum, we find

$$f_{\text{LG}} = c + 2b|t|\delta m_1^2 + \mathcal{O}(\delta m^4)$$

– the quadratic terms are completely independent of the other N components of the fluctuations $\delta m_2 \dots \delta m_N$! We'll see in a moment that this absence of a restoring force means that those degrees of freedom have infinite correlation length, everywhere in the ordered phase. They are called *Goldstone modes*.

[End of Lecture 8]

¹⁸Dachuan Lu reminds me that for some values of n , there can sometimes be extra invariants, such as $\epsilon_{i_1 \dots i_n} m_{i_1} \dots m_{i_n}$.

‘Microscopic’ Landau-Ginzburg Theory. In our variational derivation of mean field theory, we actually derived a stronger bound, since we allowed for spatially-varying magnetization. Let’s combine the Landau point of view with the knowledge that the free energy is extensive¹⁹ to learn the answer without doing any work. Because F is extensive, we can write the free energy as a sum over a contribution associated to each lattice site, or patch of the lattice $F = \sum_i f_i$, where f_i depends on the magnetization m_i at site i and nearby sites. (Think about assembling the system from big enough chunks.) If the correlation length is not so small, f_i will vary smoothly and we can approximate this as an integral: $\sum_i f(x_i) \simeq a^{-d} \int d^d x f(x)$. The integrand, in turn, depends locally on the field and its derivatives. Translation invariance forbids any explicit dependence on x :

$$F[m] = \int d^d x f(m(x), \vec{\nabla} m(x), \nabla^2 m(x) \dots).$$

Symmetries further constrain the form of f : \mathbb{Z}_2 symmetry forbids terms odd in m and h , parity symmetry forbids terms with an odd number of derivatives, rotation invariance requires us to dot all the vector indices together. So, under the crucial analyticity assumption, we have

$$f_{\text{LG}} = V(m) + \kappa \vec{\nabla} m \cdot \vec{\nabla} m + \kappa' (\nabla^2 m)^2 + \dots \quad (4.7)$$

where $V(m) = a + Bm^2 + cm^4 + dm^6 + \dots$ is the value when m is constant in space – it contains all the information about the mean field treatment of phase transitions, some of which we discussed above.

We will have a lot more to say about how to organize this expansion. So far it is an expansion in powers of m (since know that in the neighborhood of the critical point m is small). It is also an expansion in the number of derivatives, something like the dimensionless quantity $a \nabla m$, where a is the lattice spacing. If this quantity is not small then we are asking the wrong question, because the ‘field’ we are treating as continuous is varying rapidly on the scale of the lattice spacing a . The RG will give us a better understanding of this expansion: we’ll see that operators with more derivatives are more irrelevant (near any of the fixed points under discussion here).

The equation (4.7) contains an enormous amount of information. To better appreciate it, let’s first discuss the mean-field treatment of the correlation function.

By the way, what exactly *is* the LG free energy? It is not convex in m , so how can it be the actual free energy?

¹⁹I owe you some discussion of why this is the case. This happens in §5.1.

[Goldenfeld §5.6] The answer to this is that it is the free energy with the constraint that the (coarse-grained) magnetization is fixed to be $m(r)$:

$$e^{-\beta F_{\text{LG}}[m]} \equiv \sum_s e^{-\beta H(s)} \prod_{\text{blocks}, r} \delta \left(\sum_{i \in r} s_i - m(r) N_{\Lambda}(r) \right). \quad (4.8)$$

Here r denotes a block, and $N_{\Lambda}(r)$ is the number of sites in the block r . This is just like the construction of the block-spin effective Hamiltonian. It is only more ambitious in that we are hoping that $m(r)$ is smoothly varying in r , which will be true if $\xi > a$. So the LG free energy S can be regarded as (a parametrization of) the coarse-grained free energy.

It is indeed analytic in m , since we need to do only a finite number of sums in (4.8). And, also because there is only a finite number of sums, it need not be convex.

How do we get the actual, thermodynamic free energy from F_{LG} (which is convex and need not be analytic in its arguments)? We have to do the rest of the sums, the ones over m :

$$e^{-\beta F} = \sum_{\{s\}} e^{-\beta H(s)} = \sum_m e^{-\beta F_{\text{LG}}[m]}.$$

Because $m(r)$ is a continuous variable, ‘ \sum_m ’ is actually an integral, one for every block, r :

$$\sum_m = \int \prod_r dm(r) \equiv \int [Dm]$$

where the right equation defines what we mean by such a ‘functional integral.’

Altogether, we have

$$Z = \int [Dm] e^{-\beta F_{\text{LG}}[m]}$$

– we have rewritten the partition function (in a regime of moderately large correlation length) in terms of a field theory functional integral. The quantity appearing in the exponent of such an integral

$$Z = \int [Dm] e^{-S[m]}$$

is usually called the (euclidean) action, $S[m] = \beta F_{\text{LG}}[m]$.

Doing this integral over m (which is our job for the next few weeks) is what restores convexity of F , and what can allow F to be non-analytic, and what can produce non-mean-field critical behavior.

4.2 Correlations; Ginzburg criterion for MFT breakdown

[Goldenfeld §5.7] You might think that the spirit of mean field theory is antithetical to obtaining information about correlations between the spins, since after all that was precisely what we ignored to do the sums. Not so!

Here's a first pass. The connected correlator (assume translation invariance) is

$$G(r_i - r_j) \equiv \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle.$$

The magnetization is related to $\langle s_i \rangle$ by

$$m = \frac{1}{N} \sum_i \langle s_i \rangle = \frac{1}{\beta N} \partial_h \log Z$$

and the (isothermal magnetic) susceptibility is

$$\chi_T = \partial_h m = \frac{1}{N\beta} \partial_h^2 \log Z = \frac{1}{NT} \sum_{ij} G(r_i - r_j) = \frac{1}{T} \sum_i G(r_i). \quad (4.9)$$

This is called the static susceptibility sum rule. It relates a thermodynamic quantity χ_T to a (integrated) correlation function. If the correlation length is big enough, $\xi > a$, then we can approximate the sum by an integral

$$\chi_T = \frac{1}{T a^d} \int d^d r G(r).$$

Is the integral well-defined? The lower limit of integration, the UV, is fine because we are talking about a lattice model. When ξ is finite, the fact that the correlations fall off rapidly $G(r) \stackrel{r \gg a}{\sim} e^{-r/\xi}$ means that the integral converges in the IR (the upper limit of integration) as well.

But: $\chi_T \rightarrow \infty$ at the critical point, in fact we saw above that $\chi_T \stackrel{\text{MFT}}{\sim} \frac{1}{T-T_c} + \text{regular terms}$ ²⁰ as $T \rightarrow T_c$. The only way this can happen consistently with the susceptibility sum rule is if $\xi \rightarrow \infty$ as well at the transition. We'll see in a moment with what power it diverges.

MFT for $G(r)$. We can actually do better and find the form of $G(r)$ *within* the mean field approximation. This is because $G(r)$ is a *response function*. Here's what this means.

When $h = 0$, the correlation function is

$$\langle s_r s_0 \rangle = \frac{\sum_s s_r s_0 e^{-H(s)} \cdot 1}{\sum_s e^{-H(s)} \cdot 1}$$

²⁰If I keep chanting 'γ = 1' maybe I will remember these letters someday.

where we can write 1 cleverly as

$$1 = \delta_{s_0,1} + \delta_{s_0,-1}.$$

Using the fact that $H(-s) = H(s)$ (for $h = 0$), we have

$$\langle s_r s_0 \rangle = \frac{\sum' s_r e^{-H(s)}}{\sum' e^{-H(s)}} \equiv \langle s_r \rangle'$$

where \sum' means we sum over all the spins but the one at 0, and we fix $s_0 = +1$, and $\langle \dots \rangle'$ denotes expectation in this ensemble. So the correlation function $\langle s_r s_0 \rangle$ is just the magnetization at r , $m(r)$ in response to an (infinite) applied field (completely) localized to $r = 0$. In the presence of this localized source, $m(r)$ will certainly depend on its distance from the source. But the mean field equation (for $r \neq 0$) still takes the form

$$\begin{aligned} m(r) &= \tanh \beta \left(\sum_{r'} J_{rr'} m(r') \right) \\ &\stackrel{m \ll 1}{\simeq} \beta \sum_{r'} J_{rr'} m(r') \quad (r \neq 0). \end{aligned}$$

In the second line, we retreated to small m , which is useful for $T > J$. (Otherwise maybe we need some numerics.) We can do better and include the corrections at the origin, by including a source:

$$m(r) = \beta \sum_{r'} J_{rr'} m(r') + A \delta_{r,0}.$$

This (linear!) equation

$$\sum_{r'} (\delta_{rr'} - \beta J_{rr'}) m(r') = A \delta_{r,0}$$

can be solved by Fourier transform (assuming translation invariant couplings $J_{rr'} = J(r - r')$). That is, the matrix we need to invert is diagonal in momentum space. That is, take $\sum_r e^{i\vec{k} \cdot \vec{r}}$ (BHS) to get:

$$(1 - \beta \tilde{J}(k)) \tilde{m}_k = +A,$$

where

$$\tilde{m}_k \equiv \sum_{r \in \Lambda} e^{i\vec{k} \cdot \vec{r}} m(r), \quad m(r) = \int_{\text{BZ}} \mathrm{d}^d k e^{-i\vec{k} \cdot \vec{r}} \tilde{m}_k.$$

In the inversion formula, the integral is over the Brillouin zone of the lattice Λ ; for a cubic lattice, this just means $k \in (-\pi/a, \pi/a]$. The Fourier transform of the coupling is

$$\tilde{J}(k) \equiv \sum_r e^{i\vec{k}\cdot\vec{r}} J_{r,0}.$$

For example, for a cubic lattice, this is $\tilde{J}_{\text{cubic}}(k) = \sum_{\mu=x,y,z} 2 \cos k_\mu a$, where a is the lattice spacing. For small k , the general case is

$$\tilde{J}(k) = zJ(1 - R^2 k^2) + \mathcal{O}(k^4)$$

where $R^2 \equiv \frac{\sum_r r^2 J_{r,0}}{\sum_r J_{r,0}}$ is the *range-squared* of the coupling. In the last expression we assumed the lattice had an inversion symmetry so that there are no terms with odd powers of k . We'll be interested in small k because it determines the long-distance behavior of $G(r)$.

Therefore,

$$\tilde{m}_k \simeq \frac{A}{1 - \beta z J (1 - R^2 k^2)}$$

and (using the mean-field relation $bt = 1 - \beta z J$)

$$\tilde{G}_k = \frac{A/(R^2 \beta z J)}{k^2 + \xi_{\text{MF}}^{-2}}, \quad \xi_{\text{MF}} = \frac{R}{\sqrt{bt}} \sim t^{-1/2}.$$

You can check that $\int d^d r G(r) = \tilde{G}_{k=0} = \chi_T = \frac{\mu}{bt}$, independent of R as we found above, consistent with the susceptibility sum rule.

Why is ξ in this formula the correlation length? Fourier transform back:

$$G(r) \sim \frac{e^{-r/\xi_{\text{MF}}}}{r^{\frac{d-1}{2}}}$$

which is a formula named after Ornstein-Zernicke, I don't know why. So we've found the rate at which the correlation length diverges as we approach the critical temperature from above (in the mean field approximation) $\xi_{\text{MF}} \sim \frac{1}{\sqrt{t}}$; This scaling relation $\xi \sim t^{-\nu}$, defines another critical exponent ν whose mean-field value is $\nu_{\text{MF}} = \frac{1}{2}$.

Right at $t = 0$, we have

$$G(r) = \int d^d k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} \sim r^{-d+2}$$

which says $\eta_{\text{MF}} = 0$.

Ginzburg criterion for breakdown of MFT. [Goldenfeld §6] So, is mean field theory right? To get it, we had to ignore the following term in the hamiltonian

$$\Delta H = \sum_{r,r'} J_{rr'} \delta s_r \delta s_{r'}.$$

A necessary condition for its self-consistency is that the expected value of this term, calculated within MFT, is small compared to the MFT energy:

$$\langle \Delta H \rangle_{\text{MFT}} \stackrel{!}{<} E_{\text{MF}}.$$

The right hand side divided by N is

$$\frac{1}{N} E_{\text{MF}} = -\partial_{\beta} (\beta f_{\text{MF}}) \sim Jt,$$

where we used $f_{\text{MF}} \stackrel{t \ll 1}{\approx} J(1 - \beta J)^2$. The LHS/ N is

$$\frac{1}{N} \langle \Delta H \rangle_{\text{MFT}} = \frac{1}{N} \sum_{rr'} J_{rr'} \langle \delta s_r \delta s_{r'} \rangle_{\text{MF}} = \sum_{r'} J_{rr'} G_{rr'}.$$

[End of Lecture 9]

We assume that $J_{rr'}$ has a smaller range than $G_{rr'}$ (*i.e.* $R < \xi$), so that we may approximate the RHS as

$$zJG(0) = A \int_{\text{BZ}} \frac{\mathrm{d}^d k}{1 - \beta \tilde{J}(k)} \simeq \frac{A}{R^2 \beta} \int_{|k| < a^{-1}} \frac{\mathrm{d}^d k}{k^2 + \xi^{-2}}. \quad (4.10)$$

In a lattice model, the integral is over the Brillouin zone. The dangerous bit, where the RHS can become big, though, comes from $k \rightarrow 0$, which doesn't care about your lattice details. We used this in replacing \tilde{G}_k with its long-wavelength approximation in the last step of (4.10). In making this approximation, we may as well replace the BZ integral with a simple cutoff $|k| < a^{-1}$ since the form of the integrand is wrong for $|k| \sim a^{-1}$ anyway.

To separate out the UV physics ($k \sim \frac{2\pi}{a}$) from the IR physics ($k \sim \frac{2\pi}{L}$), let's use the partial-fractions trick familiar from calculus:

$$\frac{1}{k^2 + \xi^{-2}} = \frac{1}{k^2} - \frac{\xi^{-2}}{k^2(k^2 + \xi^{-2})}$$

so that

$$I \equiv \int_{|k| < a^{-1}} \frac{\mathrm{d}^d k}{k^2 + \xi^{-2}} = \underbrace{\int_{|k| < a^{-1}} \frac{\mathrm{d}^d k}{k^2}}_{\text{ind. of } T} - \xi^{-2} \int_{|k| < a^{-1}} \frac{\mathrm{d}^d k}{k^2(k^2 + \xi^{-2})}.$$

The first term is a (possibly big, honking) constant, which doesn't care about the temperature or the correlation length. The second term is finite as $a \rightarrow 0$ if $d < 4$ (finding that this integral is infinite as $a \rightarrow 0$ just means that the short-distance stuff

at the lattice matters). (Note that the integral is finite as $L \rightarrow \infty$ if $d > 2$.) When the integral is finite, we can scale out the dependence on ξ (define $x \equiv |k|\xi$):

$$I \stackrel{\xi \gg a}{\equiv} \text{const} + \xi^{2-d} K_d \int_0^\infty \frac{x^{d-3} dx}{x^2 + 1}$$

where

$$K_d \equiv \frac{\Omega_{d-1}}{(2\pi)^d}$$

is a ubiquitous combination of angular factors; Ω_d is the volume of the unit d -sphere.

So: the demand that the things we ignored be small corrections to the MFT energy *computed within MFT* requires

$$\frac{AT_c \xi^{2-d}}{R^2} \ll Jt$$

Remembering that we derived $\xi_{MF} = Rt^{-1/2}$, we can write this condition purely in terms of the mean field correlation length. If the condition

$$\xi^{4-d} \ll R^4$$

is violated then mean field theory is wrong. (The R^4 on the RHS stands in for some quantities with the right dimensions which do not vary with t near the transition)

So for sure this condition is violated if ever $\xi \rightarrow \infty$ in $d < 4$. (Remember that d is the number of space dimensions.)

Note that the condition depends on the range R of the interactions: MFT works better for longer-range interactions, and in more dimensions.

Why does mean field theory get better in more dimensions?

Mean field theory is valid if in the energy depending on a spin s_i

$$H_i \equiv J s_i \left(\sum_{\langle i|j \rangle} s_j + h_i \right)$$

we can approximate the values of the neighboring spins by their average $s_j \stackrel{?}{=} \langle s_j \rangle$, and treat the coefficient of s_i as an effective ‘molecular’ field $h_i^{\text{eff}} = \sum_{\langle i|j \rangle} \langle s_j \rangle + h_i$.

More dimensions or longer range means more neighbors (for example, for the hypercubic lattice in d dimensions, each site has $2d$ neighbors); more neighbors means that there are more terms in the sum $\sum_{\langle i|j \rangle} s_j + h_i$. If the correlations between the terms in the sum are small enough, the central limit theorem tells us that the fractional error

decays with the number of terms in the sum. And this assumption is self-consistent, since in MFT the spins s_j are statistically independent (the probability distribution factorizes).

The preceding argument says that at asymptotically large d , MFT becomes more and more correct. You saw on the homework that when the number of neighbors grows with N (namely with all-to-all interactions), then MFT is exact. When $d = 1$ MFT is completely wrong, since there is no ordering at all at finite T . So something must happen somewhere in between. We've just learned that that somewhere is $d = 4$.

$d = 4$ is maybe not so exciting for statistical mechanics applications. However, the same machinery can be used with one of the dimensions interpreted as time. For more on this, I refer you to references on QFT (such as my 215C notes).

$d = 4 = d_c$ is called the *upper critical dimension* (in the sense that mean field theory is correct for larger dimensions) for the Ising critical behavior (since we've been talking about the case with Ising symmetry). More generally, the upper critical dimension can be efficiently determined from the zoo of critical exponents as follows. The fractional error in mean field theory can be rewritten as

$$\text{error} \sim \frac{\int_V d^d r G(r)}{\int_V d^d r m(r)^2} \quad (4.11)$$

where V is a 'correlation volume', a region of space whose linear size is ξ . The numerator is $\int_V d^d r G(r) = T\chi_T \sim t^{-\gamma}$. The denominator is $\xi^d |t|^{2\beta} \sim t^{2\beta - \nu d}$, so the condition that (4.11) is small is

$$1 \gg t^{-\gamma - 2\beta + \nu d} \quad \implies \quad d_c = \frac{2\beta + \gamma}{\nu}.$$

Continuum field theory

Along the way in the preceding discussion of correlation functions in mean field theory, we showed the following, which is a useful summary of the whole discussion, and makes contact with the microscopic Landau-Ginzburg theory. Consider the simple case where

$$J_{ij} = \begin{cases} J, & r_{ij} \leq R \\ 0, & r_{ij} > R \end{cases}.$$

Then we showed that the contribution to the mean-field free energy from the interaction term is

$$-\Delta f_{MF}[m] = \sum_{ij} J_{ij} m_i m_j$$

$$\begin{aligned}
&= -J \frac{a^2}{4} \sum_i \sum_{|\delta| \leq R} \left(\left(\frac{m_{i+\delta} - m_i}{a} \right)^2 - \left(\frac{m_{i+\delta} + m_i}{a} \right)^2 \right) \\
&= -J \frac{a^2}{4} \sum_i \sum_{|\delta| \leq R} \left(\frac{m(r_i + \delta) - m(r_i)}{a} \right)^2 + \underbrace{\mathcal{O}(m^2)}_{\text{correction to } V(m)} \\
&\stackrel{\text{Taylor}}{\simeq} -J \frac{a^2}{4} \sum_i \sum_{|\delta| \leq R} \left(\frac{\vec{\delta} \cdot \vec{\nabla} m(r_i)}{a} \right)^2 + \mathcal{O}(m^2) \\
&\simeq -\frac{zJR^2}{4} \int \frac{d^d r}{a^d} (\vec{\nabla} m)^2 + \mathcal{O}(m^2)
\end{aligned}$$

where z is the coordination number of the lattice. Comparing this to our ‘local’ Landau-Ginzburg expression (4.7), we’ve learned that the constant in front is

$$\kappa \simeq R^2 \left(\frac{Jz}{4a^d} \right) = \frac{R^2 T_c^{MF}}{a^d}.$$

The equations of motion for m coming from extremizing

$$f_{\text{LG}} = \int d^d r \left(\kappa \vec{\nabla} m \cdot \vec{\nabla} m + V(m(x)) \right)$$

in this continuum approximation, are²¹

$$0 = \frac{\delta F_{\text{LG}}}{\delta m(x)} = -2\kappa \nabla^2 m + \frac{\partial V}{\partial m} \Big|_{m=m(x)} + \dots \quad (4.12)$$

If V contains a source term $\int dx h(x)m(x)$, then this is

$$0 = -2\kappa \nabla^2 m(x) + h(x) + 2btm(x) + \mathcal{O}(m^2).$$

For the case of a localized source, $h(x) = \delta(x)$, (and ignoring the interaction terms $m^{n>1}$) the solution in Fourier space

$$\tilde{m}_k = \frac{(2\kappa)^{-1}}{k^2 + bt/\kappa}$$

gives back $\xi^{-1} = \sqrt{bt/\kappa}$. You might think that ignoring the higher powers of m is OK near the critical point, since m is small; this assumption gives back mean field theory (which we’ve already seen is not always correct).

In case you’re not comfortable with this derivation of the continuum field theory description of Ising models with large correlation length, another approach is outlined on the problem set.

²¹For those of you who are not at home with variational calculus, please see the sidebar on the subject at §4.2.1.

Return for a moment to our discussion of the LG theory of a system with an $O(n)$ symmetry. Recall that in the ordered phase, we found that $n - 1$ of the modes did not appear in the quadratic term of the LG free energy. Now you can see why I said that the existence of these Goldstone modes implied that the correlation length was infinite everywhere in the ordered phase.

4.2.1 Sidebar on Calculus of Variations

Let us spend a moment thinking about functionals – a functional is a monster that eats a function and returns a number – and how they vary as we vary their arguments. I'll describe this in the context where the function in question is $x(t)$, the trajectory of a particle in time, but you can substitute $m(x)$.

The basic equation of the calculus of variations is:

$$\frac{\delta x(t)}{\delta x(s)} = \delta(t - s).$$

This the statement that $x(t)$ and $x(s)$ for $t \neq s$ are independent. From this rule and integration by parts we can get everything we need. For example, let's ask how does the potential term in the action $S_V[x] = \int dt V(x(t))$ vary if we vary the path of the particle. Using the chain rule, we have:

$$\delta S_V = \int ds \delta x(s) \frac{\delta \int dt V(x(t))}{\delta x(s)} = \int ds \delta x(s) \int dt \partial_x V(x(t)) \delta(t-s) = \int dt \delta x(t) \partial_x V(x(t)).$$

We could rewrite this information as:

$$\frac{\delta \int dt V(x(t))}{\delta x(s)} = V'(x(s)).$$

²² What about the kinetic term $S_T[x] \equiv \int dt \frac{1}{2} m \dot{x}^2$? Here we need integration by parts:

$$\frac{\delta}{\delta x(s)} S_T[x] = \frac{2}{2} m \int dt \dot{x}(t) \partial_t \frac{\delta x(t)}{\delta x(s)} = m \int dt \dot{x}(t) \partial_t \delta(t-s) = -m \int dt \ddot{x}(t) \delta(t-s) = -m \ddot{x}(s).$$

If we set the total variation to zero, we get Newton's equation:

$$\frac{\delta}{\delta x(s)} \left(\int dt \left(\frac{1}{2} m \dot{x}^2 - V(x(t)) \right) \right) = -m \ddot{x}(s) + V'(x(s))$$

²²If you are unhappy with thinking of what we just did as a use of the chain rule, think of time as taking on a discrete set of values t_i (this is what you have to do to define calculus anyway) and let $x(t_i) \equiv x_i$. Now instead of a functional $S_V[x(t)]$ we just have a function of several variables $S_V(x_i) = \sum_i V(x_i)$. The basic equation of calculus of variations is even more obvious now:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

and the manipulation we did above is

$$\delta S_V = \sum_j \delta x_j \partial_{x_j} S_V = \sum_j \delta x_j \partial_{x_j} \sum_i V(x_i) = \sum_j \sum_i \delta x_j V'(x_i) \delta_{ij} = \sum_i \delta x_i V'(x_i).$$

5 Festival of rigor

Let us pause in our assault on field theory to collect some Facts that we know for sure about the free energy of short-ranged lattice models. As with any rigorous, formal results in physics, it will be crucial to understand the hypotheses.

5.1 Extensivity of the free energy

[Parisi pp. 41-42] The Ising model free energy is extensive, $F/N = f +$ terms which go to zero as the number of sites $N \rightarrow \infty$. In particular, in the thermodynamic limit, the bulk free energy density f doesn't care about boundary conditions. This assumes that J is short-ranged: $J_{r,0}$ is either of finite support (system-size-independent range), or falling off sufficiently rapidly in r .

Here is an RG-inspired proof of this result. We begin with a finite system, with N sites.

First, notice that the hamiltonian $H(s)$ is bounded

$$-ND < H(s) < ND$$

for some constant D (for the near-neighbor Ising model on a cubic lattice it's J for each link, so $D = dJ$).

We can bound the *free* energy, too, by realizing that the number of configurations is finite – for a finite lattice with N sites, there are only 2^N of them. Each one contributes an energy below the maximum value, and above the minimum value. If all 2^N configurations achieved the max/min value, we get the smallest/biggest possible values of the partition function:

$$2^N e^{-\beta ND} \leq Z_N \leq 2^N e^{\beta ND}.$$

Taking log of the BHS gives

$$-\infty < -D - \frac{\log 2}{\beta} \leq f_N \leq D + \frac{\log 2}{\beta} < \infty$$

the important thing being that the free energy density is [bounded on either side](#), independently of N .

Now here comes the RG bit. For definiteness, take free boundary conditions on an $L \times L \times \dots \times L$ chunk of cubic lattice. (Free boundary conditions means that we just stop writing terms in the hamiltonian when the sites that would participate in them don't

exist.) Take $L \gg R$, the range of the interactions. Let Z_L^F be the partition function for this chunk.

Now we try to double the (linear) size of the system, by gluing together the right number (2^d) of smaller chunks of size L . Gluing just means that we add the terms in the hamiltonian which couple the sites across the interface. The number of terms we have to add is $L^{d-1}R$ for each interface (each pair of chunks) we glue, and we have to glue 2^d interfaces. The magnitude of the contribution of each term is bounded by D . Therefore

$$(Z_L^F)^{2^d} (2e^{-\beta D})^{2^d L^{d-1}R} \leq Z_{2L}^F \leq (Z_L^F)^{2^d} (2e^{+\beta D})^{2^d L^{d-1}R}.$$

Taking the log and dividing by $(2L)^d$ gives

$$f_L + \tilde{D}L^{-1}R \geq f_{2L} \geq f_L - \tilde{D}L^{-1}R$$

(where $\tilde{D} \equiv D + T \ln 2$), which can be written as

$$|f_{2L} - f_L| \leq \frac{c}{L}$$

for some positive number c which does not depend on L .

This means that the sequence $\{f_{2^n L}\}_n$ converges as $n \rightarrow \infty$ (for a big enough initial L). The same argument can be used to show that the effect of changing boundary conditions can be bounded on either side. Suppose we change terms in the hamiltonian in a way which is localized near the boundary and where the magnitude of the change of each term is bounded by some Δ . Then if Z^B is the resulting partition function,

$$Z_L^B e^{-\beta L^{d-1}\tilde{\Delta}} \leq Z_L^F \leq Z_L^B e^{\beta L^{d-1}\tilde{\Delta}}.$$

Again when we take the log and divide by the volume L^d , the terms proportional to $\tilde{\Delta} \equiv \Delta + T \ln 2$ are suppressed by a factor of L .

Thermodynamic limit

We conclude that in a system in d dimensions of linear size L , with short-range interactions, the free energy takes the form:

$$F = L^d f_b + L^{d-1} f_\partial + \mathcal{O}(L^{d-2})$$

$$f_b = \lim_{L \rightarrow \infty} \frac{F}{L^d}, \quad f_\partial = \lim_{L \rightarrow \infty} \frac{F - L^d f_b}{L^{d-1}}.$$

f_∂ is a *boundary free energy density*.

Two questions to ponder:

1. What should we hold fixed in the limit $L \rightarrow \infty$? In a fluid, we might want to fix the density of particles, $\rho = N_{\text{particles}}/L$. If we instead fix $N_{\text{particles}}$, we get a boring fluid.
2. How can the thermodynamic limit fail to exist? We consider a class of examples where it might fail next.

5.2 Long-range interactions

Charged system. Consider a bunch of stuff with a net electric charge, at $T = 0$. Imagine we can fix the charge density ρ , and take $d = 3$ so that the inter-charge potential is $U(r) = A/r$. The self-energy of a sphere of this stuff is (integrating the contributions from shells of radius r , which only care about contributions from inside)

$$E(R) = \int_0^R \left(\frac{4}{3} \pi r^2 \rho \right) \frac{A}{r} (4\pi r^2 \rho dr) \sim A\rho^2 R^5.$$

So the ‘bulk free energy density’ is

$$E_b = \frac{E(R)}{V(R)} \sim A\rho^2 R^2 \xrightarrow{R \rightarrow \infty} \infty. \quad (5.1)$$

So a Coulomb-like force is too long-ranged for the thermodynamic limit to exist. More physically, the conclusion (5.1) means (for $A > 0$, repulsive interactions) that the system (with a fixed number of particles or fixed charge) can lower its energy by expanding into a larger volume – it explodes.

But wait: there are Coulomb-like forces in the world, and here we are in the thermodynamic limit. A fatal assumption above is that there was only one sign of the charge. If we allow charge of both signs, we can have the phenomenon of *screening*. Screening makes a microscopically long-range force short-ranged. That last sentence has a lot of RG physics in it, and it’s worth absorbing more. This is an opportunity to say something about “running couplings”.

Screening: mean field theory of Debye and Huckel.

[McComb] We take a uniform background of $+$ charges, fixed in place. (This is sometimes called ‘jellium’.) Their number density is n_∞ . We add to this *mobile* $-$ charges (‘electrons’), with equal average number density.

Suppose we stick in a test $(-)$ charge at the origin. At temperature T , what is the probability of finding an electron a distance r from the test charge? If we knew the electrostatic potential $\phi(r)$, the classical equilibrium probability would be

$$p(r) = Z^{-1} e^{-e\phi(r)/T}.$$

In vacuum, $\phi(r)$ would be $\frac{e}{r}$. We will determine it self-consistently. The electron number density is proportional to the probability $p(r)$, and must approach the average density far away (where $\phi \rightarrow 0$), so

$$n(r) = n_\infty e^{-\beta e \phi(r)}.$$

Now we can impose Gauss' law:

$$\begin{aligned} -\nabla^2 \phi(r) &= 4\pi \rho(r) \\ &= 4\pi e(n(r) - n_\infty) \\ &= 4\pi e n_\infty (e^{-\beta e \phi(r)} - 1) \\ &\stackrel{T \gg e\phi}{\simeq} -4\pi \beta e^2 n_\infty \phi(r) + \mathcal{O}(e\beta\phi)^2. \end{aligned} \tag{5.2}$$

This is just the equation we solved in (4.12) to find the correlation function $G(r)$ away from the critical point, at finite $\xi^{-2} = 4\pi \beta e^2 n_\infty$, and the solution is

$$\phi(r) = \frac{e}{r} e^{-r/\ell_D} \equiv \frac{e_{\text{eff}}(r)}{r}. \tag{5.3}$$

The name of the correlation length in this case is

$$\ell_D \equiv \sqrt{\frac{T}{4\pi e^2 n_\infty}},$$

the Debye screening length. In the second equality in (5.3) I introduced a distance-dependent effective charge $e_{\text{eff}}(r)$: how much charge you see depends how closely you look.

The continuum approximation we've used here is consistent with classical corpuscles if the average interparticle distance is small compared to the screening length:

$$n_\infty^{-1/3} \ll \ell_D$$

which is true when $e^3 \sqrt{N} \ll T^{3/2}$, *i.e.* at high enough temperature, consistent with our approximation in (5.2).

You might worry that a collection of charges of both signs, once we let them all move around, might either implode or explode. [This paper by Lieb, called *The Stability of Matter*](#), is very interesting and not too forbidding. The early sections are about the stability of matter to *implosion*, which is a short-distance issue (whose resolution crucially involves quantum mechanics and the Pauli principle and hence is off-limits here); but Section V contains a 'rigorous version of screening' which removes the concern that matter should want to explode like in (5.1).

Other power laws. Suppose instead of Coulomb interactions in $d = 3$, we have particles interacting pairwise via a potential $U(r) = \frac{A}{r^\sigma}$ in d dimensions. Then the energy of a collection of particles with density $\rho(r)$, in a ball of radius R , B_R is

$$\begin{aligned} E(R) &= \frac{1}{2} \int_{B_R} d^d r \int_{B_R} d^d r' \rho(r) U(r - r') \rho(r') \\ &\stackrel{\text{uniform } \rho}{\simeq} A \frac{\rho^2}{2} \int_{B_R} d^d r \int_{B_R} d^d r' \frac{1}{|r - r'|^\sigma} \\ &= A \frac{\rho^2}{2} R^{2d-\sigma} C(d, \sigma) \end{aligned} \tag{5.4}$$

where

$$C(d, \sigma) \equiv \int_{B_1} \frac{d^d x d^d y}{|x - y|^\sigma}$$

In the last step we scaled out the system-size dependence of the integral by defining $r \equiv Rx, r' \equiv Ry$. This C is just a dimensionless number – if it's finite. In that case, the ‘bulk energy density’ (free energy density at $T = 0$) is

$$\varepsilon_{\text{bulk}} \equiv \frac{E(R)}{V(R)} = \frac{R^{2d-\sigma} A \rho^2 C / 2}{R^d V_d} \sim R^{d-\sigma}$$

which is finite as $R \rightarrow \infty$ (the would-be thermodynamic limit) if $\sigma > d$. (V_d is the volume of the unit d -ball.) So $\sigma > d$ is a sufficiently fast falloff of the interactions to allow for a thermodynamic limit.

When is $C(d, \sigma)$ finite? What I really mean by this is: the power law form of $U(r) \sim r^{-\sigma}$ surely only holds for $r \gg a$, some UV cutoff – for example the size of the particles. The real question is: when can we ignore this approximation for purposes of computing C ? Changing integration variables to $u \equiv x - y, v = x + y$,

$$C = V_d \Omega_{d-1} \int_{a/R}^1 \frac{u^{d-1} du}{u^\sigma} \begin{cases} = \frac{V_d \Omega_{d-1}}{d-\sigma} \left(1 - \left(\frac{a}{R}\right)^{d-\sigma}\right), & d \neq \sigma \\ \propto \log \frac{R}{a}, & d = \sigma \end{cases}$$

5.3 (Anti-)convexity of the free energy

[Goldenfeld §2.6] We're going to prove some facts about the nearest-neighbor Ising model, with Hamiltonian

$$H(s) = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i. \tag{5.5}$$

Many of them are true more generally.

(1) With the additive normalization in (5.5), the bulk free energy density is negative:

$$f < 0.$$

This statement is sensitive to the additive normalization of H – if we add a big positive constant to H , we can change this fact. The normalization (5.5) is special because there is no constant term, in the sense that

$$\sum_s H(s) = 0 \tag{5.6}$$

– it is normal-ordered.

Proof of (1): Begin with $N < \infty$ sites. The free energy density is $f = F/N = -\frac{T}{N} \log Z$, so the claim $f < 0$ means $Z > 1$. The partition function $Z = \sum_s e^{-\beta H(s)}$ is a sum of 2^N positive terms (for $0 < T < \infty$). And $Z > 1$ because there exists a configuration s^* which by itself contributes a term $e^{-\beta H(s^*)} > 1$. For example, for $J > 0, h > 0$, it happens when $s_i^* = 1, \forall i$. But more generally, it follows from the normal-ordering condition (5.6) since $H(s)$ is not identically zero, so there must be configurations with both signs of $H(s)$, and at least one which has $H(s^*) < 0$. ■

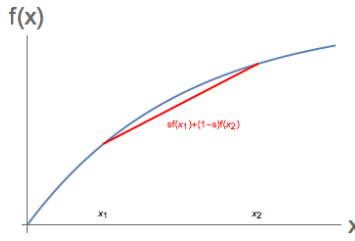
(2) The entropy density is

$$s = -\partial_T f \geq 0.$$

The proof of this statement follows from the identity

$$-\partial_T F = -\sum_s \rho_\beta(s) \log \rho_\beta(s)$$

where $\rho_\beta(s) \equiv e^{-\beta H(s)} Z^{-1}$ is the equilibrium probability distribution at temperature T . Since $0 \leq \rho \leq 1$, this is a sum of positive terms. ■

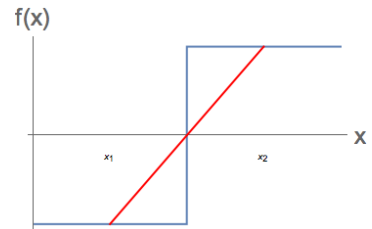


Here is a **Definition**: A function $f(x)$ is anti-convex (aka concave) in x if

$$f(tx_1 + (1-t)x_2) \geq sf(x_1) + (1-t)f(x_2), \quad \forall t \in [0, 1];$$

that is, if the graph of the function is above any chord. If f is anti-convex then I'll call $-f$ convex.

Convex implies continuous, as you can see from the picture at right of a function which jumps. If f is anti-convex,



$\partial_x f$ is non-increasing (and in particular the derivative exists almost everywhere). f can have cusps.

(3) $\ln Z(\beta, h, J)$ is anti-convex in its arguments.

[End of Lecture 10]

Proof of (3): The proof relies on a Hölder inequality. Given two sequences $\{g_k\}, \{h_k\}$ of non-negative numbers $g_k, h_k \geq 0$, and $t \in [0, 1]$,

$$\sum_k (g_k)^t (h_k)^{1-t} \leq \left(\sum_k g_k \right)^t \left(\sum_k h_k \right)^{1-t}.$$

This follows from the convexity of the logarithm²³.

Here's the idea for $\ln Z(\beta)$:

$$\begin{aligned} Z(t\beta_1 + (1-t)\beta_2) &= \sum_s e^{-t\beta_1 H(s)} e^{-(1-t)\beta_2 H(s)} \\ &\stackrel{\text{Hölder}}{\leq} \left(\sum_s e^{-\beta_1 H(s)} \right)^t \left(\sum_s e^{-\beta_2 H(s)} \right)^{1-t} \\ &= Z(\beta_1)^t Z(\beta_2)^{1-t}. \end{aligned}$$

Taking logs gives:

$$\ln Z(t\beta_1 + (1-t)\beta_2) \leq t \ln Z(\beta_1) + (1-t) \ln Z(\beta_2).$$

The limit as $N \rightarrow \infty$ of a family of convex functions is also convex. ■

Note that I could have said $f(h, J)$ is anti-convex in its arguments, but $f(\beta)$ is not necessarily so, since $f(\beta) = -\ln Z/\beta$, and the β in the denominator can mess things

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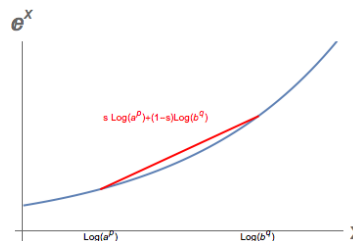
Here's the idea:

$$\begin{aligned} ab &= e^{\log ab} = e^{\frac{1}{p} \log a^p + \frac{1}{q} \log b^q} \\ &\leq \frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{a^p}{p} + \frac{b^q}{q} \end{aligned}$$

for $\frac{1}{p} + \frac{1}{q} = 1$, where we used the fact that e^x is anticonvex ($e^{tx+(1-t)y} \leq te^x + (1-t)e^y$), as illustrated at right. Apply this inequality with

$$a = \left(\frac{g_k}{\sum g} \right)^{\frac{1}{p} \equiv t}, \quad b = \left(\frac{h_k}{\sum h} \right)^{\frac{1}{q} \equiv 1-t}$$

and sum the BHS over k .



up. On the other hand, as a function of $T = 1/\beta$, the free energy $f(T) = -T \ln Z(T)$ is indeed anti-convex.

A useful alternative viewpoint: anticonvexity follows by showing that all second derivatives of f are negative. For example,

$$\partial_\beta^2 f = -\frac{1}{\beta N} \langle (H - \langle H \rangle)^2 \rangle \leq 0$$

is proportional to minus the specific heat, aka the variance of the energy. Similar statements hold for other variations, such as the magnetic susceptibility

$$\partial_h^2 f = -c \langle (s - \langle s \rangle)^2 \rangle \leq 0.$$

So the condition of convexity is related to the stability of the equilibrium.

Note that f being anticonvex in β means f is convex as a function of T .

Gibbs' inequality [Kardar, particles] Here's an application of the anticonvexity of $\ln Z$ as a function of couplings in the hamiltonian. Suppose that computing expectations in the system with hamiltonian H is hard, but with another hamiltonian H_0 (defined on the same configuration space) it is easy. Then let

$$Z(t) \equiv \text{tr} e^{-\beta(H_0(1-t)+tH)},$$

which interpolates between the two ensembles. By similar steps as above, $\ln Z(t)$ is convex in t . Convexity of a function implies that it lies above any of its tangents, and in particular²⁴,

$$\ln Z(t) \geq \ln Z(0) + t \partial_t \ln Z|_{t=0} = \ln Z(0) + \beta \langle H_0 \rangle_0 - \beta \langle H \rangle_0 .$$

On the right hand side we then have a bound on the free energy in terms only of easy-to-compute quantities. (Consider what happens in the case of the ising model, if we take $H_0 = \sum_i s_i h_i$.)

5.4 Spontaneous symmetry breaking

[Goldenfeld p. 56, Parisi p. 15]

Orders of limits. I made a big deal earlier (in §3) about the impossibility of spontaneous symmetry breaking (SSB) in finite volume. There is more to say about this. What does the free energy density (in the thermodynamic limit) look like as a function of h near $h = 0$? It must be

$$f(h) = f(0) - m_s |h| + \mathcal{O}(h^{\sigma > 1})$$

²⁴Alternatively, we could just show that $\partial_t^2 \ln Z(t) = \beta^2 \langle (H - H_0)^2 \rangle_c \geq 0$

so that the magnetization is

$$m = -\partial_h f = \begin{cases} m_s + \mathcal{O}(h^{\sigma-1}), & h > 0, \\ -m_s + \mathcal{O}(h^{\sigma-1}), & h < 0 \end{cases}.$$

(If σ were not larger than one, the magnetization would diverge as $h \rightarrow 0$ and that's not happening, since it's bounded ($|m| \leq 1$). I also imposed $f(h) = f(-h)$ by Ising symmetry.)

But before the thermodynamic limit, $f(h)$ is a smooth function. This means the two limits $h \rightarrow 0, N \rightarrow \infty$ are clashing violently:

$$\lim_{N \rightarrow \infty} \lim_{h \rightarrow 0} \frac{1}{N} \partial_h F = 0 \quad \text{but} \quad \lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \partial_h F = \pm m_s.$$

Yang-Lee singularities. Here is a toy model of how this can come about. Suppose our system of volume V is so tightly bound that only two configurations matter, the one where all N spins point up, $m = +V$, and the one where they all point down, $m = -V$. (All the rest of the configurations have such a large energy that we can ignore their contributions to Z .) So a single spin $s = \pm 1$ determines the whole configuration.

Then, in a field, we have

$$Z(h) = \sum_{s=\pm 1} e^{-\beta h V s} = 2 \cosh \beta h V$$

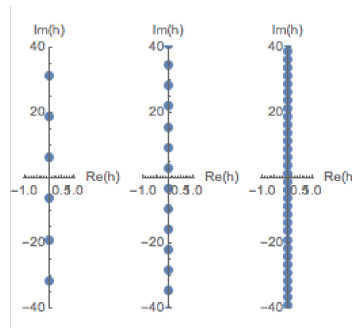
and

$$f(h) = -\frac{T}{V} \log(2 \cosh \beta h V), \quad m(h) = \partial_h f = \tanh \beta h V \xrightarrow{V \rightarrow \infty} m(h) = \text{sign}(h).$$

Where in h is the free energy $F \propto \log(Z)$ singular? When $Z(h) = 0$. For $V < \infty$, the zeros of the partition function happen at pairs of imaginary values of h

$$Z(h_n = 0) \quad \text{at} \quad h_n = \frac{(2n+1)\pi i}{2\beta V}$$

which in the thermodynamic limit $V \rightarrow \infty$ accumulate and pinch the real axis. (They are shown for $\beta V = 1, 2, 5$ in the figure at right.) These zeros are named after Yang and Lee.



Ergodicity breaking. There are many points of view from which SSB seems paradoxical. For example if the equilibrium probability density is

$$p_0(s) = Z^{-1} e^{-\beta H(s)}$$

then the Ising symmetry $H(s) = H(-s)$ implies directly that the magnetization vanishes:

$$m = \langle s \rangle \stackrel{?}{=} \langle s \rangle_0 \equiv \sum_s P_0(s) s = 0.$$

What gives? Consider, at small $h > 0$ and finite N , the ratio of the probabilities of two configurations: a reference configuration s , and the one related to it by a global spin reversal. If $m(s) \equiv \frac{1}{N} \sum_i s_i$ is the magnetization in this configuration, then

$$\frac{p(s)}{p(-s)} = \frac{e^{-\beta(hNm(s))}}{e^{+\beta(hNm(s))}} = e^{-2\beta hNm(s)} \xrightarrow{N \rightarrow \infty, h > 0, m(s) > 0} 0.$$

In the thermodynamic limit, if $h \neq 0$ one of these configurations is infinitely more probable than the other! This is true no matter how small h is, even if $h = 0^+$. If we reverse the sign of h , the other configuration wins. We've learned that

$$\lim_{h \rightarrow 0^\pm} \lim_{N \rightarrow \infty} p_0(s) \equiv p_\pm(s)$$

is a different, restricted ensemble compared to the symmetric distribution p_0 . It is restricted in the sense that $p_+(s|m(s) < 0) = 0$ – states with the wrong sign of magnetization have no weight. So in this limit, our distribution only samples a subset of the configuration space – it is not *ergodic*. This is a toy example of ergodicity breaking, a concept which is much more useful in the physics of glassy systems. Very roughly, from the usual point of view of ergodicity as underlying statistical mechanics, in terms of time evolution, the claim is that starting from an initial configuration, the probability of evolving to a configuration with the opposite sign of the magnetization goes like $e^{-\beta \Delta F}$ where the change in free energy is $\Delta F \sim N^{\sigma > 0} \xrightarrow{N \rightarrow \infty} \infty$. So we are also claiming that SSB means that the $N \rightarrow \infty$ limit and the $t \rightarrow \infty$ limit do not commute. (If we take $t \rightarrow \infty$ first, the system will explore all the configurations.) For a bit more about ergodicity-breaking, see Goldenfeld §2.10.

Cluster decomposition failure. If we prepare the system in an initial configuration with a mixture of \pm (or average over possible initial conditions with the appropriate weight), as

$$p_q(s) = qp_+(s) + (1 - q)p_-(s), \quad q \in [0, 1]$$

then our expectation for the connected correlations are

$$\langle s_i s_j \rangle_q^c \equiv \langle s_i s_j \rangle_q - \langle s_i \rangle_q \langle s_j \rangle_q \xrightarrow{|r_i - r_j| \rightarrow \infty} m^2 - ((1 - 2q)^2 m^2 = 4q(1 - q)m^2 \neq 0. \quad (5.7)$$

They don't vanish for arbitrarily-separated points!²⁵ The demand that they should is called the *cluster decomposition principle*; it is an implementation of the notion of

²⁵For some intuition for the sense in which arbitrarily-separated points are correlated in these ensembles, see the homework.

‘locality’. SSB means that cluster decomposition fails for the symmetric distribution. Only the non-symmetric ‘pure states’ with $q = 0, 1$ satisfy this demand (this is the definition of ‘pure state’ in this context).

5.5 Phase coexistence

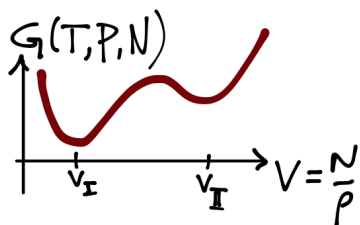
[Goldenfeld, §4] First, let’s recall some thermodynamics facts. I will speak in the language of fluids, but with appropriate substitutions of letters, it can be translated into physics of magnets or other examples. At fixed volume, the free energy which is minimized in equilibrium is the Hemholtz one (the one we’ve been talking about), $F(T, V, N) = E - TS$. If instead we fix the pressure P , the quantity which is minimized in equilibrium is the Legendre transform of F , named for Gibbs:

$$G(T, P, N) = F + PV,$$

in terms of which the first law of thermodynamics is

$$dG = -SdT + VdP + \mu dN.$$

The Gibbs-Duhem relation (basically, integrating the first law) says $E = -PV + TS + \mu N$, so that in fact $G = \mu N$ is just proportional to the chemical potential.



Let’s consider a situation at fixed P where there is a first order transition, between two phases I, II (for example, liquid and gas) where the order parameter is the volume, or the density (equivalently at fixed N , since $V = N/\rho$). Along the phase boundary, where they exchange dominance, we must have

$$G_I = G_{II}. \quad (5.8)$$

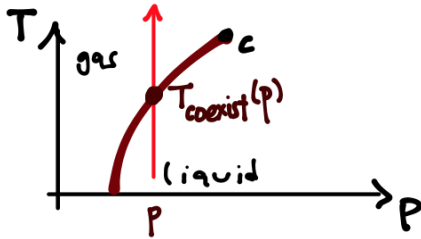
Hence also $\mu_I = \mu_{II}$; this is a condition for chemical equilibrium of the two phases.

Moving along the phase boundary, the condition (5.8) says

$$\begin{aligned} G_I(T + dT, P + dP, N) &= G_{II}(T + dT, P + dP, N) \\ &= \underbrace{dT \partial_T G_I|_P}_{-S_I} + \underbrace{dP \partial_P G_I|_P}_{V_I} \end{aligned}$$

and therefore we get the Clausius-Clapeyron equation for the slope of the coexistence curve

$$\left. \frac{dP}{dT} \right|_{\text{coexistence}} = \frac{S_I - S_{II}}{V_I - V_{II}}.$$



be a *critical endpoint* of the line of first-order transitions, where $\Delta S \rightarrow 0, \Delta V \rightarrow 0$, at some (T_c, P_c) .

The consequence of a first-order transition depends on what is held fixed as the transition is traversed. If we heat a fluid at constant pressure $P < P_c$ (for example atmospheric pressure), starting from $T < T_c$ (moving along the red vertical line in the figure, and doing so slowly enough that we stay in the equilibrium phase diagram at every step) then first the fluid expands and warms up. When it reaches the coexistence curve $T_{\text{coexistence}}(P)$, it *starts* to boil. While this happens, the energy goes into the latent heat converting I into II , and the temperature stays fixed: we are sitting at the point $(T_{\text{coexistence}}(P), P)$ on the coexistence curve in the (P, T) phase diagram, while the fraction x of the fluid which is gas grows:

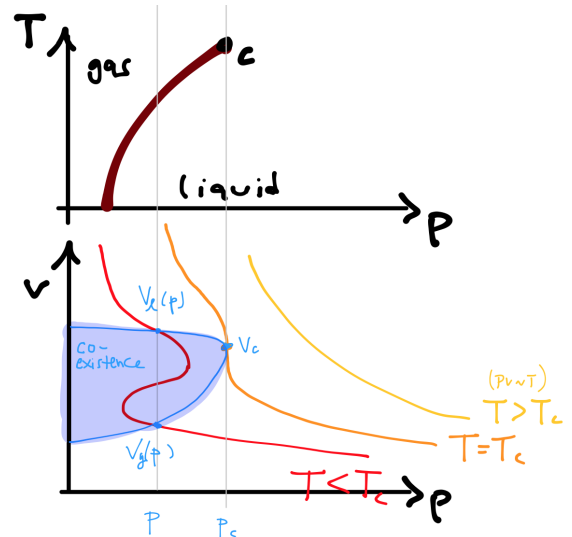
$$V = xV_l + (1 - x)V_g, \quad x = x(t)$$

is some protocol-dependent function. Although $V_l \neq V_g$, the volume of fluid itself does not jump. How do I know this? Bear with me a moment, the proof is at Eq. (5.9).

If instead we compress the fluid at constant T , starting at $T > T_c$ in the gas phase:

$$-\frac{1}{V} \partial_V P|_T \equiv \kappa_T > 0$$

a positive compressibility says that it fights back. It fights back until the volume reaches $V = V_l(T)$, which is when $P = P_{\text{coexistence}}(T)$, beyond which the fluid starts to condense.



What do these isothermal curves look like? Let $v = V/N = 1/\rho$ be the volume fraction per particle. For an ideal gas, recall that $Pv = T$. This is correct in general at high temperature. For lower temperatures, van der Waals suggests some appealing simple corrections which account for an interparticle interaction described by a potential like we discussed in §3.6:

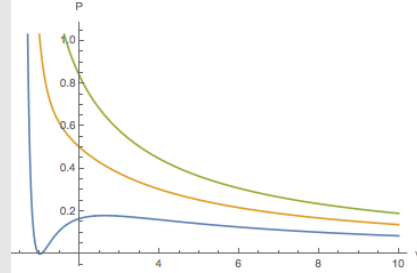
- each particle wants some amount of personal space, and therefore excludes some fixed volume b : $v \rightarrow v - b$.
- the energy per particle is decreased by the long-range attractive part of the potential by an amount proportional to the density:

$$\frac{E}{N} \rightarrow \frac{E}{N} - a\rho \quad \implies \quad P = \partial_V F \rightarrow P - \frac{a}{v^2}.$$

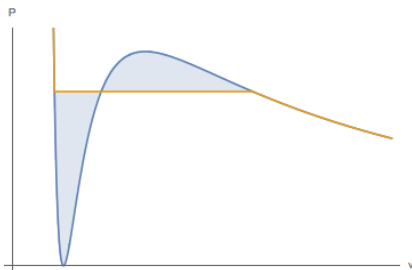
So the vdW equation of state is

$$P = \frac{T}{v - b} - \frac{a}{v^2}$$

for some constants a, b (in the plot at right we see $a = 2, b = 5$ and $T = 1, 1.5, 2$ from bottom to top). This has two nice new features for our present purposes:



- It has a critical $T = T_c$ below which there is a line of first order phase transitions. The critical point appears when $P(v) = \text{const}$ goes from having one solution ($T > T_c$, like the ideal gas), to having three. When this happens, $\partial_v P = \partial_v^2 P = 0$, so that locally $P \sim (v_c - v)^3$ is locally cubic. In fact, for the vdW equation of state, this condition is exactly a cubic equation for v : $P_0 v^3 - v^2(bP_0 + T) + av - ab = 0$.
- (Relatedly), it has regions where $\kappa_T = -\frac{1}{V} \partial_V P|_T < 0$ which says that if you try to squeeze it, it doesn't fight back, but rather tries to help you squeeze it further. Creepy! (The same thing happened in our study of the Landua-Ginzburg free energy in §4.1 and this led to the picture of the swallowtail.)
- Note by the way that the vdW equation is a masterpiece of estimation: a, b can be determined from high-temperature data and they give a (not bad) estimate of the location of the critical point.



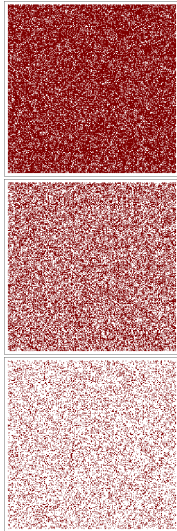
What is the free energy doing while this is going on? At coexistence, in equilibrium, $\mu_l = \mu_g$, and the first law says

$$d\mu = -\frac{S}{N}dT + \frac{V}{N}dP$$

so

$$0 = \mu_l - \mu_g = \int_{\text{gas}}^{\text{liquid}} d\mu \stackrel{\text{isotherm}}{=} \int_{\text{gas}}^{\text{liquid}} \frac{V(P)}{N} dP \quad (5.9)$$

so the area under the $V(P)$ curve is zero (and is the change in the Gibbs free energy), along any path in equilibrium. This is true even for infinitesimal paths. Therefore, the actual equilibrium trajectory of the free energy is a straight line between V_g and V_p . This is the Maxwell construction. It saves the convexity of the free energy.



The creepy self-squeezing regions of the equation-of-state curve are exactly the ones which are removed by the phase-coexistence region.

At left here, I've made some pictures where a decreasing fraction of the dots are colored red, in an attempt to depict the history of the volume fraction of one phase in the other as the coexistence region is traversed. What's wrong with this picture? How could you make it more realistic?

Notice that we are making a strong demand of equilibrium here, effectively taking $t \rightarrow \infty$ before $N \rightarrow \infty$. This failure of commutativity of these limits is the same issue as in our discussion of ergodicity-breaking above.

6 Field Theory

Now we are going to try to see where Landau and Ginzburg could have gone wrong near the critical point.

Here is a hint, from experiment. The hard thing about the critical point, which mean field theory misses, is that fluctuations at all scales are important. I know this because I've seen it, *e.g.* [here](#) and (with better soundtrack) [here](#). *Critical opalescence* is a phenomenon whereby a two-fluid mixture which is otherwise transparent becomes opaque at a continuous phase transition. (The difference in densities of the two fluids plays the role of the order parameter.) It is explained by the scattering of light by the density fluctuations *at all scales*, at least at all the wavelengths in the visible spectrum. These are the fluctuations we're leaving out in mean field theory.

At this point I want to remind you about the derivation of field theory that you made for homework 5. There, you studied the *Legendre transform* of the free energy $F[h]$ at fixed field:

$$S[m] = F[h] - \sum_r m_r h_r |_{m=+\partial_h F}.$$

In the thermodynamic limit, I claim that this is the same thing.

It's easy to get confused about Legendre transforms and all that stuff, so it's very helpful to appeal to a simpler narrative of the origin of field theory, by exploiting universality. Recall at the beginning of our discussion of Ising models in §3, I mentioned the many avatars of the Ising model. One I mentioned arose by considering a real-valued variable ϕ_x at each point in space (or on some lattice).

That is: suppose we replace each spin s_x by such a real variable, a factor in whose probability distribution is

$$p_0(\phi_x) \propto e^{-\beta V(\phi_x)} \tag{6.1}$$

where $V(\phi) \sim g(\phi^2 - 1)^2$ for large g . This probability distribution is basically zero unless $\phi = \pm 1$, so this is no change at all if g is big enough. Important piece of foreshadowing: we are going to see that a large g at the lattice scale is not at all the same as a large $g\phi^4$ term in the coarse-grained action.

So we replace

$$\sum_s \dots \equiv \prod_x \sum_{s_x = \pm 1} \dots \rightsquigarrow \int \prod_x d\phi_x p_0(\phi_x) \dots \equiv \int D\phi e^{-\beta \int_x V(\phi(x))} \dots$$

The nearest-neighbor ferromagnetic Ising Hamiltonian becomes (up to an additive constant, using $s^2 = 1$)

$$-J \sum_x \sum_{\mu=1}^d (s_{x+\mu} s_x - 2) = \frac{1}{2} J \sum_x \sum_{\mu=1}^d (s_{x+\mu} - s_x)^2 \rightsquigarrow \frac{1}{2} J \underbrace{\sum_x}_{\simeq a^{-d} \int d^d x} \sum_{\mu=1}^d \underbrace{(\phi_{x+\hat{\mu}} - \phi_x)^2}_{\simeq a^{-1} \partial_\mu \phi}.$$

That is: the ferromagnetic coupling makes the nearby spins want to agree, so it adds a term to the energy which grows when the nearby ϕ_x s disagree.

Altogether, we are going to replace the Ising partition function with

$$Z = \sum_s e^{-\beta H(s)} \rightsquigarrow \int [D\phi] e^{-\int d^d x \mathcal{L}(\phi)}$$

where (I am calling the LG free energy density \mathcal{L} for ‘Landau’ or for ‘Lagrangian’.)

$$\mathcal{L}(\phi) = \frac{1}{2} \kappa (\nabla \phi)^2 + \frac{1}{2} r \phi^2 + \frac{g}{4!} \phi^4 + h \phi + \dots$$

Our hope is that the operation \rightsquigarrow does not take us out of the basin of attraction of the Ising critical point. The constants κ, r, g are related in some way (roughly determinable but not the point here) to the microscopic parameters. For some physical situations (such as high energy particle physics!) this is a better starting point than the lattice model. There is some coarse-graining involved in the \rightsquigarrow operation, and therefore the dependence of κ, r, g on β needn’t be linear, but it should be analytic. After all, the miraculous phenomenon we are trying to understand is how physics can be non-analytic in T at some finite value of T ; we don’t want to assume the answer.

6.1 Beyond mean field theory

[Brezin, §9] So we want to understand the integral

$$Z \equiv \int [D\phi] e^{-S[\phi]}. \quad (6.2)$$

Mean field theory arises by making a saddle point approximation: find m which minimizes $S[\phi]$, $0 = \left. \frac{\delta S}{\delta \phi} \right|_{\phi=m}$, and make a (functional) Taylor of the exponent about the minimum:

$$\begin{aligned} Z &= \int [D\phi] e^{-S[\phi=m+\varphi]} \\ &= \int [D\varphi] e^{-\left(S[m] + \left. \frac{\delta S}{\delta \phi_x} \right|_{\phi=m} \varphi_x + \frac{1}{2} \left. \frac{\delta^2 S}{\delta \phi_x \delta \phi_y} \right|_{\phi=m} \varphi_x \varphi_y + \dots \right)} \end{aligned} \quad (6.3)$$

In the second line I used the fact that the change of variables $\phi = m + \varphi$ has unit Jacobian. I also used a matrix notation, where the position indices x, y are repeated indices, and hence are summed. The saddle point condition means that the term in the exponent linear in φ_x vanishes.

The mean field theory answer is just $Z_0 = e^{-S[m]}$. The first correction to mean field theory comes by keeping the quadratic term and doing the gaussian integral:

$$Z_1 = Z_0 \int [D\varphi] e^{-\frac{1}{2} \int_x \int_y \varphi_x K_{xy} \varphi_y}$$

[End of Lecture 11]

where the quadratic kernel K is

$$K_{xy} \equiv \left. \frac{\delta^2 S}{\delta\phi_x \delta\phi_y} \right|_{\phi=m} = \left(r + \frac{g}{2} m^2 - \kappa \nabla^2 \right) \delta^d(x - y) .$$

Notice that it depends on the *background field* m .

How do we do the (gaussian!) φ integral?

$$\int \prod d\varphi_x e^{-\frac{1}{2} \varphi_x K_{xy} \varphi_y} = \frac{1}{\sqrt{\det K}} = C \prod_{\lambda} \lambda^{-1/2} = e^{-\frac{1}{2} \sum_{\lambda} \log \frac{\lambda}{\lambda_0}}$$

where λ are the eigenvalues of K :

$$K_{xy} u_y^{(\lambda)} = \lambda u_x^{(\lambda)}. \quad (6.4)$$

I absorbed the constant C into the $-\log \lambda_0$ which we can choose to our advantage. So the leading correction to the mean-field free energy gives

$$F^{(1)}[h] = F_{\text{MF}}[h] + \frac{1}{2} \frac{1}{2} \sum_{\lambda} \log \frac{\lambda}{\lambda_0}.$$

Who are the eigenvalues of the kinetic operator K ? If h and hence m are constant, the problem is translation invariant, and they are plane waves, $u_q(x) = \frac{1}{\sqrt{V}} e^{i\vec{q}\cdot\vec{x}}$ – the eigenvalue equation (6.4) is

$$\int_y \delta(x - y) \left(r + \frac{g}{2} m^2 - \nabla^2 \right) u_q(y) = \underbrace{\left(r + \frac{g}{2} m^2 + q^2 \right)}_{=\lambda_q} u_q(x).$$

Therefore, the free energy is

$$F^{(1)}[h] = F_{\text{MF}}[h] + \frac{1}{2} V \int d^d q \log \left(\frac{r + \frac{g}{2} m^2(h) + q^2}{r + q^2} \right)$$

where I made a choice of λ_0 to be $\lambda(m=0)$.

Making the Legendre transform (a little tricky, and requiring us to ignore terms of the same size as the corrections to the first order approximation), we have $\Gamma[m] = V\gamma(m)$ with the answer to this order

$$\gamma^{(1)} = \frac{1}{2}rm^2 + \frac{g}{4!}m^4 + \frac{1}{2} \int \mathrm{d}^d q \log \left(\frac{r + \frac{g}{2}m^2 + q^2}{r + q^2} \right). \quad (6.5)$$

Shift of critical point, Ginzburg criterion revisited. So what? First let's use this to recover the Ginzburg criterion. The susceptibility, at $h=0$, for $T > T_c$ is $\chi = \partial_h m|_{h=0}$ which (as you'll verify on the homework) is related to the curvature of the effective potential γ by

$$\frac{1}{\chi}|_{m=0} = \partial_m^2 \gamma|_{m=0} = r + \frac{g}{2} \int \mathrm{d}^d q \frac{1}{q^2 + r}.$$

The phase transition happens when the correlation length goes to infinity; we showed by the susceptibility sum rule (4.9) that $\xi \rightarrow \infty$ is required by $\chi \rightarrow \infty$. So, while in mean field theory the critical point occurs when $r \rightarrow 0$, the fluctuation corrections we've just derived shift the location of the critical point to

$$0 \stackrel{!}{=} \chi^{-1}(T_c) = r(T_c) + \frac{g}{2} \int \mathrm{d}^d q \frac{1}{q^2 + r(T_c)}.$$

You'll show on the homework that we can eliminate the (annoying, non-universal anyway) parameter r from the discussion and relate the susceptibility near the transition to the non-dimensionalized temperature $t = \frac{T-T_c}{T_c}$:

$$\frac{1}{\chi} = c_1 t \left(1 - \frac{g}{4} \int \mathrm{d}^d q \frac{1}{q^2 (q^2 + r)} \right).$$

for some constant c_1 . Everywhere here we are ignoring terms which are as small as the corrections to the gaussian approximation. Since if g were zero, the integral would be exactly gaussian (ignoring even higher order terms like ϕ^6 for now), the corrections must come with powers of g .

When is the correction to MFT actually small? The shift in the critical point is of order $gG(0) = g \int \mathrm{d}^d q \frac{1}{q^2(q^2+t)} + \text{const}$, which is the same quantity we found in our earlier discussion of the Ginzburg criterion for short-ranged interactions. As $t \rightarrow 0$, the integral (the $q \rightarrow 0$ limit of the integral) is finite for $d > 4$, but for $d \leq 4$ it blows up at $t \rightarrow 0$. More specifically, the corrections to MFT are small when $gt^{\frac{d-4}{2}}$ is small. This determines the *size of the critical region*.

Now wait a minute: when we introduced the coupling g (at (6.1)) we said it had to be big to give a good approximation to the Ising spins, but now I'm using an approximation relying on small g . What gives? The answer is that coarse-graining can make g shrink. Here we go:

6.2 Momentum shells

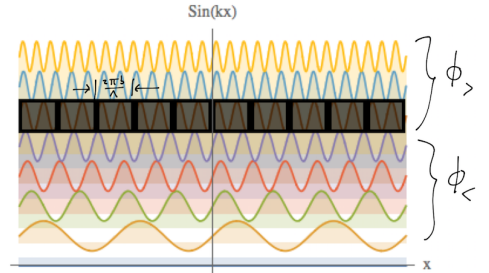
[Zee, *Quantum Field Theory in a Nutshell*, §VI.8 (page 362 of 2d Ed.)]

The continuum functional integral I've written in (6.2) is defined (to the extent that it is) by taking a limit where the lattice spacing goes to zero as the number of sites goes to infinity. This limit is dangerous and is the origin of the bad reputation of the subject of quantum field theory. In its application to the lattice Ising model, this isn't a real problem, because the lattice spacing is a real thing. It provides an *ultraviolet (UV) cutoff* on our field theory. To remind ourselves of this let me decorate our expressions a little bit:

$$Z_\Lambda \equiv \int_\Lambda [D\phi] e^{-\int d^d x \mathcal{L}(\phi)}. \quad (6.6)$$

Here the specification \int_Λ says that we integrate over field configurations $\phi(x) = \int d^d k e^{ikx} \phi_k$ such that $\phi_k = 0$ for $|k| \equiv \sqrt{\sum_{i=1}^D k_i^2} > \Lambda$. Think of $2\pi/\Lambda$ as the lattice spacing²⁶ – there just aren't modes of shorter wavelength.

We want to understand (6.6) by some coarse-graining procedure. Let us imitate the block spin procedure. Field variations *within* blocks of space of linear size na have wavenumbers greater than $\frac{2\pi}{na}$. (These modes average to zero on larger blocks; modes with larger wavenumber encode the variation *between* these blocks.)



So the analog of the partition function after a single blocking step is the following: Break up the configurations into pieces:

$$\phi(x) = \int d^d k e^{ikx} \phi_k \equiv \phi^< + \phi^> .$$

Here $\phi^<$ has nonzero fourier components only for $|k| \leq \Lambda/b$ for some $b > 1$ and $\phi^>$ has nonzero fourier components only in the shell $\Lambda/b \leq |k| \leq \Lambda$. These two parts of the field could be called respectively 'slow' and 'fast', or 'light' and 'heavy', or 'smooth' and 'wiggly'. We want to do the integral over the heavy/wiggly/fast modes to develop an effective action for the light/smooth/slow modes:

$$e^{-S_{\text{eff}}[\phi^<]} \equiv e^{-\int d^d x \mathcal{L}(\phi^<)} \int [D\phi^>] e^{-\int d^d x \mathcal{L}_1(\phi^<, \phi^>)}, \quad Z_\Lambda = \int_{\Lambda/b} [D\phi^<] e^{-S_{\text{eff}}[\phi^<]}$$

where \mathcal{L}_1 contains all the dependence on $\phi^>$ (and no other terms).

²⁶This cutoff is not precisely the same as have a lattice; with a lattice, the momentum space is periodic: $e^{ikx_n} = e^{ik(na)} = e^{i(k + \frac{2\pi}{a})(na)}$ for $n \in \mathbb{Z}$. Morally it is the same.

6.3 Gaussian fixed point

In the special case where the action is quadratic in ϕ , not only can we do the integrals, but the quadratic action is form-invariant under our coarse-graining procedure.

Consider

$$S_0[\phi] = \int d^d x \frac{1}{2} \phi(x) (r_0 - r_2 \partial^2) \phi(x) = \int_0^\Lambda d^d k \phi(k) \phi(-k) \frac{1}{2} (r_0 + r_2 k^2).$$

The coefficient r_2 of the kinetic term (I called it κ earlier) is a book-keeping device that we may set to 1 by rescaling the field variable ϕ if we choose. Why set this particular coefficient to one? One good reason is that then our coarse-graining scheme will map Ising models to Ising models, in the sense that the kinetic term is the continuum representation of the near-neighbor Ising interaction $J \sum_{\langle ij \rangle} s_i s_j$.

We can add a source $\sum_q^\Lambda h_q \phi_{-q}$ to compute

$$Z[h] = \left\langle e^{-\sum_q^\Lambda h_q \phi_{-q}} \right\rangle = Z[0] e^{-\frac{1}{2} \sum_q \frac{h_q h_{-q}}{q^2 + r}}$$

and

$$\langle \phi_q \phi_{q'} \rangle = \frac{1}{Z} \frac{\partial}{\partial h_{-q}} \frac{\partial}{\partial h_{-q'}} Z[h] |_{h=0} = \frac{1}{q^2 + r} \delta_{q+q'} = G(q) \delta_{q+q'}.$$

We can relate the parameter r to a physical quantity by our friend the susceptibility sum rule:

$$\chi = \int d^d x G(x) = G(q=0) \stackrel{\text{Gaussian}}{=} \frac{1}{r}.$$

Here's what I mean by form-invariant: because S_0 does not mix modes of different wavenumber, the integrals over the fast and slow modes simply factorize:

$$e^{-S_{\text{eff}}[\phi^<]} = \int [D\phi^>] e^{-S_0[\phi^>] - S_0[\phi^<]} = Z_{>} e^{-S_0[\phi^<]}$$

– the effective action for the slow modes doesn't change at all, except that the cutoff changes by $\Lambda \rightarrow \Lambda/b$. To make the two systems comparable, we do a change of rulers:

$$\Lambda' \equiv b\Lambda, \phi'_q \equiv b^{\frac{d-2}{2}} \phi_{bq}$$

so that

$$S_{\text{eff}} = \int_0^\Lambda d^d q \phi'_q \phi'_{-q} \frac{1}{2} (q^2 + r')$$

where $r' = b^2 r$.

What we just showed is that this RG we've constructed maps the quadratic action to itself. There are two fixed points, $r_0 = \infty$ and $r_0 = 0$. The former is the high-temperature disordered state. Near this fixed point, the parameter r_0 is relevant and grows as we iterate the RG. No other terms (besides a constant) are generated. We could say there is another fixed point at $r_0 = -\infty$, which could describe the ordered phase, but with $g = 0$, the integral is not well-defined with $r_0 < 0$.

This is the same calculation we did of the random walk, the very first calculation we did, with a lot more labels! The linear term in ϕ (the external magnetic field here) would be relevant, just like the bias term in the random walk that we introduced in §2.1. It is forbidden by the Ising symmetry.

Following the general RG strategy, once we find a fixed point, we must study the neighborhood of the fixed point.

6.4 Perturbations of the Gaussian model

Just as with the spin sums, the integrals are hard to actually do, except in a gaussian theory. But again we don't need to do them to understand the form of the result. We use it to make an RG. As usual there are two steps: coarse-graining and rescaling.

First give it a name:

$$e^{-\int d^d x \delta \mathcal{L}(\phi^<)} \equiv \int [D\phi^>] e^{-\int d^d x \mathcal{L}_1(\phi^<, \phi^>)} \quad (6.7)$$

so once we've done the integral we'll find

$$Z_\Lambda = \int_{\Lambda/b} [D\phi^<] e^{-\int d^d x (\mathcal{L}(\phi^<) + \delta \mathcal{L}(\phi^<))} . \quad (6.8)$$

To get a feeling for the form of $\delta \mathcal{L}$ let's parametrize the LG integrand:

$$\mathcal{L} = \frac{1}{2} \gamma (\partial \phi)^2 + \sum_n g_n \phi^n + \dots \quad (6.9)$$

where we include *all* possible terms consistent with the symmetries ($\phi^{\lessgtr} \rightarrow -\phi^{\lessgtr}$, $h \rightarrow -h$, rotation invariance²⁷). Then we can find an explicit expression for \mathcal{L}_1 :

$$\int d^d x \mathcal{L}_1(\phi^<, \phi^>) = \int d^d x \left(\frac{1}{2} \kappa (\partial \phi^>) ^2 + \frac{1}{2} m^2 (\phi^>)^2 + g_4 (\phi^>)^3 \phi^< + \dots \right)$$

²⁷Why impose rotation invariance here? For now, it's for simplicity. But (preview) we will see that the fixed points we find are stable to rotation-symmetry breaking perturbations. It's an *emergent symmetry*.

(I write the integral so that I can ignore terms that integrate to zero, such as $\partial\phi^<\partial\phi^>$.) This is the action for a scalar field $\phi^>$ interacting with itself and with a (slowly-varying) background field $\phi^<$. But what can the result $\delta\mathcal{L}$ of integrating out $\phi^<$ be but something of the form (6.9) again, with different coefficients?²⁸ The result is to shift the couplings $g_n \rightarrow g_n + \delta g_n$. (This includes the coefficient of the kinetic term and also of the higher-derivative terms which are hidden in the ... in (6.9). You will see in a moment the logic behind which terms I hid.)

Finally, so that we can compare steps of the procedure to each other, we rescale our rulers. We'd like to change units so that $\int_{\Lambda/b}$ is a \int_{Λ} with different couplings; we accomplish this by changing variables: $k' = bk$ so now $|k'| < \Lambda$. So $x' = x/b$, $\partial' \equiv \partial/\partial x' = b\partial_x$ and the Fourier kernel is preserved $e^{ikx} = e^{ik'x'}$. Plug this into the action²⁹

$$S_{\text{eff}}[\phi^<] = \int d^d x (\mathcal{L}(\phi^<) + \delta L(\phi^<)) = \int d^d x' b^d \left(\frac{1}{2} b^{-2} (\partial' \phi^<)^2 + \sum_n (g_n + \delta g_n) (\phi^<)^n + \dots \right)$$

We can make this look like \mathcal{L} again (with $r_2 = 1$) by rescaling the field variable: $b^{d-2} (\partial' \phi^<)^2 \equiv (\partial' \phi')^2$ (i.e. $\phi' \equiv b^{\frac{1}{2}(d-2)} \phi^<$):

$$S_{\text{eff}}[\phi^<] = \int d^d x' \left(\frac{1}{2} (\partial' \phi')^2 + \sum_n (g_n + \delta g_n) b^{d - \frac{n(d-2)}{2}} (\phi')^n + \dots \right)$$

So the end result is that integrating out a momentum shell of thickness $\delta\Lambda \equiv (1 - b^{-1})\Lambda$ results in a change of the couplings to

$$g'_n = b^{\Delta_n} (g_n + \delta g_n)$$

where

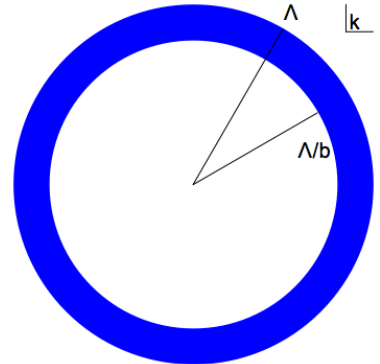
$$\Delta_n \equiv \frac{n(2-d)}{2} + d.$$

Ignore the interaction corrections, δg_n , for a moment. Then we can keep doing this and take $b \rightarrow \infty$ to reach macroscopic scales. Then, as b grows, the couplings with $\Delta_n < 0$ get smaller and smaller as we integrate out more shells. If we are interested in only the longest-wavelength modes, we can ignore these terms. They are *irrelevant*. Couplings ('operators') with $\Delta_n > 0$ get bigger and are *relevant*.

The 'mass term' $r\phi^2$ has $n = 2$ and $r' = b^2 r$ is always relevant for any $d < \infty$.

²⁸Again we apply the Landau-Ginzburg-Wilson logic. The idea is the same as in our discussion of blocking for the Ising model. The result is local in space because the interactions between the slow modes mediated by the fast modes have a range of order b/Λ . The result is analytic in $\phi^<$ at small $\phi^<$ and there is no symmetry-breaking because we only integrate the short-wavelength modes.

²⁹Really, the coefficient of $(\partial' \phi^<)^2$ should be $b^{-2}(1 + \delta\kappa)$. But $\delta\kappa$ turns out to be $\mathcal{O}(g^2)$ so let's ignore it for now.



This counting is the same as dimensional analysis: demand that βH is dimensionless, and demand that the kinetic term $(\partial\phi)^2$ stays fixed. Naive (length) dimensions:

$$[\beta H = S] = 0, \quad [x] \equiv 1, \quad [d^d x] = d, \quad [\partial] = -1$$

The kinetic term tells us the engineering dimensions of ϕ :

$$0 = [S_{\text{kinetic}}] = d - 2 + 2[\phi] \implies [\phi] = \frac{2-d}{2}.$$

Then an interaction term has dimensions

$$0 = [\beta g_n \phi^n] = d + [g_n] + n[\phi] \implies [g_n] = -(d + n[\phi]) = -\left(d + n\frac{2-d}{2}\right) = -\Delta_n$$

– couplings with negative length dimension are relevant. This result is the same as engineering dimensional analysis because we've left out the interaction terms. This is actually correct when $g_n = 0, n \geq 3$, which is the gaussian fixed point.

An important conclusion from this discussion is that there is only a *finite* number of marginal and relevant couplings that we must include to parametrize the physics. Further, if the interactions produce small corrections, they will not change a very irrelevant operator to a relevant operator. This should mitigate some of the terror you felt when we introduced the horrible infinite-dimensional space of hamiltonians \mathcal{M} at the beginning of the course.

Another important conclusion is that the gaussian Ising critical point is *stable* to interactions in $d > 4$. It is of course unstable in the sense that $r\phi^2$ is relevant. And it is unstable if we allow terms with odd powers of ϕ which break the Ising symmetry. But what is the smallest-dimension operator which we haven't added and which respects the Ising symmetry? According to our Gaussian counting, each derivative counts for +1, and each power of ϕ counts for $\frac{2-d}{2}$. If we demand rotation invariance (or even just parity) so we can't have a single derivative, the next most important perturbation is $g_4\phi^4$. Its dimension is $\Delta_4 = 4 - d$ – it is irrelevant if $d > 4$ and relevant if $d < 4$. We could have expected this, since it coincides with the breakdown of mean field theory – above the upper critical dimension, the interactions are irrelevant and MFT gives a correct accounting of the fixed point. In $d = 4$, the ϕ^4 term is marginal, and it is an opportunity for small interaction corrections to decide its fate.

[End of Lecture 12]

6.5 Field theory without Feynman diagrams

[Brezin, Chapter 11] Before we do so systematically, let's pursue the calculation we did in §6.1 a bit further, now that we've learned to organize the integrals over the fluctuations scale-by-scale.

Suppose we've already done the integrals over the shell: $\int_{\Lambda/b}^{\Lambda} dk$, so that the effects of the fluctuations with those wavenumbers are already incorporated into $\gamma^{(1)}[m, b]$. As we argued, $\gamma^{(1)}[m, b]$ will take the same form as the more microscopic effective potential γ , but with some new values of the couplings, depending on b . And then let's do the rest of the integrals using this action (still in the quadratic approximation) and ask how things depend on b .

If we're just integrating out the fluctuations with momenta in the shell above Λ/b , in the quadratic approximation, we can just replace (6.5) with

$$\begin{aligned}\gamma^{(1)}[m, b] &= \frac{1}{2}r_0m^2 + \frac{g_0}{4!}m^4 + \frac{1}{2} \int_{\Lambda/b}^{\Lambda} \mathrm{d}^d q \log \left(\frac{r_0 + \frac{g_0}{2}m^2 + q^2}{r_0 + q^2} \right). \\ &\equiv \frac{1}{2}r(b)m^2 + \frac{g(b)}{4!}m^4 + \dots\end{aligned}\quad (6.10)$$

I also added some subscripts on the couplings to emphasize that r_0, g_0 are parameters in some particular zeroth-order accounting we are making of the physics, not some holy symbols whose values we can measure. In the last line, we've defined *running couplings* $r(b), g(b)$.

From this expression we can read off

$$r(b) = r_0 + \frac{g_0}{2} \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d q}{q^2 + r}.$$

A slightly more useful parameter is the deviation from the critical coupling. The critical point occurs when $\chi^{-1} = \partial_m^2 \gamma|_{m=0} \rightarrow 0$, which happens when r_0 is

$$r_0^c = -\frac{g_0}{2} \int \frac{\mathrm{d}^d q}{q^2} + \mathcal{O}(g_0^2).$$

On the RHS here, we ignored the r in the denominator because it is $\mathcal{O}(g)$. This gives the deviation in temperature from the critical point, by subtracting the previous two displayed equations:

$$t(b) \equiv r_0 - r_0^c = r_0 \left(1 - \underbrace{\frac{g_0}{2} \int \frac{\mathrm{d}^d q}{q^2(q^2 + r)}}_{\equiv I_d(r, b)} + \mathcal{O}(g_0^2) \right).$$

(Note that $t = t(b)$ is merely a convenient relabelling of the coordinate r_0 ; the relation between them is analytic and t depends on our zoom factor b .)

Now we must study the integral I . We've observed that $I_d(r, b \rightarrow \infty)$ blows up (by taking $b \rightarrow \infty$ we include all the fluctuations) when $r \rightarrow 0$ for $d \leq 4$. Let's start at $d = 4$, where

$$I_4(r, b) = K_4 \int_{\Lambda/b}^{\Lambda} \frac{q^3 dq}{q^2(q^2 + r)} \stackrel{\Lambda^2 \gg r}{\approx} + K_4 \log b. \quad (6.11)$$

The running quartic coupling is

$$g(b) \equiv \partial_m^4 \gamma|_{m=0} = g_0 - \frac{3g_0^2}{2} \int \frac{d^d q}{(q^2 + r_0)^2} + \mathcal{O}(g_0^3) \quad (6.12)$$

$$\stackrel{d \rightarrow 4}{\simeq} g_0 - \frac{3g_0^2}{2} K_4 \frac{1}{2} \log \frac{b^2(r + \Lambda^2)}{b^2 r + \Lambda} \stackrel{\Lambda^2 \gg r}{\simeq} g_0 - \frac{3g_0^2}{2} K_4 \log b. \quad (6.13)$$

Combining (6.12) and (6.11), we conclude that for $d = 4$

$$t(b) = r \left(1 - \frac{g_0}{2} K_4 \log b\right).$$

$$g(b) = g - \frac{3g_0^2}{2} K_4 \log b.$$

(I may drop the subscripts on the t s sometimes.)

These expressions are useful because the b -dependence is explicit and we can derive from them an infinitesimal statement:

$$-b\partial_b t = t(b)\kappa(g(t)). \quad (6.14)$$

For the case above, $\kappa(g) = \frac{1}{2}g + \mathcal{O}(g^2)$. Similarly,

$$-b\partial_b g = \beta_g = K_4 \frac{3}{2} g^2 + \mathcal{O}(g^3). \quad (6.15)$$

These vector fields indicating the continuous flows of the couplings with the zoom factor are generally called *beta functions*.

The ordinary differential equation (6.14) is solved by

$$t(b) = t_0 e^{-\int_1^b \frac{d\mu}{\mu} \kappa(g(\mu))}, \quad (6.16)$$

where $t_0 \equiv t(b = 1)$. If there exists a fixed point, $g = g_*$ with $\kappa(g_*) \neq 0$, then its contribution to the exponent (the upper limit dominates) is

$$-\int^b \frac{d\mu}{\mu} \kappa(g_*) \stackrel{b \rightarrow \infty}{\rightarrow} -\kappa(g_*) \underbrace{\int_1^b \frac{d\mu}{\mu}}_{=\log b}.$$

Hence, in this case

$$t(b) = t_0 b^{-\kappa(g_*)} \quad (6.17)$$

– that is $\kappa(g_*)$ determines the critical exponent with which the IR value of $t(b)$ diverges. Why do we care about the IR value of $t(b)$? It determines the correlation length! We’ll come back to this.

What is the solution of the beta function equation for the coupling in $d = 4$? To save writing, let’s redefine $\tilde{g}_0 \equiv K_4 g_0$ and drop the tilde. The equation is

$$-b\partial_b g = \frac{3}{2}g_0^2 + \mathcal{O}(g_0^3)$$

which is solved by

$$g(b) = \frac{2g_0}{2 + 3g_0 \log b} \stackrel{b \gg 1}{\simeq} \frac{2}{3} \frac{1}{\log b} \stackrel{b \rightarrow \infty}{\rightarrow} 0. \quad (6.18)$$

There is an attractive IR fixed point at $g_0 = 0$. This is part of the way towards justifying my claim that perturbation theory would be useful to study the long-wavelength physics in this problem.

In the case of $d = 4$, then, the interesting physics comes from the slow approach to the free theory in the IR. To get something interesting we must include the flow, for example in the solution for t , Eq. (6.16): since the flow of g_0 (6.18) never stops, we can parametrize the flow by g_0 and use the chain rule to write $\frac{d\mu}{\mu} = \frac{dg_0}{\beta(g_0)}$ so that

$$\begin{aligned} \int^b \frac{d\mu}{\mu} \kappa(g_0(\mu)) &= \int_{g_0}^{g_0(b)} \underbrace{\frac{\kappa(g)}{\beta(g)}}_{=\frac{1}{3g}(1+\mathcal{O}(g))} dg \stackrel{b \gg 1}{\simeq} \frac{1}{3} \log \frac{g(b)}{g_0} \end{aligned}$$

From which we conclude

$$t_0(b) \stackrel{b \gg 1}{\simeq} t (\log b)^{-1/3}. \quad (6.19)$$

This equation, a power law (remember that a log is a special case of a power law) relation between the effective temperature and the zoom factor, will be useful below.

Extracting physics from the running couplings. Let’s use this information to study the susceptibility and understand when the quadratic approximation is under control.

First, physics is independent of where we start the flow:

$$\chi^{-1}(t_0, g_0, \Lambda) = \chi^{-1}(t(b), g(b), \Lambda/b) \quad (6.20)$$

– this is what I called the *Callan-Symanzik equation* during the random-walk discussion §1.3. Second, we use ordinary engineering dimensional analysis:

$$[\Lambda] = -1, [\chi^{-1}] = [t] = -2, [g_0] = 0$$

This implies that the RHS of (6.20) is

$$\chi^{-1}(t(b), g(b), \Lambda/b) = b^{-2} \chi^{-1}(t(b)b^2, g(b), \Lambda).$$

For $d \sim 4$, we found that

$$\chi^{-1} \sim t_0 \left(1 - \frac{g_0}{2} \log \frac{\Lambda}{\sqrt{t}} \right). \quad (6.21)$$

This says perturbation theory (in g_0) breaks down (even for small g_0 !) when $g_0 \log \Lambda/\sqrt{t} \gtrsim 1$. But for fixed physics (as in (6.20)), we can *choose* the zoom factor $b = b^*$ so that the argument of the logarithm is

$$1 \stackrel{!}{=} \frac{\Lambda/b^*}{\sqrt{t(b^*)}}. \quad (6.22)$$

When does the zoom factor hit the sweet spot (6.22)? The answer is different in $d = 4$ and $d < 4$.

Using (6.19), this happens when

$$\Lambda/b^* = \sqrt{t(b^*)} = \sqrt{t_0} (\log(b^*))^{-1/6} \leftrightarrow (b^*)^{-2} (\log b^*)^{1/3} = \frac{t_0}{\Lambda^2}$$

which we can solve for b^* in the limit $t \ll \Lambda$ (closer to the critical point than the lattice scale):

$$(b^*)^{-2} \stackrel{t/\Lambda^2 \ll 1}{\simeq} \frac{t}{\Lambda^2} \frac{2}{(\log(t/\Lambda^2))^{1/3}}. \quad (6.23)$$

Putting this information back into the Callan-Symanzik equation for the susceptibility (6.20), we have

$$\begin{aligned} \chi^{-1}(t, g_0, \Lambda) &= (b^*)^{-2} \chi^{-1}(t(b^*), g_0(b^*), \Lambda) \\ &\stackrel{(6.21)}{=} (b^*)^{-2} t \left(1 - \frac{g_0(b^*)}{2} \underbrace{\log \left(\frac{\Lambda}{\sqrt{t(b^*)}} \right)}_{=0, \text{ by design}} \right) \\ &\stackrel{(6.23)}{=} \frac{t}{(\log t/\Lambda^2)^{1/3}}. \end{aligned} \quad (6.24)$$

This is a scaling law for how the susceptibility diverges as we tune the knob on our thermostat towards the critical value.

A comment on active versus passive RG. I've presented the condensed-matter perspective on the RG here: there is a fixed, real cutoff, and the couplings run as we integrate out longer and longer wavelength modes, *i.e.* vary the resolution with which we look at the degrees of freedom.

Another perspective (which leads to the same conclusions!), taken by high-energy physicists, is that the cutoff Λ is an artificial device. We should be able to vary this cutoff without changing the physics, at the cost of changing the values of the couplings at the cutoff. That is we regard the couplings at the cutoff (what I called r_0, g_0 above, the ones appearing in the Lagrangian) *as depending on the cutoff* Λ . To make this precise, we must ask *how* the couplings in $\gamma(\Lambda)$ need to depend on Λ to *keep* the physics from depending on this fictional division we are making between UV and IR. We can think about the RG transformation as replacing the cutoff Λ with a new (smaller) cutoff Λ/b .

Something we *can* measure, and which should not depend on our accounting parameter b , is the susceptibility (for $T > T_c$):

$$r \equiv \chi^{-1} = \partial_m^2 \gamma|_{m=0} = r_0 + \frac{g_0}{2} \int^{\Lambda/b} \frac{d^d q}{q^2 + r_0}.$$

(Such an equation, relating a physical quantity like χ to something we can compute in terms of the running couplings $g_n(b)$, is sometimes called a *renormalization condition*.) We can invert this equation to figure out $r_0(b)$:

$$r_0 = r - \frac{g_0}{2} \int^{\Lambda/b} \frac{d^d q}{q^2 + r} + \mathcal{O}(g_0^2).$$

Again we subtract the critical value of r_0 to get

$$t_0 \equiv r_0 - r_0^c = r \left(1 + \frac{g_0}{2} \int \frac{d^d q}{q^2(q^2 + r)} + \mathcal{O}(g_0^2) \right).$$

Near $d = 4$, this is

$$t_0 = r \left(1 + \frac{g_0}{2} K_4 \log \frac{\Lambda/b}{\sqrt{r}} \right) = r \left(1 - \frac{g_0}{2} K_4 \log b + \dots \right)$$

(where the ellipsis is independent of b).

Another quantity we can imagine measuring is the coupling g , a non-linear susceptibility:

$$g \equiv \partial_m^4 \gamma|_{m=0} = g_0 - \frac{3g_0^2}{2} \int^{\Lambda/b} \frac{d^d q}{(q^2 + r_0)^2} + \mathcal{O}(g_0^3)$$

– notice that this is the same equation as (6.12), but the BHS is interpreted differently: now the LHS is a physical, fixed, measurable thing, and g_0 is a fake thing that depends on the artificial parameter b . We can invert this equation to find g_0 in terms of g and b :

$$g_0(b) = g + \frac{3g^2}{2} \int^{\Lambda/b} \frac{d^d q}{(q^2 + r_0)^2} + \mathcal{O}(g^3)$$

(where we studiously neglect higher order things). Near $d = 4$ this is

$$g_0(b) \stackrel{d \rightarrow 4}{\simeq} g + \frac{3g^2}{2} K_4 \frac{1}{2} \log \frac{\Lambda}{b\sqrt{r}} + \mathcal{O}(g^3) = g - \frac{3g^2}{2} K_4 \log b + \mathcal{O}(g^3) \quad (6.25)$$

(where the ellipsis is independent of b). This reproduces the same beta functions as above.

Two important generalizations. Now we make two easy but crucial generalizations of the $d = 4$ Ising calculation we've just done: namely $\mathbb{Z}_2 \rightarrow \mathcal{O}(n)$ and $d \rightarrow 4 - \epsilon$.

$\boxed{\mathcal{O}(n)}$: [Goldenfeld, §11.1] by the LG logic, a $\mathcal{O}(n)$ -invariant and translation-invariant free energy at fixed magnetization m^a must look like

$$S[\phi^a] = \int d^d x \left(\frac{1}{2} \vec{\nabla} \phi^a \cdot \vec{\nabla} \phi^a + \frac{1}{2} r_0 \phi^a \phi^a + \frac{g_0}{4!} (\phi^a \phi^a)^2 \right)$$

For $n > 1$, in expanding about the mean field configuration $\phi^a = m^a + \varphi^a$, we encounter a distinction between the one (longitudinal) fluctuation in the direction of $m^a \equiv m e_0^a$ and the $n - 1$ transverse fluctuations. The quadratic part of this action comes from the kernel

$$K_{xy}^{ab} = \frac{\delta S}{\delta \phi_x^a \delta \phi_y^b} \Big|_{\phi=m} = \left(\left(-\nabla^2 + r_0 + \frac{g_0}{6} m^2 \right) \delta_{ab} + \frac{g_0}{3} m_a m_b \right) \delta_{xy}.$$

[End of Lecture 13]

Its eigenvectors can be constructed using an orthonormal basis $\{e^a\}$, in terms of which we can choose $m^a = m e_0^a$ and decompose $\varphi^a \equiv \varphi^L e_0^a + \varphi_\alpha^T e_\alpha^a$, $\alpha = 1..n - 1$. (L is for longitudinal, T is for transverse, to the direction of the ordering vector m^a .) Then K is block-diagonal in this basis:

$$K_{xy}^{ab} = \left((\delta_{ab} - e_a e_b) \left(-\nabla^2 + r_0 + \frac{g_0}{6} m^2 \right) + e_a e_b \left(-\nabla^2 + r_0 + \underbrace{\frac{1}{2}}_{=1/3+1/6} g_0 m^2 \right) \right) \delta_{xy}$$

This matrix is made of one copy of the $n = 1$ Ising case with coefficient of m^2 equal to $g_0/2$, and $n - 1$ degenerate copies of the same thing with $g_0/6$. So the sum of the logs

of the eigenvalues is

$$\text{tr}_{x,a} \log K = V \int \mathrm{d}^d q \left(\log \left(r_0 + \frac{g_0}{2} m^2 + q^2 \right) + (n-1) \log \left(r_0 + \frac{g_0}{6} m^2 + q^2 \right) + \text{const} \right).$$

Redoing the steps between (6.10) and (6.12), we find

$$g = g_0 - \left(\frac{3g_0^2}{2} + (n-1) \frac{g_0^2}{6} \right) \int \frac{\mathrm{d}^d q}{(q^2 + r)^2} + \mathcal{O}(g_0^3) \stackrel{d \rightarrow 4}{\simeq} g_0 - g_0^2 \left(\frac{3}{2} + \frac{n-1}{6} \right) K_4 \log b$$

so that the beta function is

$$\beta(g_0) = K_4 \frac{n+8}{6} g_0^2 + \mathcal{O}(g_0^3)$$

The flow of the temperature is

$$t_0(b) = t_0 \left(1 - g_0 \frac{n+2}{6} K_4 \log b \right).$$

$\boxed{d = 4 - \epsilon}$: If $d \neq 4$ the self-coupling term $\int d^d x \varphi^4$ in the LG action is not dimensionless: $[\varphi] = \frac{2-d}{2}$, $\implies [g_0] = 4 - d \equiv \epsilon$. Let's extract powers of the UV cutoff to make a dimensionless coupling $g_0 \rightarrow \Lambda^\epsilon g_0$, so that the LG action is

$$\gamma[\phi] = \dots + \int d^d x \Lambda^\epsilon g_0 \phi^4.$$

Anticipating the result a bit, we are going to treat g_0 and ϵ as being of the same order in our expansion, so $\mathcal{O}(g_0) = \mathcal{O}(\epsilon)$ and $\mathcal{O}(g_0^2) = \mathcal{O}(\epsilon g_0)$ et cetera. Thinking of ϵ as small, then, the only change in (6.12) is

$$\partial_m^4 \gamma(b)|_{m=0} \equiv \Lambda^\epsilon g_0 - \mathfrak{b}_0 g_0^2 \int \frac{d^d q}{(q^2 + r)^2} \tag{6.26}$$

where $\mathfrak{b}_0 \equiv \frac{3}{2} + \frac{n-1}{6}$.

The coefficient of ϕ^4 in the effective action at scale Λ/b is

$$(\Lambda/b)^\epsilon g(b) \equiv \partial_m^4 \gamma(b)|_{m=0} = \Lambda^\epsilon g_0 - \mathfrak{b}_0 K_d \Lambda^\epsilon g_0^2 \log b + \mathcal{O}(g_0^2 \epsilon).$$

Here comes the magic: the key fact is roughly that “ $\Lambda^\epsilon = 1 + \epsilon \log \Lambda + \mathcal{O}(\epsilon^2)$ ”; I put that in quotes because it is distasteful to take the log of a dimensionful quantity. Systematically ignoring things that can be ignored (including the Λ^ϵ which is needed in the previous equation for dimensions to work), this is:

$$g(b) = g_0 (1 + \epsilon \log b) - \mathfrak{b}_0 g_0^2 \log b + \mathcal{O}(g_0^2 \epsilon)$$

(Again we absorb the factors of K_d into g, g_0 .)

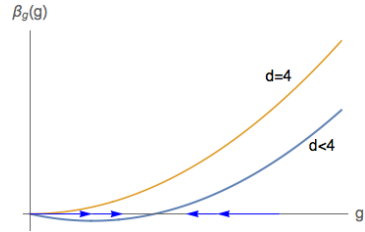
$$\beta_g \equiv -b\partial_b g(b) = -\epsilon g_0 + \mathfrak{b}_0 g_0^2 + \mathcal{O}(g_0^3).$$

The crucial extra term proportional to ϵg_0 comes from the engineering dimensions of g_0 .

Where are the fixed points? There is still one at $g_0 = 0$, our old friend the Gaussian fixed point. But **there is another**, at

$$g^* = \frac{\epsilon}{\mathfrak{b}_0} + \mathcal{O}(\epsilon^2) = \frac{6}{n+8}\epsilon + \mathcal{O}(\epsilon^2).$$

This is the Wilson-Fisher fixed point (really one for every n and $d \lesssim 4$). As was foretold, g_0 is of order ϵ .



The WF fixed point and the Gaussian critical point exchange roles as we decrease d through four. For $d > 4$, the Gaussian critical point is IR attractive and governs the critical behavior at long wavelengths: MFT is right. At $d = 4$, they collide and this produces the weird logarithms in the approach to $g = 0$ that we saw above. For $d < 4$, the Gaussian fixed point is unstable to the interaction term: the $g_0\phi^4$ term is a relevant perturbation, since g_0 grows as we zoom out.

Correlation length critical exponent. Now we can look at the behavior of the correlation length as we approach this critical point, by tuning the temperature. Again, physics is independent of where we start the flow:

$$\xi(t_0, g_0, \Lambda) = \xi(t(b), g(b), \Lambda/b) \tag{6.27}$$

– this is what I called the *Callan-Symanzik equation* during the random-walk discussion §1.3. Second, we use ordinary engineering dimensional analysis:

$$[\Lambda] = -1, [\xi] = 1, [g_0] = 0, [t] = -2$$

– the correlation length is a length and so zooms like a length. From this, we deduce that (the RHS of (6.27) is)

$$\xi(t(b), g(b), \Lambda/b) = b\xi(t(b)b^2, g(b), \Lambda).$$

Now we can choose a convenient zoom factor, b . Again, we choose $b = b^*$ so that the argument of the logs are all 1 and they go away:

$$\frac{t(b^*)}{(\Lambda/b^*)^2} = 1. \tag{6.28}$$

If $b^* \rightarrow \infty$, then $g(b^*) \rightarrow g_*$, the IR fixed point value, where

$$t(b^*) \stackrel{(6.17)}{\simeq} (b^*)^{-\kappa(g_*)} t.$$

We can solve this equation for b^* , using (6.28):

$$b^* = \left(\frac{t}{\Lambda^2} \right)^{-\frac{1}{2-\kappa(g_*)}} \quad t \ll \Lambda \rightarrow \infty$$

which indeed blows up in the critical region $t \ll \Lambda$ – that is: this is an IR fixed point, a fixed point we reach by zooming out.

Therefore

$$\begin{aligned} \xi(t, g_0, \Lambda) &= b^* \xi(t(b^*) (b^*)^2 = \Lambda^2, g_*, \Lambda) \sim \left(\frac{t}{\Lambda^2} \right)^{-\frac{1}{2-\kappa(g_*)}} \\ &\equiv \left(\frac{t}{\Lambda^2} \right)^{-\nu} \end{aligned} \quad (6.29)$$

Explicitly, $\kappa(g_0) = \frac{n+2}{6} g_0 + \mathcal{O}(g_0^2)$ means $\kappa(g_*) = \frac{n+2}{n+8} \epsilon + \mathcal{O}(\epsilon^2)$ so that

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)} \epsilon + \mathcal{O}(\epsilon) \quad (6.30)$$

Notice that all the information about the short-distance stuff has dropped out of (6.29) (except for the stuff hidden in the twiddle, i.e. the overall coefficient) – only the physics *at* the fixed point matters for the exponent.

We can do remarkably well by setting $\epsilon = 1$ in (6.30) and comparing to numerical simulations in $d = 3$.

6.6 Perturbative momentum-shell RG

[Kardar, *Fields*, §5.5, 5.6] I will say a bit about how to develop this perturbative RG more systematically. We'll end up at the same place, but with more context. This calculation is important enough that it's worth doing many ways.

We'll do n -component fields, ϕ^a , $a = 1..n$ with $O(n)$ symmetry, in $d = 4 - \epsilon$ dimensions. Let's decompose the action as

$$S[\phi] = S_0[\phi] + \mathcal{U},$$

with S_0 the gaussian terms, as above. For n component fields, the gaussian term looks like

$$S_0[\phi] = \int_0^\Lambda \int d^d k \underbrace{\phi^a(k)\phi^a(-k)}_{\equiv |\phi|^2(k)} \frac{1}{2} (r_0 + r_2 k^2).$$

(If it is not diagonal, do a field redefinition to make it so.) We assume the model has a $O(n)$ symmetry which acts by $\phi^a \rightarrow R_b^a \phi^b$, with $R^t R = \mathbb{1}_{n \times n}$. The most relevant, symmetric interaction term (non-Gaussian perturbation) is the ϕ^4 term

$$\mathcal{U} = \int d^d x u_0 (\phi^a(x)\phi^a(x))^2 = u_0 \int \prod_{i=1}^4 \int d^d k_i \sum_{a_1,2,3,4=1}^n \phi^{a_1}(k_1)\phi^{a_2}(k_2)\phi^{a_3}(k_3)\phi^{a_4}(k_4) \delta(\sum_i k_i) \delta^{a_1 a_2} \delta^{a_3 a_4}.$$

(I've defined $\delta(q) \equiv (2\pi)^d \delta^d(q)$.)

We'll show that it's not actually necessary to ever do any momentum integrals to derive the RG equations.

Again we break up our fields into slow and fast, and we want to integrate out the fast modes first:

$$Z_\Lambda = \int [D\phi_{<}] e^{-\int_0^{\Lambda/b} \int d^d k |\phi_{<}(k)|^2 \left(\frac{r_0 + r_2 k^2}{2}\right)} Z_{0,>} \langle e^{-\mathcal{U}[\phi_{<}, \phi_{>}]} \rangle_{0,>}.$$

The $\langle \dots \rangle_{0,>}$ means averaging over the fast modes with their Gaussian measure, and $Z_{0,>}$ is an irrelevant normalization factor, independent of the objects of our fascination, the slow modes $\phi_{<}$.

The corrections to the effective action for $\phi_{<}$ can be organized as a cumulant expansion:

$$\log \langle e^{-\mathcal{U}} \rangle_{0,>} = - \underbrace{\langle \mathcal{U} \rangle_{0,>}}_1 + \frac{1}{2} \underbrace{\left(\langle \mathcal{U}^2 \rangle_{0,>} - \langle \mathcal{U} \rangle_{0,>}^2 \right)}_2 + \mathcal{O}(\mathcal{U}^3)$$

Let's focus on the first-order term first:

$$1 = \langle \mathcal{U}[\phi_{<}, \phi_{>}] \rangle_{0,>} = u_0 \int \prod_{i=1}^4 \int d^d k_i \delta^d \left(\sum_i k_i \right) \left\langle \prod_i (\phi_{<} + \phi_{>})_i \right\rangle_{0,>}$$

It is useful to introduce a diagrammatic notation in which these 16 terms decompose as in Fig. 1.

We can compute the averages over the fast modes by doing Wick contractions. This is a fact about Gaussian integrals, which can be summarized by noting that

$$\langle e^{h_A \phi_A} \rangle_0 = e^{\frac{1}{2} h_A \langle \phi_A \phi_B \rangle_0 h_B}$$

where A is a multi-index over space and flavor labels and whatever else (to prove it, complete the square). Then expand both sides to learn that

$$\langle \phi_{A_1} \cdots \phi_{A_m} \rangle_0 = \begin{cases} 0, & \text{if } m \text{ is odd} \\ \text{sum of all pairwise contractions,} & \text{if } m \text{ is even} \end{cases} .$$

By ‘pairwise contraction’ I just mean a way of replacing a pair of ϕ s on the LHS with $\langle \phi_A \phi_B \rangle$. Each pairwise contraction is given by the ‘propagator’, which in our case is

$$\langle \phi_{>}^a(q_1) \phi_{>}^b(q_2) \rangle_{0,>} = \frac{\delta^{ab} \not{k}(q_1 + q_2)}{r_0 + q_1^2 r_2} = \text{wiggly line with } a, b \text{ labels} .$$

In the figure, these are denoted by wiggly lines. The slow modes are denoted by straight lines. The 4-point interaction is denoted by a dotted line connecting two pairs of lines (straight or wiggly).

$$u_0 \delta^{a_1 a_2} \delta^{a_3 a_4} \not{k} \left(\sum_i q_i \right) = \text{4-point interaction diagram} .$$

Although the four fields must be at the same point in space we separate the two pairs whose flavor indices are contracted, so that we can follow the conserved flavor index around the diagrams.

Let’s analyze the results of the first order correction: The interesting terms are

$$\mathbf{1}_3 = -u_0 \underbrace{2}_{\text{symmetry}} \underbrace{n}_{=\delta^{aa}} \int_0^{\Lambda/s} d^d k |\phi_{<}(k)|^2 \int_{\Lambda/s}^{\Lambda} d^D q \frac{1}{r_0 + r_2 q^2}$$

$$\mathbf{1}_4 = \frac{4 \cdot 1}{2 \cdot n} \mathbf{1}_3$$

has a bigger symmetry factor but no closed flavor index loop. The result through $\mathcal{O}(u)$ is then just what we found previously:

$$\boxed{r_0 \rightarrow r_0 + \delta r_0 = r_0 + 4u_0(n+2) \int_{\Lambda/b}^{\Lambda} d^d q \frac{1}{r_0 + r_2 q^2} + \mathcal{O}(u^2)} .$$

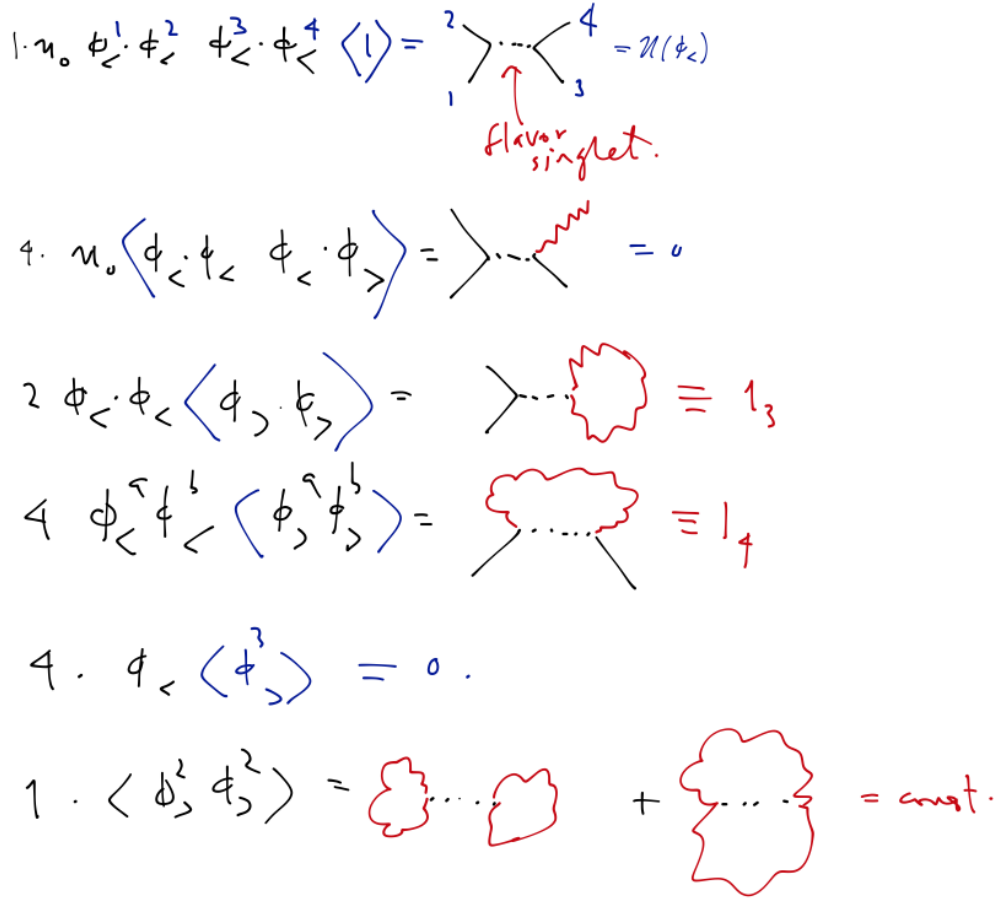


Figure 1: 1st order corrections from the quartic perturbation of the Gaussian fixed point of the $O(N)$ model. Naturally, wiggly lines denote propagation of fast modes $\phi_>$, straight lines denote (external) slow modes $\phi_<$. A further refinement of the notation is that we split apart the 4-point vertex to indicate how the flavor indices are contracted; the dotted line denotes a direction in which no flavor flows, i.e. it represents a coupling between the two flavor singlets, $\phi^a\phi^a$ and $\phi^b\phi^b$. The numbers at left are multiplicities with which these diagrams appear. (The relative factor of 2 between 1_3 and 1_4 can be understood as arising from the fact that 1_3 has a symmetry which exchanges the fast lines but not the slow lines, while 1_4 does not.) Notice that closed loops of the wiggly lines produce of n , since we must sum over which flavor is propagating in the loop – the flavor of a field running in a closed loop is not determined by the external lines, just like the momentum.

r_2 and u are unchanged. The second part of the RG step is rescaling

$$\tilde{q} \equiv bq, \quad \tilde{\phi}(k) \equiv \zeta^{-1}\phi_<(k)$$

to restore the original action: we must choose $\zeta = b^{1+d/2}$ to keep $\tilde{r}_2 = r_2$ (the unfamiliar power is because $\phi(k) = \int d^d x \phi(x) e^{ikx}$ scales differently from $\phi(x)$).

The second-order-in- u_0 terms are displayed in Fig. 2. The interesting part of the

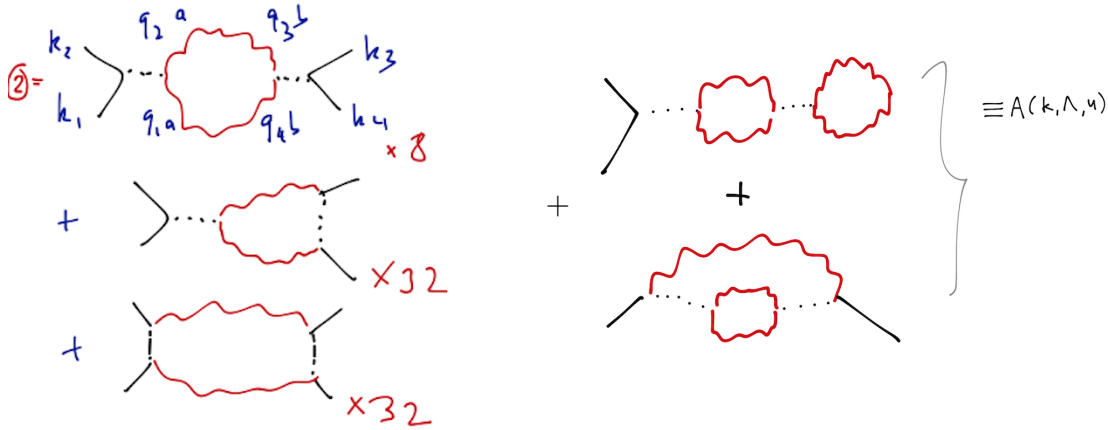


Figure 2: 2nd order corrections from the quartic perturbation of the Gaussian fixed point of the $O(N)$ model. The left column of diagrams are corrections to the quartic interaction, and the right column correct quadratic terms. In fact the top right diagram is independent of the external momentum and hence only corrects r_0 ; the bottom right diagram (that looks like a sheep) also corrects the kinetic term.

Notice that the diagram at top right has two closed flavor loops, and hence goes like n^2 , and it comes with two powers of u_0 . You can convince yourself by drawing some diagrams that this pattern continues at higher orders. If you wanted to define a model with large n you could therefore consider taking a limit where $n \rightarrow \infty$, $u_0 \rightarrow 0$, holding $u_0 n$ fixed. The quantity $u_0 n$ is often called the 't Hooft coupling.

second order bit

$$\delta_2 = \frac{1}{2} \langle \mathcal{U}[\phi_<, \phi_>]^2 \rangle_{0, >, \text{connected}}$$

is the correction to $\mathcal{U}[\phi_<]$. There are less interesting bits which are zero or constant.

[End of Lecture 14]

The correction to the quartic term at 2nd order is

$$\delta_2 S_4[\phi_<] = u_0^2 (4n + 32) \int_0^{\Lambda/b} \prod_i^4 (d^d k_i \phi_<(k_i)) \delta(\sum k_i) f(k_1 + k_2)$$

with

$$f(k_1 + k_2) = \int d^d q \frac{1}{(r_0 + r_2 q^2)(r_0 + r_2(k_1 + k_2 - q)^2)} \simeq \int d^d q \frac{1}{(r_0 + r_2 q^2)^2} (1 + \mathcal{O}(k_1 + k_2))$$

– the bits that depend on the external momenta give irrelevant derivative corrections, like $\phi_<^2 \partial^2 \phi_<^2$. We ignore them. This leaves behind just the correction to u we found before.

There are also two-loop corrections to the quadratic term (diagrams with two straight lines sticking out). Altogether, the full result through $\mathcal{O}(u_0^2)$ is then the original action, with the parameter replacement

$$\begin{pmatrix} r_2 \\ r_0 \\ u_0 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{r}_2 \\ \tilde{r}_0 \\ \tilde{u}_0 \end{pmatrix} = \begin{pmatrix} b^{-d-2}\zeta^2(r_2 + \delta r_2) \\ b^{-d}\zeta^2(r_0 + \delta r_0) \\ b^{-3d}\zeta^4(u_0 + \delta u_0) \end{pmatrix} + \mathcal{O}(u_0^3).$$

The shifts are:

$$\begin{cases} \delta r_2 = u_0^2 \frac{\partial_k^2 A(0)}{r_2} \\ \delta r_0 = 4u_0(n+2) \int_{\Lambda/b}^{\Lambda} \mathbb{d}^d q \frac{1}{r_0+r_2q^2} - A(0)u_0^2 \\ \delta u_0 = -\frac{1}{2}u_0^2(8n+64) \int_{\Lambda/b}^{\Lambda} \mathbb{d}^d q \frac{1}{(r_0+r_2q^2)^2} \end{cases} .$$

Here A is a new ingredient that we didn't notice earlier: it produces a correction to the kinetic term for $\phi_{<}$ (which is called 'wavefunction renormalization'): $A(k) = A(0) + \frac{1}{2}k^2\partial_k^2 A(0) + \dots$. We can choose to keep $\tilde{r}_2 = r_2$ by setting

$$\zeta^2 = \frac{b^{d+2}}{1 + u_0^2 \partial_k^2 A(0)/r_2} = b^{d+2} (1 + \mathcal{O}(u_0^2)).$$

Now let's make the RG step infinitesimal:

$$\begin{aligned} b &= e^\ell \simeq 1 + \delta\ell \\ \begin{cases} \frac{dr_0}{d\ell} = 2r_0 + \frac{4(n+2)K_d\Lambda^d}{r_0+r_2\Lambda^2}u_0 - Au_0^2 + \mathcal{O}(u_0^3) \\ \frac{du_0}{d\ell} = (4-d)u_0 - \frac{4(n+8)K_d\Lambda^d}{(r_0+r_2\Lambda^2)^2}u_0^2 + \mathcal{O}(u_0^3) \end{cases} \end{aligned} \quad (6.31)$$

To see how the previous thing arises, and how the integrals *all went away*, let's consider just the $\mathcal{O}(u_0)$ correction to the mass:

$$\begin{aligned} \tilde{r}_0 &= r_0 + \delta\ell \frac{dr_0}{d\ell} = s^2 \left(r_0 + 4u_0(n+2) \int_{\Lambda/b}^{\Lambda} \frac{\mathbb{d}^d q}{r_0 + r_2q^2} + \mathcal{O}(u_0^2) \right) \\ &= (1 + 2\delta\ell) \left(r_0 + 4u_0(n+2)K_d\Lambda^d \frac{1}{r_0 + r_2\Lambda^2} \delta\ell + \mathcal{O}(u_0^2) \right) \\ &= \left(2r_0 + \frac{4u_0(n+2)}{r_0 + r_2\Lambda^2} K_d\Lambda^d \right) \delta\ell + \mathcal{O}(u_0^2). \end{aligned} \quad (6.32)$$

Now we are home. (6.31) has two fixed points. One is the free fixed point at the origin where nothing happens. The other (Wilson-Fisher) fixed point is at

$$\begin{cases} r_0^* = -\frac{2u_0^*(n+2)K_d\Lambda^d}{r_0^*+r_2\Lambda^2} \stackrel{d=4-\epsilon}{=} -\frac{1}{2} \frac{n+2}{n+8} r_2\Lambda^2 \epsilon + \mathcal{O}(\epsilon^2) \\ u_0^* = \frac{(r_0^*+r_2\Lambda^2)^2}{4(n+8)K_d\Lambda^d} \epsilon \stackrel{d=4-\epsilon}{=} \frac{1}{4} \frac{r_2^2}{(n+8)K_4} \epsilon + \mathcal{O}(\epsilon^2) \end{cases}$$

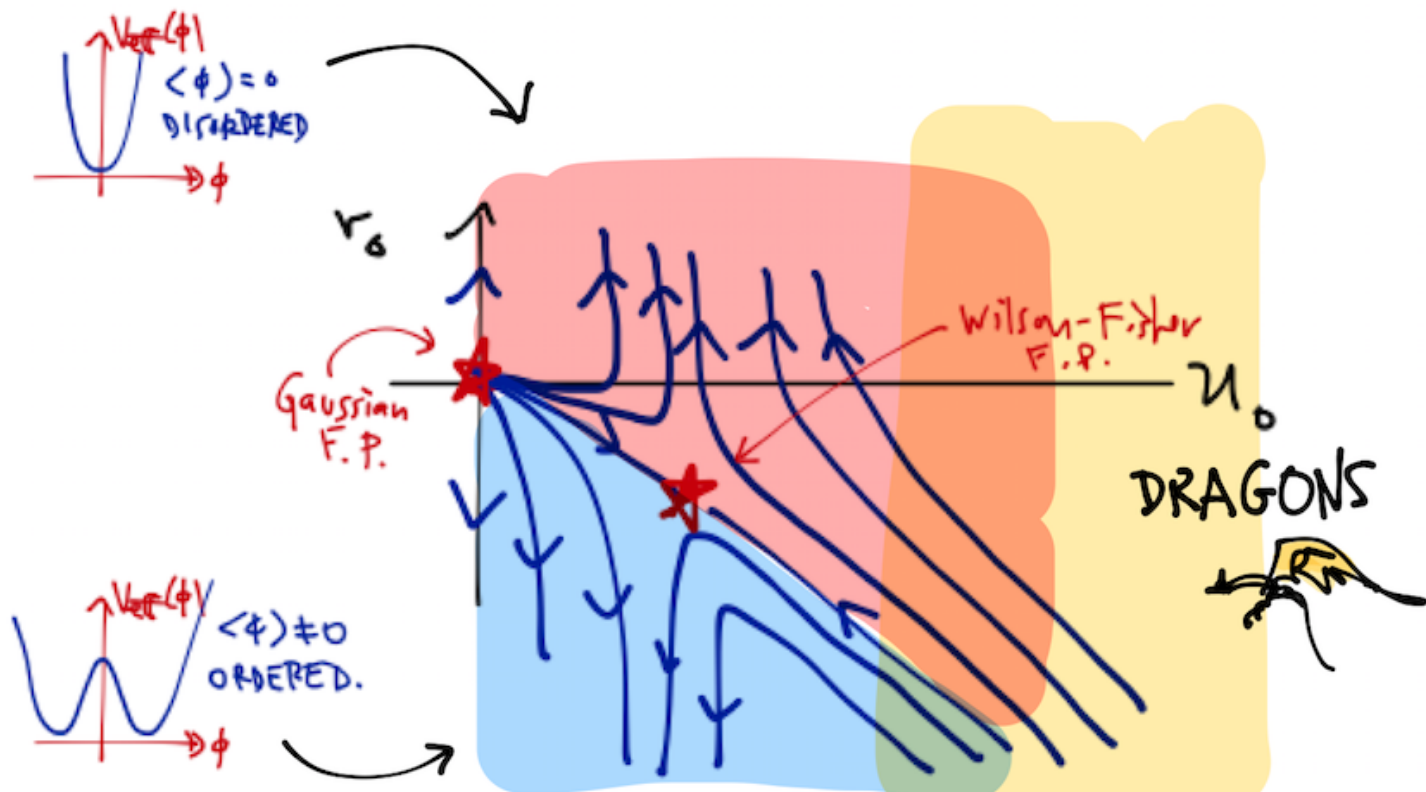


Figure 3: The ϕ^4 phase diagram, for $\epsilon > 0$. If $r_0(\ell = \infty) > 0$, the effective potential for the uniform ‘magnetization’ has a minimum at the origin; this is the disordered phase, where there is no magnetization. If $r_0(\ell = \infty) = V''_{\text{eff}} < 0$, the effective potential has minima away from the origin, and the groundstate breaks the $O(n)$ symmetry; this is the ordered phase. Too far to the right, u_0 is too large for us to trust our perturbative analysis. Experimental and numerical evidence suggests, however, that there are no other fixed points nearby, *i.e.* that there are actually no dragons.

which is at positive u_0^* if $\epsilon > 0$. In the second step we keep only leading order in $\epsilon = 4 - d$.

Now we follow protocol and *linearize* near the W-F fixed point:

$$\frac{d}{d\ell} \begin{pmatrix} \delta r_0 \\ \delta u_0 \end{pmatrix} = M \begin{pmatrix} \delta r_0 \\ \delta u_0 \end{pmatrix}$$

The matrix M is a 2×2 matrix whose eigenvalues describe the flows near the fixed point. It looks like

$$M = \begin{pmatrix} 2 - \frac{n+2}{n+8}\epsilon & \dots \\ \mathcal{O}(\epsilon^2) & -\epsilon \end{pmatrix}$$

Its eigenvalues (which don’t care about the off-diagonal terms because the lower left

entry is $\mathcal{O}(\epsilon^2)$ are

$$y_r = 2 - \frac{n+2}{n+8}\epsilon + \mathcal{O}(\epsilon^2) > 0$$

which determines the instability of the fixed point and

$$y_u = -\epsilon + \mathcal{O}(\epsilon^2) < 0 \quad \text{for } d < 4$$

which is a stable direction.

So y_r determines the correlation length exponent. Its eigenvector is δr_0 to $\mathcal{O}(\epsilon^2)$. This makes sense: r_0 is the relevant coupling which must be tuned to stay at the critical point. The correlation length can be found as follows ξ is the value of $s = s_1$ at which the relevant operator has turned on by an order-1 amount, *i.e.* by setting $\xi \sim s_1$ when $1 \sim \delta r_0(s_1)$. According to the linearized RG equation, close to the fixed point, we have $\delta r_0(s) = s^{y_r} \delta r_0(0)$. Therefore³⁰

$$\xi \sim s_1^{-\frac{1}{y_r}} = (\delta r_0(0))^{-\nu} .$$

This last equality is the definition of the correlation length exponent (how does the correlation length scale with our deviation from the critical point $\delta r_0(0)$). Therefore

$$\nu = \frac{1}{y_r} = \left(2 \left(1 - \frac{1}{2} \frac{n+2}{n+8} \epsilon \right) \right)^{-1} + \mathcal{O}(\epsilon^2) \simeq \frac{1}{2} \left(1 + \frac{n+2}{2(n+8)} \epsilon \right) + \mathcal{O}(\epsilon^2).$$

7 Scaling

Scaling functions from the RG. [Goldenfeld, §9.4, 9.2] Consider a renormalization group near a fixed point with one relevant parameter, which transforms as

$$T' = \mathcal{R}_b(T).$$

At the fixed point $T_\star = \mathcal{R}_b(T_\star)$. Near the fixed point

$$T' - T_\star = \mathcal{R}_b(T) - \mathcal{R}_b(T_\star) = R_b(T - T_\star) + \mathcal{O}(T - T_\star)^2$$

where as in §1.5, $R_b \equiv \partial_T \mathcal{R}_b|_{T_\star}$, which is already diagonalized, so that

$$R_b = b^{y_t} \quad y_t = \frac{1}{b} \log R_b.$$

³⁰I find this kind of argument very slippery; the next section is an attempt to make it and its ilk more systematic.

Letting $t_0 = \frac{T-T_*}{T_*}$, the reduced temperature transforms under a single RG step as $t' = t_0 b^{y_t}$ and under n RG steps as

$$t^{(n)} = t (b^{y_t})^n.$$

Since the correlation length ξ is a length, it transforms as $\xi' = \xi/b$ (think of a blocking transformation whereby b^d lattice sites are blocked, so a fixed length is smaller in units of the new lattice spacing). Under n steps,

$$\xi^{(n)} \equiv \xi(t^{(n)}) = \xi(t_0)/b^n$$

which says

$$\xi(t_0) = b^n \xi(t^{(n)}) = b^n \xi(t_0 b^{ny_t}).$$

Now *choose* b so that

$$b^n = \left(\frac{\mathfrak{b}}{t_0}\right)^{1/y_t}, \quad \mathfrak{b} \sim 1$$

where \mathfrak{b} is some order-one number. Then we have

$$\xi(t_0) = (\mathfrak{b}^{-1} t_0)^{-1/y_t} \xi(\mathfrak{b}) \quad \text{as } t_0 \rightarrow 0.$$

Here $\xi(\mathfrak{b})$ is the high-temperature correlation length, far from the critical point, which we can regard as a constant, independent of t_0 . Comparing to the definition of the correlation length critical exponent $\xi(t_0) \sim t_0^{-\nu}$ this says

$$\nu = \frac{1}{y_t}. \tag{7.1}$$

Similarly, the (singular part³¹ of the) free energy density transforms as

$$f(t) = b^{-d} f(t') = b^{-nd} f(t^{(n)}) = \left(\frac{t_0}{\mathfrak{b}}\right)^{d/y_t} f(\mathfrak{b}).$$

From this we learn, for example, that the specific heat

$$c_V \sim \partial_t^2 f \sim t^{-\alpha}$$

has $\alpha = 2 - \frac{d}{y_t}$. Comparing to the expression (7.1) for ν , we have

$$2 - \alpha = \nu d$$

which is called the Josephson scaling relation.

³¹That is, not including the constant C in (3.12), which doesn't scale.

All the variables. More generally, there are many more couplings. For example, for the Wilson-Fisher fixed point (in $2 < d < 4$) there are two relevant couplings t, h (the latter of which breaks symmetries), and a long list of irrelevant operators which I'll call K_3, K_4 . Let us suppose that we have already diagonalized the matrix $R_\alpha^\beta = \partial_{K^\alpha} \mathcal{R}^\beta|_{K=K_*}$ at the fixed point, and $t, h, K_3, K_4 \dots$ are the coordinates in the eigendirections, with scaling exponents $y_t, y_h, y_3, y_4 \dots$. The statements about relevance above say $y_t, y_h > 0, y_3, y_4 < 0$. (Note that the quartic coupling is in the list of *irrelevant* perturbations of the WF fixed point for $2 < d < 4$.) Then

$$f(t, h, K_3, K_4 \dots) = b^{-d} f(tb^{y_t}, hb^{y_h}, K_3 b^{y_3}, K_4 b^{y_4} \dots) \quad (7.2)$$

$$= b^{-nd} f(tb^{ny_t}, hb^{ny_h}, K_3 b^{ny_3}, K_4 b^{ny_4} \dots) \quad (7.3)$$

$$= t^{d/y_t} \mathfrak{b}^{-d} f\left(\mathfrak{b}, h (t/\mathfrak{b})^{-y_h/y_t}, K_3 (t/\mathfrak{b})^{-y_3/y_t}, K_4 (t/\mathfrak{b})^{-y_4/y_t} \dots\right) \quad (7.4)$$

$$\xrightarrow{t \rightarrow 0} t^{d/y_t} \mathfrak{b}^{-d} f\left(\mathfrak{b}, h (t/\mathfrak{b})^{-y_h/y_t}, 0, 0, \dots\right) \quad (7.5)$$

$$\equiv t^{2-\alpha} F(h/t^\Delta), \quad \Delta \equiv y_h/y_t. \quad (7.6)$$

The last expression for the free energy density is in terms of a *scaling function* F —which basically just means a function of dimensionless arguments. The existence of this function implies all the so-called *hyperscaling relations* (such as the Josephson one above) which relate various exponents.

An important disclaimer about the $t \rightarrow 0$ limit in (7.5): the limit $f(t, h, K_3)$ as $K_3 \rightarrow 0$ may not exist. In that case, K_3 is called a *dangerous irrelevant variable*. An example where this happens is at the gaussian fixed point in $d > 4$, with $K_3 = g$, the quartic coupling. If the quartic coupling is zero, then for $t \leq 0$, the partition function blows up, so the limit $t \rightarrow 0$ does not commute with $g \rightarrow 0$. Despite the fact that g is an irrelevant perturbation, $g > 0$ is crucial for determining the saddle point configuration of m when $t \leq 0$.

Corrections to scaling. In experiments or simulations, t isn't really zero. So the contributions from irrelevant operators are not exactly zero. For example, by a similar argument to (7.5), the susceptibility is

$$\chi_T(t, h, \dots) = |t|^{-\gamma} F_\chi^\pm\left(\frac{h}{t^\Delta}, K_3 t^{-y_3/y_t}, \dots\right)$$

and even though $y_3/y_t < 0$, the second argument of the scaling function is not actually zero. On the other hand, if K_3 is not a dangerous irrelevant variable which affects the vacuum structure, then $F(x, y)$ will be analytic in y near 0, so we can Taylor expand:

$$\chi_T(t, h, \dots) = |t|^{-\gamma} (A_\pm + B_\pm K_3 |t|^{-y_3/y_t} + \dots).$$

For $h = 0$, A_{\pm}, B_{\pm} are non-universal constants. The term with A gives the leading singularity $|t|^{-\gamma}$, but the term with B , which goes like $|t|^{-\gamma-y_3/y_t}$ can also be singular at $t \rightarrow 0$ and if so must be included in a comparison with experiment or simulation.

Two relevant couplings. If we also keep track of the external field, the singular part of the free energy is

$$f_s(t, h) = |t|^{2-\alpha} F_{\pm} \left(\frac{h}{|t|^{\Delta}} \right), \quad \Delta \equiv y_h/y_t.$$

The \pm label is to allow for the possibility of different scaling functions for $t > 0$ and $t < 0$. $F_{\pm}(0)$, which describes $h = 0$, fixed t , must be a constant so that *e.g.* $c_V \sim \partial_t^2 f_s|_{h=0} \sim |t|^{-\alpha}$. $F_{\pm}(\infty)$ describes the behavior $t \rightarrow 0$ at fixed h , which is constrained by

$$M = -T^{-1} \partial_h f_s \sim t^{2-\alpha-\Delta} F'_{\pm} \left(\frac{h}{|t|^{\Delta}} \right). \quad (7.7)$$

When $h = 0$, we require $M \sim t^{\beta}$, so we learn that

$$\beta = 2 - \alpha - \Delta = 2 - \alpha - y_t/y_h. \quad (7.8)$$

When $t = 0$, we must have $M \sim h^{1/\delta}$. If $F'_{\pm}(x) \stackrel{x \rightarrow \infty}{\sim} x^{\lambda}$ then (7.7) is

$$M \sim h^{\lambda} |t|^{2-\alpha-\Delta-\Delta\lambda} \stackrel{(7.8)}{=} h^{\lambda} |t|^{\beta-\Delta\lambda}.$$

which requires both $\beta = \Delta\lambda$ and $\lambda = \frac{1}{\delta}$.

[End of Lecture 15]

Scaling for the correlation function. A similar scaling argument can be made for the spin-spin correlation function, $G(r, \{K\}) \equiv \langle m(r)m(0) \rangle_c$. On the one hand, G transforms as

$$G' = G(r/b, \{K'\}) = G(r/b, tb^{y_t}, hb^{y_h} \dots).$$

Notice that the separation between the points is just like another coupling with dimensions of length. On the other hand, I claim that

$$G' = b^{2(d-y_h)} G(r, \{K\}).$$

This follows if we regard G' as the correlation function of the block spins (see Goldenfeld §9.8). The input is: $Z(K') = Z(K)$, $G(r, K) = \delta_{h_r} \delta_{h_0} \ln Z(h)$ (hence the factor of b^{-2y_h} comes from the rescaling of h) and finally the factor of b^{2d} comes from the fact that a block spin contains b^d spins.

If we choose $b = t^{-1/y_t}$ then

$$G(r, t, h \dots) = t^{-2(d-y_h)/y_t} \underbrace{G(rt^{1/y_t}, 1, ht^{-y_h/y_t})}_{\equiv (rt^{1/y_t})^{-2(d-y_h)} F_G(rt^{1/y_t}, ht^{-y_h/y_t})} \quad (7.9)$$

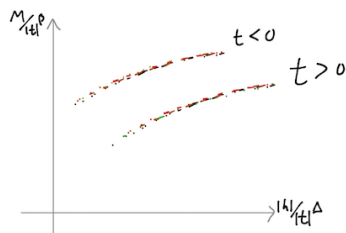
$$r^{-2(d-y_h)} F_G(rt^{1/y_t}, ht^{-y_h/y_t}). \quad (7.10)$$

from which we learn that $2(d - y_h) = d - 2 + \eta$ (as long as $F_G(x, y)$ is smooth as $x, y \rightarrow 0$).

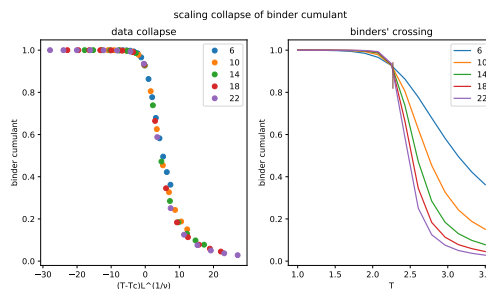
Of all the greek letters we defined $(\alpha, \beta, \gamma, \delta, \nu, \eta, \Delta)$ only *two* combinations are independent – they all depend on the two exponents y_t, y_h associated with the *two* relevant perturbations of the fixed point in question.

Data collapse. The main reason to care about scaling functions is the phenomenon of data collapse. If we plot, say, the magnetization as a function of temperature, for various values of the external field, we'll get a different curve $M(t)$ for each h . On the other hand, (7.7) presents $M(t, h)$ as t^β times a function of the single variable $h/|t|^\Delta$.

This formula is valid for small $|h|, |t|$, but arbitrary h/t . It implies that if we plot $M/|t|^\beta$ as a function of $|h|/|t|^\Delta$, all of the data will lie on two curves, one for $t < 0$ and one for $t > 0$. At right is a cartoon of what this looks like.



Even better, if we plot an observable which has scaling dimension zero, then we don't need to divide by powers of t . An example of such a variable is the *Binder cumulant*, $g = \frac{1}{2} \left(1 - \frac{\langle M^4 \rangle}{3 \langle M^2 \rangle^2} \right)$. Here is some data from some small monte carlo simulations of the 2d ising model the data collapse for the Binder cumulant, as well as the crossing at T_c (see the discussion below on finite-size scaling).



7.1 Crossover phenomena

[Cardy, chapter 4, Goldenfeld §9.9]. A *crossover* refers to a *smooth* change between two behaviors as a parameter is varied, the opposite of a phase transition. There are many reasons why critical behavior might be absent; examples include symmetry breaking, finite volume, disorder.

Suppose we take a critical system with some symmetry and perturb it by a small symmetry-breaking term. For example, consider the Ising fixed point, perturbed by a small magnetic field. There will no longer be a sharp transition between high and low temperature, but what does the critical theory say about the behavior of physical quantities?

Our scaling function expression for

$$f_s(h, t) = |t|^{2-\alpha} F_{\pm} \left(\frac{h}{|t|^{\Delta}} \right)$$

tells us that when $h = 0$ we see the expected critical behavior, *e.g.* $c_v \sim |t|^{-\alpha}$. On the other hand, when $h \sim t^{\Delta}$, something else happens – the dependence on t in the argument of the scaling function matters; $t_X \equiv h^{1/\Delta}$ is called the *crossover temperature*. When $h \gg t^{\Delta}$, instead we see the limit $x \rightarrow \infty$ of $F_{\pm}(x) \stackrel{x \rightarrow \infty}{\sim} x^{1+\frac{1}{\delta}}$, which gives instead $c_V \sim t^{-\alpha-\Delta(1+1/\delta)}$, a completely ‘wrong’ critical exponent:

$$c_V \sim \begin{cases} t^{-\alpha}, & t_X \ll t \ll 1 \\ t^{-\alpha-\Delta(1+1/\delta)}, & t \ll t_X \end{cases} .$$

So watch out for residual magnetic field when trying to measure critical exponents.

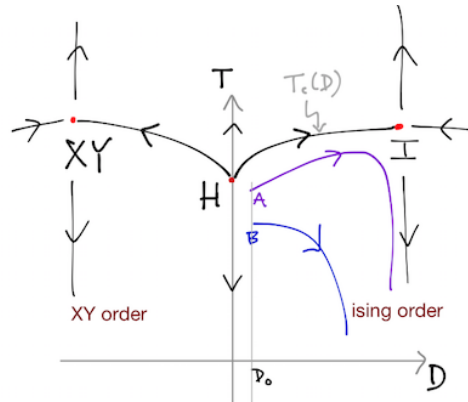
$\mathbf{O}(3) \rightarrow \mathbf{O}(2) \times \mathbb{Z}_2$. [Here I am really just re-typing Cardy §4.2] Above we considered a \mathbb{Z}_2 -symmetric fixed point, perturbed by something which broke all the symmetry. Consider instead an $\mathbf{O}(3)$ -symmetric fixed point which breaks the symmetry down to $\mathbf{O}(2) \times \mathbb{Z}_2$. A definite lattice model is provided by 3-component rotors $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ with $\vec{S}_i \cdot \vec{S}_i = 1$ on a lattice in $2 < d < 4$, with

$$-H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j + \sum_i D (S_i^z)^2 .$$

The $\mathbf{O}(3)$ -symmetric fixed point is named after Heisenberg. The Landau theory looks like

$$\mathcal{L} = \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} t \phi^a \phi^a + u (\phi^a \phi^a)^2 + D \phi_z^2 .$$

For large $D > 0$, the perturbation encourages large $S_z = \pm 1$, and we can just forget $S^{x,y}$, it’s an Ising model. For large $D < 0$, $|S^z| \ll 1$, and we forget S_z , it’s just an XY model – an $\mathbf{O}(n)$ model with $n = 2$. For finite D , we don’t know a priori, but we can connect the dots of the phase diagram as at right.



Near $D = 0, T = T_c^H$, let $t = \frac{T - T_c^H}{T_c^H}$, and we can write a scaling function using the scaling near the Heisenberg fixed point

$$f_s(t, D) = b^{-nd} f_s \left(t b^{ny_t^H}, D b^{ny_D} \right) = |t|^{2-\alpha_H} \Psi \left(D |t|^{-y_D/y_t^H} \right) \quad (7.11)$$

where y_t^H is the dimension of t as a perturbation of the Heisenberg fixed point, H , and $y_D > 0$ is the dimension of D , which is relevant at that fixed point. In the second expression we did the familiar step of choosing $tb^{ny_t^H} = \text{some order-one number}$. $\phi \equiv y_D/y_t^H$ is called the *crossover exponent*; notice that it is determined by data of the Heisenberg (UV) fixed point.

So at $D = 0$, $c_V \sim \partial_t^2 f_s \sim |t|^{-\alpha_H}$ where α_H is the specific heat exponent at H . The crossover happens when $D_0|t|^{-\phi} \sim 1$, *i.e.* $|t| \sim t_X \equiv D^{1/\phi}$, the crossover temperature. What happens for larger D_0 ?

Here's a piece of physics input: for $|t| \ll t_X$, we should see Ising behavior, $c_V \sim |t|^{-\alpha_I}$. This is because the trajectory (A in the figure above) will spend a long time near the Ising fixed point. This input constrains the scaling function Ψ in (7.11):

$$c_V \sim |t|^{-\alpha_H} \Psi(D|t|^{-\phi}) = D^{-\alpha_H/\phi} (D|t|^{-\phi})^{\alpha_H/\phi} \Psi(D|t|^{-\phi}) \equiv D^{-\alpha_H/\phi} \tilde{\Psi}(tD^{-1/\phi})$$

where $\tilde{\Psi}$ is another scaling function. Demanding the Ising singularity at $t_c(D)$, we have

$$c_V \stackrel{!}{\sim} A(D) (t - t_c(D))^{-\alpha_I}$$

which requires

$$\tilde{\Psi}(tD^{-1/\phi}) \sim a (tD^{-1/\phi} - b)^{-\alpha_I}, \quad (a, b \text{ constants})$$

which in turn implies

$$c_V \sim aD^{(\alpha_I - \alpha_H)/\phi} (t - bD^{1/\phi})^{-\alpha_I}.$$

This tells us two interesting things about measurable quantities: (1) the peak in c_V across the ising transition varies with D as $c_V \sim D^{(\alpha_I - \alpha_H)/\phi}$ (grows, since $\alpha_I > \alpha_H$). And (2), $t_c \sim D^{1/\phi}$ is the shape of the phase boundary; notice that this is determined completely by properties of the UV fixed point H . Since it turns out $\phi < 1$, it is shaped like a cusp, as in the cartoon above.

7.2 Finite-size scaling

[Cardy §4.4, Goldenfeld §9.11] Another important example of a crossover phenomenon comes from departing from the thermodynamic limit. Simulations happen in finite volume, say $V = L^d < \infty$. This means that there are no sharp phase transitions. Nevertheless, we can use the RG to predict what the physics will look like if we simulate a system in a critical regime of its parameters. The prediction is in terms of the infinite-system critical temperature and the critical exponents, and therefore can be used to extract this information from simulations.

The system size L enters through the combination $L = Na$ where N is the number of sites on a side. An RG step replaces $a \rightarrow ba$, holding L fixed. Therefore $N \rightarrow N/b$. Assuming that N does not appear explicitly in the RG map (this is violated by long-range interactions), we can add the parameter N to the list of arguments of the singular free energy density:

$$f_s(t, h, K_3, \dots, N^{-1}) = b^{-d} f_s(tb^{yt}, hb^{yh}, K_3b^{y_3}, \dots, N^{-1}b),$$

i.e. N^{-1} acts just like a relevant parameter with dimension $y_N = +1$. Goldenfeld gives a nice definition: a relevant parameter is one that an experimenter has to adjust to reach the critical point; the inverse system size N^{-1} certainly must be adjusted to zero to have any critical behavior.

Then, for simplicity at $h = 0$, we can write a scaling function by the hopefully-standard-by-now trick of running b to an appropriate value:

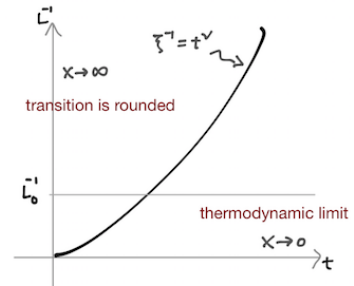
$$f_s(t, N^{-1}) = |t|^{2-\alpha} F_{\pm}(N^{-1}|t|^{-y_N/y_t}).$$

The argument of the scaling function is

$$N^{-1}|t|^{-y_N/y_t} = N^{-1}|t|^{-1/y_t} = N^{-1}|t|^{-\nu} = N^{-1}\xi_{\infty}$$

where ξ_{∞} is the would-be correlation length at $L = \infty$ for the given couplings.

Having done the RG business, I can write $L = Na$ instead of N , where a is the microscopic lattice spacing. If $L \gg \xi$, we are in the thermodynamic limit. Around $|t| = t_X \sim L^{-1/\nu}$, the crossover temperature, something (gradually!) happens, and below this temperature, the answer depends on the finite-size geometry and boundary conditions. For a finite system, the dependence on t is analytic.



To understand this regime, let's rewrite the scaling function so that the analytic dependence on t is manifest:

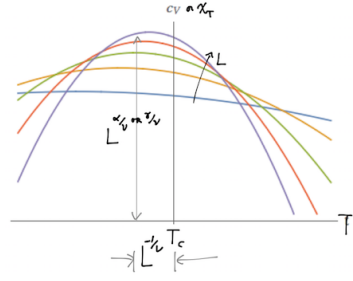
$$f_s(t, L^{-1}) = |t|^{2-\alpha} F(L^{-1}|t|^{-\nu}) \quad (7.12)$$

$$= |t|^{2-\alpha} (L^{-1}|t|^{-\nu})^{\frac{2-\alpha}{\nu}} \tilde{F}(tL^{1/\nu}) = L^{\frac{\alpha-2}{\nu}} \tilde{F}(tL^{1/\nu}) \quad (7.13)$$

where \tilde{F} is a new scaling function. The fact that N is finite means that $\tilde{F}(x)$ is analytic in its argument. This leads to powerful conclusions. In particular, it enormously constrains the functions which blow up at a critical point, such as the susceptibility $\chi_T(t, L^{-1}) \sim \partial_h^2 f_s \sim L^{\gamma/\nu} \psi(tL^{1/\nu})$, or the correlation length ξ , or the specific heat

$$c_V(t, L^{-1}) \sim \partial_t^2 f_s \sim L^{\alpha/\nu} \tilde{F}''(tL^{1/\nu}).$$

Instead of a divergence, each of these functions will have a maximum at some value of t , determined entirely by the scaling function \tilde{F}'' – say x_0 is the location of the maximum of $\tilde{F}''(x)$. This means that the location of the peak in t is $t_0 = x_0/L^{1/\nu} \sim L^{-1/\nu}$ – we know its dependence on system size! Furthermore, the height of the (finite!) peak as a function of L is determined by the prefactor $L^{\alpha/\nu}$.



For example, consider the correlation length itself:

$$\xi(t, N^{-1}) = b\xi(tb^{y_t}, bN^{-1}) \quad (7.14)$$

$$= t^{-\nu} F_\xi(N^{-1}t^\nu) \quad (7.15)$$

$$= t^{-\nu} (Lt^{-\nu}) \tilde{F}(L^{1/\nu}t) \quad (7.16)$$

$$= L\tilde{F}(tL^{1/\nu}) . \quad (7.17)$$

For $L \rightarrow \infty$ at fixed $t \ll 1$, $\xi \sim t^{-\nu}$ requires $\tilde{F}(x) \stackrel{x \rightarrow \infty}{\sim} x^{-\nu}$. For fixed L , $t \rightarrow 0$, we have $\xi \sim L$, and $\tilde{F}(x) = A + Bx + \dots$ is analytic near 0. But this implies that

$$\xi(t, L^{-1}) = A + BtL^{1/\nu} + \dots$$

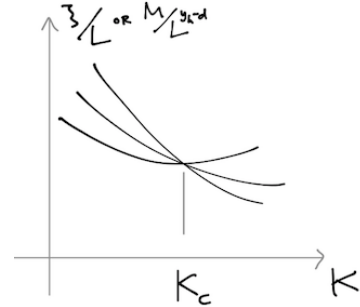
so that at the critical value of the couplings, K , for any L , this is $\xi(0, L^{-1}) = A$. If we plot ξ/L as a function of K for various L , all the curves will cross at the critical coupling.

A similar argument applies to the magnetization (which is easier to measure)

$$M(t, h = 0, L^{-1}) = t^{\frac{d-y_h}{y_t}} F_M(t^{-\nu} L^{-1}) = L^{-d+y_h} \tilde{F}_M(tL^{y_t})$$

where $\tilde{F}_M(y) \stackrel{y \rightarrow 0}{\sim} A + By + \dots$ is analytic near 0 since this is the finite-size limit. Thus

$$\frac{M(t, L^{-1})}{L^{y_h-d}} = A + BtL^{y_t} + \dots$$



– the curves will cross at the critical coupling $K = K_c$ (*i.e.* $t = 0$).

This allows us to determine the critical value of the coupling, and y_h . (Probably, keeping track of the leading irrelevant operator is a good idea.) We can then determine $\nu = 1/y_t$ as well by

$$\partial_K \left(\frac{M}{L^{y_h-d}} \right) \sim BL^{y_t}$$

i.e. $\log(LHS) = \log B + \frac{1}{\nu} \log L$.

- The finite-size scaling analysis (up to (7.13)) applies equally well if the system geometry is $L \times L \times \infty$ or $L \times \infty \times \infty$. The difference is that the scaling function F'' will be different. In the former case, the system is effectively one-dimensional, and F'' is still smooth. It can be determined by a transfer matrix calculation. In the latter case, F'' is determined by the critical behavior of an auxiliary 2d system. Cardy has a bit more detail on this point.
- This account of scaling and scaling functions is completely ahistorical. The idea of data collapse was known and used long before it was justified by the RG in the way I've described.

[End of Lecture 16]

8 The operator product expansion and conformal perturbation theory

[Cardy, chapter 5] Some of the information in the beta functions depends on our choice of renormalization scheme and on our choice of regulator. Some of it does not: for example, the topology of the fixed points, and the critical exponents associated with them. Next we discuss a point of view which makes clear some of the data in the beta functions is also universal. It also gives a more general perspective on the epsilon expansion and why it works. And it leads to the modern viewpoint on conformal field theory.

Operator product expansion (OPE). Suppose we want to understand a correlation function of local operators like

$$\langle \phi_i(x_1) \phi_j(x_2) \Phi \rangle$$

where $\{\Phi\}$ is a collection of other local operators at locations $\{x_l\}$; suppose that the two operators we've picked out are closer to each other than to any of the others:

$$|x_1 - x_2| \ll |x_{1,2} - x_l|, \forall l.$$

Then from the point of view of the collection Φ , $\phi_i \phi_j$ looks like a single local operator. But which one? Well, it looks like some sum over all of them:

$$\langle \phi_i(x_1) \phi_j(x_2) \Phi \rangle = \sum_k C_{ijk}(x_1 - x_2) \langle \phi_k(x_1) \Phi \rangle$$

where $\{\phi_k\}$ is some basis of local operators. By Taylor expanding we can move all the space-dependence of the operators to one point, *e.g.*:

$$\phi(x_2) = e^{(x_2-x_1)^\mu \frac{\partial}{\partial x_1^\mu}} \phi(x_1) = \phi(x_1) + (x_2 - x_1)^\mu \partial_\mu \phi(x_1) + \dots$$

A shorthand for this collection of statements (for any Φ) is the OPE

$$\phi_i(x_1)\phi_j(x_2) \sim \sum_k C_{ijk}(x_1 - x_2)\phi_k(x_1) \quad (8.1)$$

which is to be understood as an operator equation: true for all states, but only up to collisions with other operator insertions (hence the \sim rather than $=$).

This is an attractive concept, but is useless unless we can find a good basis of local operators. At a fixed point of the RG, it becomes much more useful, because of scale invariance. This means that we can organize our operators according to their scaling dimension. Roughly it means two wonderful simplifications:

- We can find a special basis of operators $\{\mathcal{O}_i\}$ where

$$\langle \phi_i(x)\phi_j(0) \rangle_\star = \frac{\delta_{ij}}{r^{2\Delta_i}} \quad (8.2)$$

(here, for the simple case of scalar operators) where Δ_i is the *scaling dimension* of ϕ_i . The \star indicates that this correlator is evaluated at the fixed point. (8.2) defines the multiplicative normalizations of the ϕ_k . This basis is the same as the operators multiplying eigenvectors of the scaling matrix R_\star in (1.6), and the Δ_k are related to the eigenvalues (by $y_k = d - \Delta_k$).

Given (8.2), we can order the contributions to \sum_k in the OPE (8.1) by increasing Δ_k , which means smaller contributions to $\langle \phi\phi\Phi \rangle$.

- Further, the form of C_{ijk} is fixed up to a number. Again for scalar operators,

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \sim \sum_k \frac{c_{ijk}}{|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k}} \mathcal{O}_k(x_1) \quad (8.3)$$

where c_{ijk} is now a set of pure numbers, the *OPE coefficients* (or *structure constants*).

The structure constants are universal data about the fixed point: they transcend perturbation theory. How do I know this? Because they can be computed from correlation functions of scaling operators *at the fixed point*: multiply the BHS of (8.3) by $\mathcal{O}_k(x_3)$ and take the expectation value at the fixed point:

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3) \rangle_\star \stackrel{(8.3)}{=} \sum_{k'} \frac{c_{ijk'}}{|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_{k'}}} \langle \mathcal{O}_{k'}(x_1)\mathcal{O}_k(x_3) \rangle_\star$$

$$\stackrel{(8.2)}{=} \frac{c_{ijk}}{|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k}} \frac{1}{|x_1 - x_3|^{2\Delta_k}} \quad (8.4)$$

(There is a better way to organize the RHS here, but let me not worry about that here.) The point here is that by evaluating the LHS at the fixed point, with some known positions $x_{1,2,3}$, we can extract c_{ijk} .

Confession: I (and Cardy) have used a tiny little extra assumption of *conformal invariance* to help constrain the situation here. It is difficult to have scale invariance without conformal invariance, so this is not a big loss of generality. We can say more about this later but for now it is a distraction.

Conformal perturbation theory. Suppose we find a fixed point of the RG, H_* . (For example, it could be the gaussian fixed point of N scalar fields.) Let us study its neighborhood. (For example, we could seek out the nearby interacting Wilson-Fisher fixed point in $D < 4$ in this way.) For definiteness and simplicity let's think about the equilibrium partition function

$$Z = \text{tre}^{-H}$$

– we set the temperature equal to 1 and include it in the couplings, so H is dimensionless. We can parametrize it as

$$H = H_* + \sum_x \sum_i g_i a^{\Delta_i} \mathcal{O}_i(x) \quad (8.5)$$

where a is the short distance cutoff (*e.g.* the lattice spacing), and \mathcal{O}_i has dimensions of length $^{-\Delta_i}$ as you can check from (8.2). So g_i are de-dimensionalized couplings which we will treat as small and expand in³².

Then

$$\begin{aligned} Z &= \underbrace{Z_*}_{\equiv \text{tre}^{-H_*}} \left\langle e^{-\sum_x \sum_i g_i a^{\Delta_i} \mathcal{O}_i(x)} \right\rangle_* \\ &\stackrel{\sum_x \approx \frac{1}{a^d} \int d^d r}{\simeq} Z_* \left(1 - \sum_i g_i \int \langle \mathcal{O}_i(x) \rangle_* \frac{d^d x}{a^{d-\Delta_i}} \right. \\ &\quad + \frac{1}{2} \sum_{ij} g_i g_j \int \frac{d^d x_1 d^d x_2}{a^{2d-\Delta_i-\Delta_j}} \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle_* \\ &\quad \left. - \frac{1}{3!} \sum_{ijk} g_i g_j g_k \int \int \int \frac{\prod_{a=1}^3 d^d x_a}{a^{3d-\Delta_i-\Delta_j-\Delta_k}} \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \rangle_* + \dots \right). \end{aligned}$$

Comments:

³²Don't be put off by the word 'conformal' in the name 'conformal perturbation theory' – it just means doing perturbation theory about a general fixed point, not necessarily the gaussian one.

- We used the fact that near the fixed point, the correlation length is much larger than the lattice spacing to replace $\sum_x \simeq \frac{1}{a^d} \int d^d x$.
- There is still a UV cutoff on all the integrals – the operators can't get within a lattice spacing of each other: $|x_i - x_j| > a$.
- The integrals over space are also IR divergent; we cut this off by putting the whole story in a big box of size L . This is a physical size which should be RG-independent.
- The structure of this expansion does *not* require the initial fixed point to be a free fixed point; it merely requires us to be able to say something about the correlation functions. As we will see, the OPE structure constants c_{ijk} are quite enough to learn something.

Now let's do the RG dance. We'll take the high-energy point of view here: while preserving Z , we make an infinitesimal change of the cutoff,

$$a \rightarrow ba = (1 + \delta\ell)a, \quad 0 < \delta\ell \ll 1 .$$

The price for preserving Z is letting the couplings run $g_i = g_i(b)$. Where does a appear?

- (1) in the integration measure factors $a^{d-\Delta_i}$.
- (2) in the cutoffs on $\int dx_1 dx_2$ which enforce $|x_1 - x_2| > a$.
- (3) *not* in the IR cutoff – L is fixed during the RG transformation, independent of b .

The leading-in- $\delta\ell$ effects of (1) and (2) are additive and so may be considered separately:

$$(1) \quad \tilde{g}_i = (1 + \delta\ell)^{d-\Delta_i} g_i \simeq g_i + (d - \Delta_i)g_i\delta\ell \equiv g_i + \delta_1 g_i$$

The effect of (2) first appears in the $\mathcal{O}(g^2)$ term, the *change* in which is

$$(2) \quad \sum_{i,j} g_i g_j \int_{|x_1-x_2| \in (a(1+\delta\ell), a)} \int \frac{d^d x_1 d^d x_2}{a^{2d-\Delta_i-\Delta_j}} \underbrace{\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle}_* \\ = \sum_k c_{ijk} \int_{|x_1-x_2| \in (\Delta_k - \Delta_i - \Delta_j)} \langle \mathcal{O}_k \rangle_*$$

$$= \delta\ell \sum_{ijk} g_i g_j c_{ijk} \Omega_{d-1} a^{-2d+\Delta_k} \int \langle \mathcal{O}_k \rangle_*$$

So this correction can be absorbed by a change in g_k according to

$$\delta_2 g_k = -\delta\ell \frac{1}{2} \Omega_{d-1} \sum_{ij} c_{ijk} g_i g_j + \mathcal{O}(g^3)$$

where the $\mathcal{O}(g^3)$ term comes from triple collisions which we haven't considered here.

Therefore we arrive at the following expression for evolution of couplings: $\frac{dg}{d\ell} = (\delta_1 g + \delta_2 g) / \delta\ell$

$$\frac{dg_k}{d\ell} = (d - \Delta_k)g_k - \frac{1}{2} \Omega_d \sum_{ij} c_{ijk} g_i g_j + \mathcal{O}(g^3) . \quad (8.6)$$

³³ At $g = 0$, the linearized solution is $dg_k/g_k = (d - \Delta_k)d\ell \implies g_k \sim e^{(d-\Delta_k)\ell}$ which translates our understanding of relevant and irrelevant at the initial fixed point in terms of the scaling dimensions Δ_k : g_k is relevant if $\Delta_k < d$.

(8.6) says that to find the interaction bit of the beta function for g_k , we look at all the OPEs between operators in the perturbed hamiltonian (8.5) which produce g_k on the RHS.

Let's reconsider the Ising model from this point of view:

$$\begin{aligned}
H &= -\frac{1}{2} \sum_{x,x'} J(x-x') S(x) S(x') - h \sum_x S(x) \\
&\simeq -\frac{1}{2} \sum_{x,x'} J(x-x') S(x) S(x') - h \sum_x S(x) + \lambda \sum_x (S(x)^2 - 1)^2 \\
&\simeq \int d^d x \left(\frac{1}{2} (\vec{\nabla} \phi)^2 + r_0 a^{-2} \phi^2 + u_0 a^{d-4} \phi^4 + h a^{-1-d/2} \phi \right) \tag{8.7}
\end{aligned}$$

In the first step I wrote a lattice model of spins $S = \pm 1$; in the second step I used the freedom imparted by universality to relax the $S = \pm 1$ constraint, and replace it with a potential which merely discourages other values of S ; in the final step we took a continuum limit.

In (8.7) I've temporarily included a Zeeman-field term hS which breaks the $\phi \rightarrow -\phi$ symmetry. Setting it to zero it stays zero (*i.e.* it will not be generated by the RG) because of the symmetry. This situation is called *technically natural*.

Now, consider for example as our starting fixed point the Gaussian fixed point, with

$$H_{\star,0} = \int d^d x \frac{1}{2} (\vec{\nabla} \phi)^2 .$$

Since this is quadratic in ϕ , all the correlation functions (and hence the OPEs, which we'll write below) are determined by Wick contractions using

$$\langle \phi(x_1) \phi(x_2) \rangle_{\star,0} = \frac{\mathcal{N}}{|x_1 - x_2|^{d-2}} .$$

³³ To make the preceding discussion we considered the partition function Z . If you look carefully you will see that in fact it was not really necessary to take the expectation values $\langle \rangle_{\star}$ to obtain the result (8.6). Because the OPE is an operator equation, we can just consider the running of the operator e^{-H} and the calculation is identical. A reason you might consider doing this instead is that expectation values of scaling operators on the plane actually vanish $\langle \mathcal{O}_i(x) \rangle_{\star} = 0$. However, if we consider the partition function in finite volume (say on a torus of side length L), then the expectation values of scaling operators are not zero. You can check these statements explicitly for the normal-ordered operators at the gaussian fixed point introduced below. Thanks to Sridip Pal for bringing these issues to my attention.

It is convenient to rescale the couplings of the perturbing operators by $g_i \rightarrow \frac{2}{\Omega_{d-1}} g_i$ to remove the annoying $\Omega_{d-1}/2$ factor from the beta function equation. Then the RG equations (8.6) say

$$\begin{cases} \frac{dh}{d\ell} = (1 + d/2)h - \sum_{ij} c_{ijh} g_i g_j \\ \frac{dr_0}{d\ell} = 2r_0 - \sum_{ij} c_{ijr_0} g_i g_j \\ \frac{du_0}{d\ell} = \epsilon u_0 - \sum_{ij} c_{iju_0} g_i g_j \end{cases}$$

So we just need to know a few numbers, which we can compute by doing Wick contractions with free fields.

Algebra of scaling operators at the Gaussian fixed point. It is convenient to choose a basis of *normal-ordered* operators, which are defined by subtracting out their self-contractions. That is

$$\mathcal{O}_n \equiv: \phi^n := \phi^n - (\text{self-contractions})$$

so that $\langle : \phi^n : \rangle = 0$, and specifically³⁴

$$\mathcal{O}_2 = \phi^2 - \langle \phi^2 \rangle, \quad \mathcal{O}_4 = \phi^4 - 6 \langle \phi^2 \rangle \phi^2 + \langle \phi^4 \rangle. \quad (8.10)$$

This amounts to a shift in couplings $r_0 \rightarrow r_0 + 3u \langle \phi^2 \rangle_*$. The benefit of this choice of basis is that we can ignore any diagram where an operator is contracted with itself. Note that the contractions $\langle \phi^2 \rangle$ discussed here are defined on the plane. They are in fact quite UV sensitive and require some short-distance cutoff.

³⁴The coefficients in (8.10) disagree with Cardy's book. Here's where these numbers come from. The self-contractions are annoying both because they are more terms, and also because they are infinite. We want to define the \mathcal{O}_n so that they are both orthonormal and finite. When I write $\langle \phi^2 \rangle$, you can imagine that I am separating the locations of the two operators by some cutoff ϵ , so $\langle \phi^2 \rangle = \langle \phi(\epsilon)\phi(0) \rangle = \epsilon^{2-d}$; the goal is to subtract off all the bits which are singular as $\epsilon \rightarrow 0$, and then take the limit.

We can do this inductively. In particular, \mathcal{O}_n is orthogonal to the identity operator $\mathcal{O}_0 = \mathbb{1}$ says $\langle \mathcal{O}_n \rangle = 0$. This fixes $\mathcal{O}_2 = \phi^2 - \langle \phi^2 \rangle$. To save writing let $G_0 \equiv \langle \phi^2(x) \rangle$. Now let

$$\mathcal{O}_4 = \phi^4 + a\phi^2 \langle \phi^2 \rangle + b \langle \phi^4 \rangle = \phi^4 + a\phi^2 G_0 + 3bG_0^2.$$

First we demand

$$0 \stackrel{!}{=} \langle \mathcal{O}_4 \rangle = 3G_0^2 + aG_0^2 + 3bG_0^2$$

which requires $0 = 3 + a + 3b$. The next demand is that

$$0 \stackrel{!}{=} \langle \mathcal{O}_4(x)\mathcal{O}_2(0) \rangle = \langle \phi^4(x)\phi^2(0) \rangle - \langle \phi^4 \rangle G_2 + a \langle \langle \phi^2(x)G_0\phi^2(0) \rangle - \langle \phi^2 \rangle G_2^2 \rangle + 3bG_0^2 \langle \mathcal{O}_2(0) \rangle \quad (8.8)$$

$$= 3G_0^3 + 12G_x G_0 - 3G_0^3 + a(2G_x G_0 + G_0^3 - G_0^3) = G_x G_0(12 + 2a) + G_0^3. \quad (8.9)$$

which requires $a = -6$, and hence $b = +1$. Notice, however, that this changes nothing about the operational definition (omit self-contractions). Thanks to Aria Yom for questioning the expression in (8.10).

To compute their OPEs, we consider a correlator of the form above:

$$\langle \mathcal{O}_n(x_1) \mathcal{O}_m(x_2) \Phi \rangle \quad \begin{array}{c} \phi_2(x_1) \text{ (circle with two dots)} \\ \phi_2(x_2) \text{ (circle with two dots)} \end{array} \quad \Phi \text{ (circle with a vertical line through it)}$$

We do wick contractions with the free propagator, but the form of the propagator doesn't matter for the beta function, only the combinatorial factors.

$$\begin{array}{c} \phi_2(x_1) \text{ (circle with two dots)} \\ \phi_2(x_2) \text{ (circle with two dots)} \end{array} \xrightarrow{\text{wick}} \mathbb{1} \cdot 2$$

If we can contract all the operators making up \mathcal{O}_n with those of \mathcal{O}_m , then what's left looks like the identity operator to Φ ; that's the leading term, if it's there, since the identity has dimension 0, the lowest possible. More generally, some number of ϕ s will be left over and will need to be contracted with bits of Φ to get a nonzero correlation function.

$$\begin{array}{c} \phi_2(x_1) \text{ (circle with two dots)} \\ \phi_2(x_2) \text{ (circle with two dots)} \end{array} \xrightarrow{\text{wick}} \phi_2 \cdot \text{f}$$

For example, the contributions to $\mathcal{O}_2 \cdot \mathcal{O}_2$ are depicted at right. In determining the combinatoric factors, note that permuting the legs on the right does not change anything, they are identical.

$$\begin{array}{c} \phi_2(x_1) \text{ (circle with two dots)} \\ \phi_2(x_2) \text{ (circle with two dots)} \end{array} \xrightarrow{\text{wick}} \phi_4$$

The part of the result we'll need (if we set $h = 0$) can be written as (omitting the implied factors of $|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k}$ necessary to restore dimensions):

$$\begin{cases} \mathcal{O}_2 \mathcal{O}_2 & \sim 2\mathbb{1} + 4\mathcal{O}_2 + \mathcal{O}_4 + \dots \\ \mathcal{O}_2 \mathcal{O}_4 & \sim 12\mathcal{O}_2 + 8\mathcal{O}_4 + \dots \\ \mathcal{O}_4 \mathcal{O}_4 & \sim 24\mathbb{1} + 96\mathcal{O}_2 + 72\mathcal{O}_4 + \dots \end{cases}$$

Notice that the *symmetric* operators (the ones we might add to the action preserving the symmetry) form a closed subalgebra of the operator algebra.

At $h = 0$, the result is (the $N = 1$ case of the result in §6.6)

$$\begin{cases} \frac{dr_0}{d\ell} = 2r_0 - 4r_0^2 - 2 \cdot 12r_0 u_0 - 96u_0^2 \\ \frac{du_0}{d\ell} = \epsilon u_0 - r_0^2 - 2 \cdot 8r_0 u_0 - 72u_0^2 \end{cases}$$

and so the ($N = 1$) WF fixed point occurs at $u_0 = u_0^* = \epsilon/72, r_0 = \mathcal{O}(\epsilon^2)$.

The difference numerical numbers in the values of the fixed point couplings come from our different parametrization (recall that we shifted the definition of r when we switched to a basis of normal-ordered operators in (8.10)) – that is not universal information. We can extract something universal and independent of our choices as

follows. Linearizing the RG flow about the *new* fixed point,

$$\frac{dr_0}{d\ell} = 2r_0 - 24u_0^*r_0 + \dots$$

gives

$$\frac{dr_0}{r_0} = \left(2 - \frac{24}{72}\epsilon\right)d\ell \quad \implies \quad r_0 \sim e^{(2 - \frac{24}{72}\epsilon)\ell} \equiv (e^\ell)^{\frac{1}{\nu}}$$

which gives $\nu = \frac{1}{2} + \frac{1}{12}\epsilon + \mathcal{O}(\epsilon^2)$.

[End of Lecture 17]

9 Lower dimensions and continuous symmetries

[Cardy §6, Goldenfeld §11]

9.1 Lower critical dimension

Mean field theory gets better as the number of dimensions grows, so naturally it gets worse when the number of dimensions shrinks. For low enough d , the fluctuations completely destroy the order at any finite temperature. For the Ising model, this *lower critical dimension* is $d = 1$; that is, T_c is zero for an Ising chain. Recall our (Peierls') understanding of this: if we fix the spins to be up at one end of the chain, then the free energy cost for making a region of down spins is

$$\Delta F_1 = E - TS \simeq 4J - 2T \ln L$$

where L is the system size – the domain walls can be in any of L places. For any $T > 0$, for large enough L , ΔF_1 is negative (hence favorable). In contrast, in $d = 2$, the energy of the domain wall is $2JL^{d-1} = 2JL$, while the entropy is of order $\log \mu^L$ (the domain wall is a self-avoiding but closed random walk of length of order L ; at each of $\sim L$ steps it has of order $\mu = z - 1$ choices of direction to go), so

$$\Delta F_2 \sim L(2J - T \ln \mu)$$

which, for small enough T , is positive for all L .

In contrast with this case of spontaneous breaking of a discrete symmetry, the domain walls of a *continuous* symmetry are floppier, and this raises the lower critical dimension. If we fix the order parameter to have one value at one end of the system and some other value at the other end (a distance L away), the transition from one to the other can be made gradually, so that the order parameter gradient is $\vec{\nabla}\phi \sim \frac{1}{L}$; the energy density is $(\vec{\nabla}\phi)^2 \sim \frac{1}{L^2}$ and the energy is $\int d^d x (\vec{\nabla}\phi)^2 \sim L^{d-2}$, which is much cheaper than JL^{d-1} for an Ising domain wall.

Rigidity of the order parameter. This notion we just defined and estimated – the free energy cost for twisting the boundary conditions of the order parameter for a spontaneously-broken continuous symmetry – is an important and useful one, called the *stiffness* or *rigidity*. The origin of the name is as follows: the Hamiltonian governing a collection of atoms has continuous translation symmetry; in a crystalline solid, this continuous symmetry is spontaneously broken down to a discrete subgroup preserved by the lattice. The discussion here applies to this case as well. Translating *the whole solid* doesn't change the spacing between atoms; it is a symmetry operation and

doesn't change the energy. Translating different parts of the solid by slightly different amounts will therefore cost a small energy, proportional to the gradient of the translation. The excitations of the solid therefore include large-correlation-length modes (Goldstone bosons) $\vec{u}^i(x)$ which appear in the energy only through their derivatives. When experiencing such an elastic deformation, the solid will exert a restoring force, encoded in the energy functional for u by terms like $K \int \partial_i u_j C^{ijkl} \partial_k u_l$ (analogous to $\int K (\vec{\nabla}\theta)^2$). The fact that a solid is rigid is a consequence of spontaneous symmetry breaking, and this concept of rigidity generalizes to other cases of SSB. We'll have more to say about the stiffness of magnets.

Hohenberg-Mermin-Wagner-Coleman Theorem. Consider an $O(n)$ model, with n -component rotors \vec{S}_r at each site, $\vec{S}_r \cdot \vec{S}_r = 1, \forall r$. If the system orders, we can write the spin as $\vec{S}_r = (\sqrt{1 - \sigma_r^2}, \vec{\sigma}_r)$, where $\vec{\sigma}_r$ is an $n - 1$ -component vector pointing *transversely* to the ordering direction, describing the fluctuations about a particular ordered state – we will assume $\sigma^2 \ll 1$. The action for these fluctuations (known as spin waves in the context of magnetism) is

$$S = -\frac{1}{2T} \sum_{rr'} J_{rr'} \sigma_r \sigma_{r'} \simeq \text{const} + \frac{K}{2} \int d^d r \left(\vec{\nabla} \sigma \right)^2 + \mathcal{O} \left(\vec{\nabla} \sigma^4 \right). \quad (9.1)$$

As we observed before, there is no ‘mass term’ $\propto \sigma^2$. This is because σ parametrizes the orbit of the chosen magnetization by the broken continuous symmetry, which must all have the same energy. The only way it can cost energy to change the magnetization is if it varies in space; therefore the action can depend on σ only through its gradient $\vec{\nabla} \sigma$. Therefore the correlation length is infinite everywhere in the ordered phase, and we are justified in using the continuum approximation in (9.1). The parameter $K \sim \frac{J}{T}$ is the spin stiffness.

The fluctuations of σ go like

$$\langle \sigma_a(r) \sigma_b(0) \rangle = \frac{1}{Z} \int [D\sigma] e^{-S[\sigma]} \sigma_a(r) \sigma_b(0) \stackrel{\text{gaussian integral}}{=} \frac{\delta_{ab}}{K} \oint_{BZ} d^d k \frac{e^{ikr}}{k^2}.$$

We care about this because it corrects the expectation value of the spin:

$$\langle S_1(r) \rangle = 1 - \frac{1}{2} \langle \sigma(r)^2 \rangle + \dots \quad (9.2)$$

$$= 1 - \frac{n-1}{K} \oint_{BZ} \frac{d^d k}{k^2} + \dots \quad (9.3)$$

And now here's the crucial point: in $d = 2$, this fluctuation correction to the magnetization goes like $\int \frac{d^2 k}{k^2} \sim \ln L\Lambda$. It diverges with system size, which clearly means it's not a small correction to the leading term. (Notice that the form of the integrand is not exactly correct far from $k = 0$ in the Brillouin zone, but it is the infrared divergence at

$k = 0$ which is the story here.) The singularity from the long-wavelength fluctuations is only worse if $d < 2$. The way out is that our assumption that there was ordering in the first place was wrong in $d \leq 2$. We conclude that it is not possible to spontaneously break a continuous symmetry in $d \leq 2$ ³⁵. A more proofy proof of this statement is on the homework.

9.2 Kosterlitz-Thouless phase and phase transition

Although there is no magnetization for $n \geq 2$ in $d \leq 2$, there can still be a low-temperature phase which is distinct from the high-temperature disordered phase, and separated from it by a continuous phase transition. This is a new kind of phase transition, where the crucial degrees of freedom at the transition are not those of the order parameter, since there *is* no order parameter. To understand this, let's focus on the case $n = 2, d = 2$ which is the most interesting.

Let $\Phi(r) = S_1(r) + \mathbf{i}S_2(r)$ in the rotor description above. $\Phi(r)$ could also be the macroscopic wavefunction of a superfluid, where the broken symmetry is the one associated with particle-number conservation. The action can be parametrized as

$$-S[\Phi] = \int d^2x \left(\frac{1}{2} |\vec{\nabla}\Phi|^2 + \frac{u_0}{4} \left(|\Phi|^2 - \frac{r_0}{u_0} \right)^2 \right) = \frac{K}{2} \int d^2x \left(\vec{\nabla}\theta \right)^2 + \dots$$

In the second step, we focussed on the universal physics by considering u_0 large, with fixed r_0/u_0 . This has the effect of making the longitudinal excitations very costly – the walls of the potential become very steep about the circle of minima. Writing $\Phi(r) = e^{i\theta(r)} \left(\sqrt{\frac{r_0}{u_0}} + \delta(r) \right)$, the longitudinal excitation $\delta(r)$ is very hard to excite and we can forget about it. We defined $K = \sqrt{\frac{r_0}{u_0}}$, but recall that the overall coefficient of the action is J/T , and this is what determines K . The angular variable θ is the Goldstone mode – it only appears in the action via its derivatives.

The spin Green's function is

$$G(r) \equiv \langle \Phi(r)\Phi(0)^\dagger \rangle \tag{9.4}$$

$$\propto \langle e^{i(\theta(r)-\theta(0))} \rangle \tag{9.5}$$

$$= e^{-\frac{1}{2}\langle(\theta(r)-\theta(0))^2\rangle} . \tag{9.6}$$

³⁵Actually there is an interesting exception to this statement, involving orientational order. For a discussion of this exception (as well as the history, an interesting extension, and a generalization of the theorem) see [this new paper by Halperin](#).

where we used Wick's theorem in the last step. This correlation function of the Goldstones is

$$\langle (\theta(r) - \theta(0))^2 \rangle = 2 (\langle \theta(0)^2 \rangle - \langle \theta(r)\theta(0) \rangle) \quad (9.7)$$

$$2 \int_0^{\Lambda=1/a} d^2k \underbrace{\langle |\tilde{\theta}_k|^2 \rangle}_{=\frac{1}{Kk^2}} (1 - e^{ikr}) \stackrel{r \gg a}{\simeq} \frac{1}{2\pi K} \log r/a. \quad (9.8)$$

Therefore

$$G(r) = r^{-\eta}, \quad \eta = \frac{1}{2\pi K} = \frac{T}{2\pi J}. \quad (9.9)$$

At the last step, we restored the factor of $1/T$ in the action. Important comments:

- η is indeed the anomalous dimension of the spin operator, defined as usual for a critical theory by $G(r) \sim r^{2-d-\eta}$. But this is not at a critical point, this behavior occurs everywhere in a whole phase.

On the other hand, at high temperatures, we know that the correlations must be short-ranged (for example, using the (convergent!) high-temperature expansion), $G(r) \stackrel{T \gg J}{\sim} e^{-r/\xi}$. The distinction between these two asymptotic behaviors of $G(r)$ is sharp, and they represent different phases.

The low-temperature phase is consistent with the Mermin-Wagner theorem – there is no disconnected piece of $G = G_{\text{connected}}$. It is called *algebraic order* or *quasi-long-range order*. In between there must be a phase transition of some kind, which we will understand below.

- K is *not* a redundant variable. In much of our previous discussion of scalar field theory, we removed the coefficient $r_2(\vec{\nabla}\phi)^2$ of the kinetic term by redefining $\phi \rightarrow \sqrt{r_2}\phi$. For the goldstone mode θ (and more generally), its normalization is fixed by its periodicity $\theta \equiv \theta + 2\pi$. Changing K really changes the theory.
- And indeed, the exponent η varies with K and hence with temperature! K is an exactly marginal perturbation of a scale-invariant theory– it parametrizes a *line* of (different!) fixed points.

Why isn't (9.9) an exact statement for all temperatures? In our computation of (9.9) we neglected the important fact that $\theta \simeq \theta + 2\pi$, θ is compact. This means that in addition to the smooth configurations which lead to (9.9), there are other, topologically distinct, configurations where as we move around in a loop in space, θ wanders around on the circle, and only returns to itself up to a multiple of 2π . That is, there can be configurations of $\theta(r)$ and loops C for which

$$\oint_C d\vec{r} \cdot \vec{\nabla}\theta = 2\pi n, \quad n \in \mathbb{Z}. \quad (9.10)$$

We say that the loop C encloses n vortices. The presence of a vortex is topological because the winding number n is an integer, which therefore cannot vary continuously.

(9.10) says that $\vec{\nabla}\theta \sim \frac{1}{r}$, from which we can estimate that the energy of a vortex is

$$E_{\text{one vortex}} = \frac{1}{2}K \int d^2r \left(\vec{\nabla}\theta\right)^2 = \pi J \log L/a$$

where L is the system size. Notice that this diverges in the thermodynamic limit: a net number of vortices is not a finite energy configuration. To have finite energy, the largest loops must contain a net number zero of vortices. However smaller regions may contain vortices ($n > 0$) and antivortices ($n < 0$). The energy of a vortex-antivortex pair (a vortex dipole) separated by distance R is

$$E_{v-\bar{v}} \simeq \int_a^R \frac{dr}{r} \sim \log R/a,$$

finite in the thermodynamic limit. We estimated above the energy of a single vortex, but what is its free energy? Its entropy is

$$S_{\text{one vortex}} = \log (\# \text{ of possible locations}) = \log \left(\frac{L}{a}\right)^2$$

so that

$$F_{\text{one vortex}} = E_{\text{one vortex}} - TS_{\text{one vortex}} = (\pi J - 2T) \log L/a \rightarrow \begin{cases} +\infty, & T < \pi J/2, \\ -\infty, & T > \pi J/2 \equiv T_{KT}. \end{cases}$$

This gives us an estimate (it turns out to be exactly correct) for the transition temperature between the low-temperature, algebraically-ordered phase, and the high-temperature disordered phase.

[\[End of Lecture 18\]](#)

KT transition. To give a more quantitative account of the transition, we must explicitly include the vortices in our calculation. To this end we deform our theory of the Goldstone field θ by introducing a *fugacity* for vortices – adding a vortex lowers the energy by y_0 . Formally we can do this by changing the action to

$$S = \int d^2x K \frac{1}{2} \left(\vec{\nabla}\theta\right)^2 - y_0 \int \frac{d^2x}{a^2} (V(x) + V^\dagger(x)).$$

In this expression $V(x)$ is an operator which creates a vortex at position x , and $V^\dagger(x)$ creates an antivortex at x . $V(x)$ is an example of a *disorder operator* – it is defined by its effects on the spins: for example

$$\langle V(x) \cdots \rangle_0 \equiv \int_{\text{configurations of } \theta \text{ with } \oint_{C_x} d\vec{r} \cdot \vec{\nabla}\theta = 2\pi} [D\theta] e^{-S[\theta]} \dots$$

where C_x is any curve containing the point x . (Here $\langle \dots \rangle_0$ denotes an expectation value in the theory with $y_0 = 0$.) By some cleverness (following Cardy), we will figure out what we need without finding an explicit expression for V ; such an expression can be found as part of a duality map (see Cardy §3.3 or Herbut §6.3). Notice that an expectation value with only a single vortex will be $\langle V(x) \dots \rangle_0 \propto e^{-E_1/T} \sim e^{-\pi K \log L/a} \xrightarrow{L \rightarrow \infty} 0$, but expectations with zero total vortex number $\langle V(x)V(0)^* \dots \rangle_0$ will be finite.

Granting this starting point, the partition sum is now a function of two variables:

$$Z(K, y_0) = \left\langle e^{y_0 \int (V+V^\dagger)} \right\rangle_0 \quad (9.11)$$

$$= \left\langle 1 + y_0 \int (V + V^\dagger) + \frac{1}{2} y_0^2 \int V \int V^\dagger + \dots \right\rangle_0 \quad (9.12)$$

$$= 1 + y_0^2 \int \frac{d^2 r_1 d^2 r_2}{a^4} \langle V(r_1) V^\dagger(r_2) \rangle_0 + \dots \quad (9.13)$$

The terms with odd powers of y_0 vanish by the fact that they have a net number of vortices, and therefore $e^{-E} = e^{-\infty} = 0$, zero Boltzmann weight.

In the last line (9.13), the vortex-antivortex correlator $\langle V(r_1) V^\dagger(r_2) \rangle_0 \equiv e^{-E(r_1, r_2)}$ is just the partition function for the spin waves in the presence of a vortex at r_1 and an antivortex at r_2 . We can find the resulting free energy $E(r_1, r_2)$ by saddle point (since the integral over smooth configurations of θ is gaussian), *i.e.* just solve the equations of motion

$$\nabla^2 \theta = 0, \quad (9.14)$$

(away from the vortices) with boundary conditions demanding the appropriate winding number around r_1, r_2 . The solution is

$$\theta(\vec{r}) = \Theta(\vec{r} - \vec{r}_1) - \Theta(\vec{r} - \vec{r}_2), \quad \text{where } \Theta(\vec{r}) \equiv \text{the angle between } \vec{r} \text{ and } \hat{x}.$$

The resulting free energy is³⁶

$$E(r_1, r_2) = 2\pi K \ln \frac{r_{12}}{a} + 2\pi K \tilde{C}.$$

³⁶To get the important and confusing factors of 2π correct, a perhaps more useful expression for θ comes from noticing that the vortices are sources of θ : integrating over a small region R_1 containing the point r_1

$$\int_{R_1} \nabla^2 \theta = \oint_{C_{r_1}} d\vec{r} \cdot \vec{\nabla} \theta \stackrel{!}{=} 2\pi.$$

Therefore, we must have $\nabla^2 \theta = 2\pi (\delta(r - r_1) - \delta(r - r_2))$ whose momentum space solution is

$$\theta(r) = 2\pi \int \frac{d^2 k}{k^2} e^{i\vec{k} \cdot \vec{r}} (e^{i\vec{k} \cdot \vec{r}_1} - e^{i\vec{k} \cdot \vec{r}_2}).$$

Plugging this into

$$E = \frac{1}{2} K \int (\vec{\nabla} \theta)^2 = (2\pi)^2 K \int \frac{d^2 k}{k^2} (1 - e^{i\vec{k} \cdot \vec{r}_{12}})$$

with $r_{12} \equiv |\vec{r}_1 - \vec{r}_2|$. The additive constant, which is associated with the energy in the core of the vortices, we can absorb into a rescaling of y_0 : $y_0 \rightarrow y_0 e^{-\pi K \tilde{C}} \equiv y$. Therefore we conclude that

$$\langle V(r_1) V^\dagger(r_2) \rangle_0 = \left(\frac{r_{12}}{a} \right)^{-2\pi K},$$

– we learn that the scaling dimension of the operator V is $\Delta_V = \pi K$. The scaling behavior of y is then $y(b) = b^{y_V} y(1)$ with $y_V = d - \Delta_V = 2 - \pi K$:

$$\beta_y = \frac{dy}{d\ell} = (2 - \pi K)y \quad (9.15)$$

– y is irrelevant for $x \equiv 2 - \pi K \propto T - T_{KT} < 0$, and relevant for $x > 0$. Here $T_{KT} \equiv \pi J/2$ as in the estimate above.

To complete the RG equations we need to know how the temperature variable x runs. Either T or x is the coupling associated with the ‘energy operator’ $(\partial\theta)^2$. We know a few things a priori: if $y = 0$, it doesn’t run. Only even powers of y can appear since the total number of vortices and antivortices must be zero (hence zero mod 2). Therefore, near $x = 0, y = 0$, the RG equation for x must have the form

$$\beta_x = \frac{dx}{d\ell} = Ay^2 + \mathcal{O}(y^4). \quad (9.16)$$

A fancier argument (in Cardy’s book) uses the OPE: Comparing (9.15) with our general form in terms of OPE coefficients (8.6), we see that the OPE between V and the energy operator has the form $V \cdot (\partial\theta)^2 \sim V + \dots$. A general fact of CFT (Cardy §11.2) says that the OPE coefficients c_{ijk} are completely symmetric, and this means that we must have $V \cdot V^* \sim (\partial\theta)^2 + \dots$. Comparing to (8.6) then implies (9.16).

Equation (9.16) has a nice physical interpretation.

Notice that the equation we solved for the behavior of the θ field in response to the vortices (9.14) is Coulomb’s law in $d = 2$, where θ plays the role of the electrostatic potential. K plays the role of the dielectric constant of the medium. The running of x (equivalently K) in (9.16) is dielectric screening of the Coulomb field θ by the charge-anticharge (vortex-antivortex) pairs.

θ	A_0
(anti)vortex	\pm charge
K	dielectric constant
$2\pi K \log r$	Coulomb potential, $\int \frac{d^d k}{k^2} e^{i\vec{k}\cdot\vec{r}}$
χ_V in (9.17)	polarizability, χ_E

which is the same integral as we saw in (9.8):

$$\int \frac{d^d k}{k^2} (1 - e^{i\vec{k}\cdot\vec{r}}) \stackrel{r \gg a}{\simeq} \frac{1}{2\pi} \ln r/a + \text{cst.}$$

The combined RG equations

$$\frac{dy}{d\ell} = xy, \quad \frac{dx}{d\ell} = Ay^2$$

imply that $\frac{dy}{dx} = \frac{x}{Ay}$ which integrates to $Ay^2 - x^2 = \text{a constant}$ determined by the initial conditions. The flow lines in the xy plane are hyperbolae, except for the special initial condition where the constant is zero, which is the lines $y = \pm x/\sqrt{A}$. The line $y = -x/\sqrt{A}$ is the critical surface of the KT critical point at $x = y = 0$. Any initial condition above this line flows off to the upper right, large x and y .

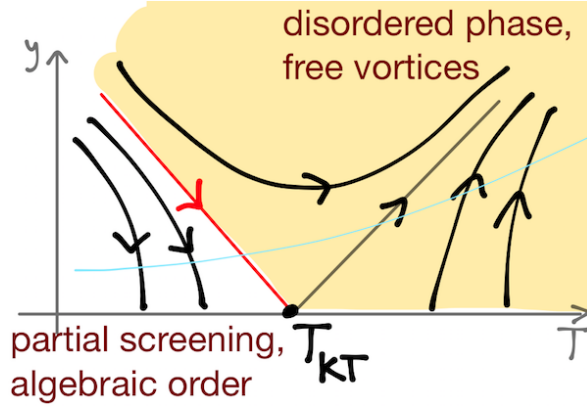


Figure 4: The Kosterlitz-Thouless phase diagram. The red line is the critical surface of the KT fixed point; to its right is the disordered phase, where the flows end up at large x, y . The thin blue line is a cartoon of a family of initial conditions for different values of the temperature, including the fact that the bare value of y goes like $e^{-\pi K \bar{C}} = e^{-\# / T}$, and hence is small for small T .

To understand what happens at large x, y , consider the vortex polarizability (the analog of the polarizability of the dielectric medium, the fluctuations of the average dipole moment):

$$\chi_V \equiv \sum_R R^2 \langle n(\vec{R})n(\vec{0}) \rangle = \frac{\partial^2 f}{\partial E_{\vec{R}} \partial E_{\vec{0}}}, \quad \text{with } H \rightarrow H + \vec{E} \cdot \sum_R \vec{R}n(R) \quad (9.17)$$

where $n(\vec{R})$ is the number of vortices at the site \vec{R} , and therefore $\sum_{\vec{R}} \vec{R}n(R)$ is the vortex dipole moment. The vortex number fluctuations satisfy

$$\langle n(\vec{R})n(\vec{0}) \rangle = 0 + y^2 2e^{-E(\vec{R},0)} + \mathcal{O}(y^4) = y^2 \left(\frac{R}{a} \right)^{-2\pi K} + \mathcal{O}(y^4)$$

where the y^2 term comes from configurations with a single vortex at R and an antivortex at 0 (or vice versa, hence the factor of 2). Therefore

$$\chi_V = y^2 \sum_R R^{2-2\pi K} + \mathcal{O}(y^4) \rightarrow \infty \text{ for } T > T_{KT}.$$

Infinite polarizability means free charges: it means an arbitrarily small electric field \vec{E} moves all the \pm charges to opposite ends of the sample. By the same token, it means an external charge is completely screened beyond the correlation length ξ . Everywhere in the previous sentences ‘charge’ means ‘vortex’.

Let’s return to the spin stiffness, the free energy cost for twisted boundary conditions. More precisely, consider the system on a torus, and consider the boundary conditions $\theta(x, L) = \theta(x, 0), \theta(L, y) = \theta(0, y) + \alpha$. We can relate this to the periodic BC problem by defining

$$\theta(x, y) = \theta_0(x, y) - \alpha x/L$$

where $\theta_0(x, y)$ is periodic. In the gaussian approximation, the free energy density is

$$\frac{K_0}{2} (\partial\theta)^2 = \frac{K_0}{2} \left(\frac{\alpha}{L}\right)^2 + \dots$$

where the \dots comes from the periodic bit, which does not care about α . The spin stiffness is defined to be

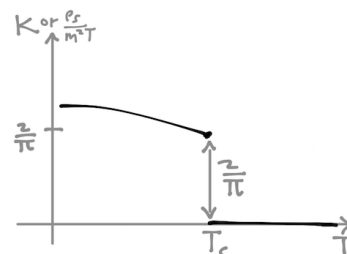
$$\kappa \equiv L^2 \partial_\alpha^2 f \stackrel{\text{gaussian}}{=} K_0.$$

Including the effects of fluctuations and vortices and all that – in the low-temperature phase – the stiffness is $\kappa = K(\ell = \infty)$, the running coupling evaluated in the far infrared. This is because, in the low temperature phase, the vortex fugacity flows to $y(\infty) = 0$, and we return to the gaussian model, with a renormalized coupling $K(\infty)$. Therefore:

$$\begin{cases} T < T_c : & \kappa = K(\infty) \text{ varies with } T \\ T = T_c : & \kappa = K(\infty) = K_c = \frac{2}{\pi}, \text{ a universal value} \\ T > T_c : & \kappa = 0 \end{cases}$$

where we know the answer for $T > T_c$ because the finite correlation length means $\partial_\alpha f \propto e^{-L/\xi}$ – the influence of the boundary conditions is short-ranged, and so the leading bit of the free energy doesn’t care about α .

We conclude that across T_{KT} , the spin stiffness *jumps* by a universal amount $\Delta K = \frac{2}{\pi}$. This has been [measured](#) in superfluid films.



Screening by vortices. Above I may have made the β function equation for x seem mysterious. Actually, it can be directly calculated by considering the renormalization of the stiffness, *i.e.* its screening by vortex-antivortex pairs. Here we go:

[Chaikin-Lubensky §9.4] We'll compute the running of the stiffness parameter K by computing the stiffness in the presence of nonzero y : that is, we compute the free energy in the presence of a uniform gradient of θ :

$$\theta(x) = \theta_0(x) + \vec{\alpha} \cdot \vec{x}$$

where $\vec{\alpha}$ parametrizes the twist around the two directions and θ_0 is periodic. We can further decompose the periodic bit

$$\theta_0(x) \equiv \theta_v(x) + \theta_s(x)$$

into a smooth piece θ_s which satisfies $0 = \epsilon_{ij} \partial_i \partial_j \theta_s$, and a vortex piece, which satisfies $\epsilon_{ij} \partial_i \partial_j \theta_v = 2\pi n_v$ (but $0 = \delta_{ij} \partial_i \partial_j \theta_v$). Then the free energy defines the renormalized stiffness K^R :

$$F(\alpha) - F(0) \equiv \frac{1}{2} L^2 K^R \alpha^2 \tag{9.18}$$

$$= -\ln \text{tr} e^{-H} - F(0) \tag{9.19}$$

$$= \frac{1}{2} L^2 K \alpha^2 - \ln \text{tr} \left(e^{-H(\alpha=0)} e^{-K \int \vec{\alpha} \cdot \vec{\nabla} \theta_0} \right) - F(0) \tag{9.20}$$

$$= \frac{1}{2} L^2 K \alpha^2 - \frac{1}{2} K^2 \int d^2 x \int d^2 x' \langle \partial_i \theta(x) \partial_j \theta(x') \rangle \alpha_i \alpha_j + \mathcal{O}(\alpha^4). \tag{9.21}$$

(Note that we are still working in the convention where $T = 1$.) In the second line we expanded out $S = \frac{1}{2} K \int d^2 x (\partial \theta_0 + \alpha)^2$. Since $\int d^2 x \partial_i \theta_s(x) = 0$ for all configurations, only θ_v contributes to (9.21).

The defining condition of θ_v (that is, $\epsilon_{ij} \partial_i \partial_j \theta_v = 2\pi n_v$) tells us that in momentum space,

$$(\partial_j \theta_v)(q) = \frac{-i \epsilon_{ji} q_i}{q^2} 2\pi n_v(q).$$

Therefore³⁷ :

$$K^R = K - (2\pi K)^2 \int d^2x \langle \partial_i \theta(x) \partial_i \theta(0) \rangle \quad (9.22)$$

$$= K - (2\pi K)^2 \lim_{q \rightarrow 0} \frac{\langle n_v(q) n_v(-q) \rangle}{q^2}. \quad (9.23)$$

Net vortex neutrality implies $\langle n_v(q) \rangle \xrightarrow{q \rightarrow 0} 0$, plus rotation invariance, implies that

$$\langle n_v(q) n_v(-q) \rangle = \frac{1}{2} \chi_v q^2 + \mathcal{O}(q^4)$$

where χ_v is exactly the vortex polarizability defined above. Therefore

$$K_R = K - \frac{1}{2} (2\pi K)^2 y^2 \int \frac{d^2x}{a^2} \left(\frac{x}{a} \right)^{-2\pi K+2} + \mathcal{O}(y^4) \quad (9.24)$$

$$= K - \frac{1}{2} (2\pi K)^2 y^2 2\pi \int_a^\infty \frac{dr r^{3-2\pi K}}{a^{4-2\pi K}}. \quad (9.25)$$

Let's take the high-energy point of view on the RG: we change the cutoff $a \rightarrow ae^\ell$ and demand that the physics (K^R) is invariant. This is accomplished by replacing

$$K \rightarrow K(\ell) = K - cy^2 \int_a^{ae^\ell} \frac{dr r^{3-2\pi K}}{a^{4-2\pi K}}$$

(where $c > 0$ is a constant) and

$$y \rightarrow y(\ell) = ye^{\ell(2-\pi K)}.$$

This reproduces our RG equations:

$$-\partial_\ell K = \partial_\ell x = Ay^2 \quad (9.26)$$

$$\partial_\ell y = (2 - \pi K(\ell))y(\ell) + \mathcal{O}(y^3) = xy. \quad (9.27)$$

³⁷A little bit more detail which justifies the first line in (9.22): Claim 1:

$$\langle (\partial_i \theta_v)(q_1) (\partial_j \theta_v)(q_2) \rangle = f(q) (2\pi)^2 \delta^2(q_1 + q_2) (\delta_{ij} - \hat{q}_i \hat{q}_j) \equiv G_{ij}.$$

Claim 2:

$$(\partial_i \theta_v)(q) (\partial_i \theta_v)(-q) = \frac{(2\pi)^2}{q^2} n_v(q) n_v(-q).$$

Claim 1 follows from $0 = \partial_i \partial_i \theta_v$, which implies $q_1^i G_{ij} = q_2^j G_{ij} = 0$. Translation invariance implies $G_{ij} \propto \delta(q_1 + q_2)$, and rotation invariance implies $G_{ij} = A(q^2) \delta_{ij} + B(q^2) q_i q_j$.

Claim 2 follows from $\epsilon_{ij} \partial_i \partial_j \theta_v = 2\pi n_v$, *i.e.* $\vec{\nabla} \times \vec{\nabla} \theta_v = 2\pi n_v \hat{z}$. Taking curl of the BHS gives $\vec{\nabla} \times (\vec{\nabla} \times \vec{\nabla} \theta_v) = -\nabla^2 (\vec{\nabla} \theta_v) = \vec{\nabla} \times 2\pi n_v \hat{z}$ which says

$$(\partial_j \theta_v)(q) = \frac{-i \epsilon_{ji} q_i}{q^2} 2\pi n_v(q).$$

10 RG approach to walking

10.1 SAWs and $O(n \rightarrow 0)$ magnets

[Brézin, ch 8; Cardy ch 9; the original reference is (brief!) P. de Gennes, [Phys. Lett. A38 \(1972\) 339.](#)]

At each site i of a lattice (actually, it could be an arbitrary graph), place an n -component vector \vec{s}_i ; we'll normalize them so that for each site i $n = \vec{s}_i \cdot \vec{s}_i \equiv \sum_{a=1}^n (s_i^a)^2$, and we'll study the hamiltonian

$$H(s) = -K \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

(I have named the coupling K to make contact with our previous discussion of SAWs). Denote by $d\Omega(s)$ the round (*i.e.* $O(n)$ -invariant) measure on an $(n-1)$ -sphere, normalized to $\int d\Omega(s) = 1$. The partition sum is

$$\begin{aligned} Z &= \int \prod_i d\Omega(s_i) e^{-H(s)} \\ &= \int \prod_i d\Omega(s_i) \sum_{k=0}^{\infty} \frac{K^k}{k!} \left(\sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j \right)^k \\ &= \sum_{\text{graphs, } G} \frac{K^{k \equiv N_l(G)}}{k!} \int \prod_i d\Omega(s_i) \prod_{\langle ij \rangle \in G} \vec{s}_i \cdot \vec{s}_j . \end{aligned} \quad (10.1)$$

Here we are doing the high-temperature expansion, and further expanding the product of factors of the Hamiltonian; we interpret each such term as a graph G covering a subset of links of the lattice. $N_l(G)$ is the number of links covered by the graph G . Now we can do the spin integrals. The integral table is

$$\begin{aligned} \int d\Omega(s) &= 1 \\ \int d\Omega(s) s_a s_b &= \delta_{ab} n \\ \int d\Omega(s) s_a s_b s_c s_d &= \frac{n}{n+2} (\delta_{ab} \delta_{cd} + 2 \text{ perms}) \end{aligned} \quad (10.2)$$

where the second follows by $O(n)$ invariance and taking partial traces. The generating function is useful:

$$f_n(x) \equiv \int d\Omega(s) e^{\vec{x} \cdot \vec{s}} = \frac{\int_0^\pi d\theta \sin^{n-2} \theta e^{x \cos \theta}}{\int_0^\pi d\theta \sin^{n-2} \theta} = \sum_{p=0}^{\infty} \frac{x^p}{p!} \underbrace{\frac{\int_0^\pi d\theta \sin^{n-2} \theta \cos^p \theta}{\int_0^\pi d\theta \sin^{n-2} \theta}}_{=0, n \text{ odd}}$$

$$= 1 + \sum_{p=1}^{\infty} \frac{x^{2p}}{p!} \frac{n^p}{2^p n(n+2) \cdots (n+2p-2)} \xrightarrow{n \rightarrow 0} f_0(x) = 1 + \frac{x^2}{2}. \quad (10.3)$$

Let's interpret this last result: it says that in the limit $n \rightarrow 0$, each site is covered either zero times or two times. This means that the graphs which contribute at $n \rightarrow 0$ avoid themselves. ³⁸

Returning to $n > 0$, since $\langle s_i^a s_i^b \rangle = 0$ if $a \neq b$, the value of the spin is conserved along the closed loops. We get a factor of n from the spin sums $\sum_{a=1}^n$ from *closed* loops. Only closed loops contribute to Z . So $Z \xrightarrow{n \rightarrow 0} 1$, yay. Less trivially, however, consider

$$G^{ab=11}(r, K) \equiv \langle s_0^{a=1} s_r^{b=1} \rangle \equiv Z^{-1} \int \prod_i d\Omega(s_i) e^{-H(s)} s_0^1 s_r^1.$$

Doing the same high-temperature expansion to the numerator, we get contributions from loops which end only at sites 0 and r . In the $n \rightarrow 0$ limit all the closed loops go away from both numerator and denominator, leaving

$$G^{ab=11}(r, K) \xrightarrow{n \rightarrow 0} \sum_p K^p M_p(\vec{r}) \quad (10.4)$$

where $M_p(\vec{r})$ is (as in (2.4)) the number of SAWs going from 0 to \vec{r} in p steps. This is the generating function we considered earlier! The quantity G in (2.6) is actually the correlation function of the $O(n \rightarrow 0)$ magnet!

Summing the BHS of (10.4) over r , the LHS is $\sum_r G^{11}(r, K) = \chi^{11}(K) \sim (K_c - K)^{-\gamma}$ near the critical point of this magnet. The RHS is $\sum_p \sum_r K^p M_p(R) = \sum_p M_p K^p \rightarrow \infty$ which $K \rightarrow K_c$ (from below), from which we concluded earlier that for large walks, $M_p \stackrel{p \gg 1}{\sim} p^{\gamma-1} a^p$ (with $a = 1/K_c$, a non-universal constant which is sometimes fetishized by mathematicians).

Furthermore, the quantity ξ in (2.7) is actually the correlation length, $G^{11}(r, K) \sim e^{-r/\xi}$. At the critical point, $\xi \sim (K_c - K)^{-\nu}$ means that $R_p \sim p^\nu$, which determines the fractal dimension of the SAW in d dimensions to be $D_{\text{SAW}} = \lim_{n \rightarrow 0} \frac{1}{\nu(n, d)}$, where $\nu(n, d)$ is the correlation-length critical exponent for the $O(n)$ Wilson-Fisher fixed point in d dimensions.

[End of Lecture 19]

³⁸Cardy has a clever way of avoiding these spherical integrations by starting with a microscopic model with a nice high temperature expansion (namely $H(s) = \sum_{\langle ij \rangle} \log(1 + K \vec{s}_i \cdot \vec{s}_j)$) and appealing to universality.

10.1.1 SAW:RW::WF:Gaussian

In the same way, the Gaussian fixed point determines the fractal dimension of the unrestricted walk. This can be seen by a high-temperature expansion of the Gaussian model. (For more on this point of view, see Parisi §4.3 - 4.4.) Alternatively, consider unrestricted walks on a graph with adjacency matrix A_{ij} , starting from the origin 0. Denote the probability of being at site r after n steps by $P_n(r)$. Starting at 0 means $P_0(r) = \delta_{r,0}$. For an unrestricted walk, we have the one-step (Markov) recursion:

$$P_{n+1}(r) = \frac{1}{z} \sum_{r'} A_{r'r} P_n(r') \quad (10.5)$$

where the normalization factor $z \equiv \sum_{r'} A_{r'r}$ is the number of neighbors (more generally, the matrix A could be a weighted adjacency matrix and z could depend on r). Defining the generating function

$$G(r|q) \equiv \sum_{n=0}^{\infty} q^n P_n(r)$$

the recursion (10.5) implies

$$\left(\delta_{r'r} - \frac{q}{z} A_{r'r} \right) G(r|q) = \delta_{r',0}. \quad (10.6)$$

In words: G is the correlation function of the Gaussian model with

$$Z = \int \prod_r d\phi_r e^{-\sum_{r'r} \phi_{r'} \left(\delta_{r'r} - \frac{q}{z} A_{r'r} \right) \phi_r}.$$

For the hypercubic lattice with spacing a , this is

$$G(r|q) = \int_{\text{BZ}} d^d p \frac{e^{i\vec{r} \cdot \vec{p}}}{1 - \frac{q}{z} \sum_{\mu} \cos ap_{\mu}}.$$

The long-wavelength properties (for which purposes the denominator may be replaced by $p^2 + r$ as $r \sim q - 1$) of the Gaussian model near its critical point at $q \rightarrow 1$ determine the behavior of large unrestricted walks, and in particular the RMS size $\sim \sqrt{n}$ and fractal dimension is 2.

And the Gaussian answer is the right answer even for a SAW in $d > 4$. We could anticipate this based on our understanding of the fate of the WF fixed point as $d \rightarrow 4$ from below. How can we see the correctness of mean field theory for SAWs in $d > 4$ directly from the walk?

There is a simple answer, and also a more involved, quantitative answer (next). The simple answer is: the random walk has fractal dimension $D = 2$ (if it is embedded in two or more dimensions and is unrestricted). Two-dimensional subspaces of

\mathbb{R}^d will generically intersect (each other or themselves) if $d \leq 4$ (generic intersection happens when the sum of the codimensions is ≤ 0 , so the condition for intersection is underdetermined). For $d \geq 4$, they generically miss each other, and the self-avoidance condition does not have a big effect.

10.1.2 Worksheet theory of the SAW

[Cardy §9.2] Consider the following statistical model of a chain of N monomers at positions \vec{r}_i in d dimensions:

$$Z = \int \prod_{i=1}^N d^d r_i \exp \left(- \sum_i \frac{(\vec{r}_{i+1} - \vec{r}_i)^2}{a^2} - ua^d \sum_i \sum_j \delta^d(\vec{r}_i - \vec{r}_j) \right).$$

The first term insists that neighboring monomers be spaced by a distance approximately a . The second term penalizes a configuration where any two monomers collide. We used factors of the chain-spacing a to render the coupling u dimensionless.

Now zoom out. Suppose that $a \ll \xi$ so that we may treat the polymer as a continuous chain, $\vec{r}(t_i \equiv ia^2) \equiv \vec{r}_i$. In taking the continuum limit we must take $t \sim a^2$ in order to keep the coefficient of the \dot{r}^2 term independent of a . The exponent becomes the Edwards hamiltonian:

$$H_E[r] = \int dt \left(\frac{dr}{dt} \right)^2 + ua^{d-4} \int \int dt_1 dt_2 \delta^d(\vec{r}_1 - \vec{r}_2).$$

This is a ‘worksheet’ point of view: it is a 1d system of size Na^2 , with a long (infinite) range interaction. a plays the role of a UV cutoff.

If $u = 0$, so the walk does not avoid itself, dimensional analysis ($[r] = 1, [t] = 2$ and demanding the kinetic term be dimensionless) gives

$$r(t) \sim a \left(\frac{t}{a^2} \right)^{1/2} \tag{10.7}$$

and r does not scale when we rescale the cutoff $a \rightarrow ba, t \rightarrow t$.

RG: The interaction strength ua^{d-4} is independent of the cutoff when $d = 4$. It is irrelevant for $d > 4$, giving back the MFT result (10.7), as promised.

For $d < 4$ it seems to grow in the IR, and we might hope for an IR fixed point u^* , and a resulting anomalous dimension for the operator r :

$$a \rightarrow ba, \quad r \rightarrow b^{-x}r, \quad t \rightarrow t.$$

Here is a clever (though approximate) argument (due to Flory) that suggests a value for x . At a fixed point, the two terms in H must conspire, and so should scale the same way. For general x , the kinetic term and the potential scale respectively as

$$\text{KE} \rightarrow \text{KE}b^{-2x}, \quad V \rightarrow Vb^{d-4+dx}$$

suggesting that $x = \frac{4-d}{2+d}$. Dimensional analysis says

$$r(t) = af \left(\frac{t}{a^2} \right) \sim t^{\frac{1+x}{2}}$$

and therefore the RMS walk size is

$$R = r(t = N) \sim N^\nu, \quad \nu = \frac{1+x}{2} \Big|_{\text{Flory}} = \frac{3}{d+2}.$$

This isn't too bad; in fact it's exactly right in $d = 2$. See Cardy Chapter 9 for more on this, and see the homework for a more quantitative approach to the value of ν .

A comment on 'power counting'.

How did we know from the engineering dimensional analysis that u was irrelevant when $d > 4$?

Let me describe the analogous argument in the case of field theory with local interactions. Consider the gaussian critical point in d dimensions $S_0[\phi] = \int d^d x (\nabla\phi)^2$, so that the length dimensions of the field are $[\phi] = \frac{2-d}{2}$. Perturb by $S_{\text{interaction}} \equiv \int d^d x g\phi^p$.

$$0 = [S_{\text{interaction}}] = d + [g] + p[\phi] \implies [g] = -(d + p[\phi]) = - \left(d + p \frac{2-d}{2} \right).$$

The coupling is dimensionless when $[g] = 0$ which happens when

$$p = p_d \equiv \frac{2d}{d-2},$$

this case is naively scale invariant, at least until we study the fluctuations. For $d > 2$, the coupling g has length dimensions

$$[g] = d \cdot \frac{p - p_d}{p_d} \begin{cases} > 0 \text{ when } p > p_d, & \text{non-renormalizable or irrelevant} \\ = 0 \text{ when } p = p_d, & \text{renormalizable or marginal} \\ < 0 \text{ when } p < p_d, & \text{super-renormalizable or relevant.} \end{cases}$$

Consider the 'non-renormalizable' case. Suppose we calculate some physical quantity f with $[f]$ as its naive dimension, in perturbation theory in g , *e.g.* by Feynman diagrams. We'll get:

$$f = \sum_{n=0}^{\infty} g^n c_n$$

with c_n independent of g . So

$$[f] = n[g] + [c_n] \implies [c_n] = [f] - n[g]$$

So if $[g] > 0$, c_n must have more and more powers of some inverse length as n increases. What dimensionful quantity makes up the difference?? The dimensions are made up by dependence on the short-distance cutoff $\Lambda = \frac{2\pi}{a}$, which has $[\Lambda] = -1$. Generically: $c_n = \tilde{c}_n (\Lambda)^{n[g]}$, where \tilde{c}_n is dimensionless, and $n[g] > 0$ – it's higher and higher powers of the cutoff. But this means that if we integrate out shells down to Λ/b , in order for physics to be independent of the zoom parameter b , the microscopic coupling $g(b)$ will have to depend on b to cancel this factor. In particular, we'll have to have

$$g(b) = g_0 b^{-n[g]} \xrightarrow{b \rightarrow \infty} 0.$$

10.2 RG approach to unrestricted lattice walk

We showed above that the generating function $G(r|q)$ for unrestricted walks on a lattice (from 0 to r) satisfies (10.6), which says that it's a Green's function for the lattice laplacian. The data of the Green's function is encoded in the spectrum of the adjacency matrix

$$A_{ij}v_j^\epsilon = \epsilon v_i^\epsilon. \quad (10.8)$$

This determines G via

$$G(i|q) = \sum_{\epsilon} v_0^\epsilon v_i^\epsilon \frac{1}{1 - \epsilon q/z}.$$

The eigensystem of A encodes the solution to many physics problems. For example, we could consider a continuous-time random walk, where the probability $p_i(t)$ for a walker to be at site i at time t satisfies

$$\partial_t p_i = (\delta_{ij} z_j - A_{ij}) p_j \quad (10.9)$$

where $z_j \equiv \sum_i A_{ij}$ is coordination number at site j , which addition guarantees $0 = \sum_i \partial_t p_i$, the conservation of probability. The solution is then

$$p_i(t) = \sum_{\epsilon, j} e^{-(z-\epsilon)t} v_i^\epsilon v_j^\epsilon p_j(0).$$

Alternatively, we could think of these as the equations for the normal modes of the lattice vibrations of a collection of springs stretched along the bonds of the lattice. In that case, this spectrum determines the (phonon contribution to the) heat capacity of a solid with this microstructure.

Previously, we solved this problem using translation symmetry of the lattice, by going to momentum space. Here I would like to illustrate an RG solution to this eigenvalue problem which is sometimes available. It takes advantage of the scaling symmetry of the lattice. Sometimes both scaling symmetry and translation symmetry are both present, but they don't commute.

Sometimes, as for most fractals, only the self-similarity is present. So this method is useful for developing an analytic understanding of walks on fractal graphs, or more generally the spectrum of their adjacency matrix. I believe the original references are [this paper](#) and [this one](#). Roughly, we are going to learn how to compute the phonon contribution to the heat capacity of the broccoflower!



Let's solve (10.8) for the case of a chain, with $A_{ij} = t(\delta_{i,j+1} + \delta_{i,j-1})$. I've introduced a 'hopping amplitude' t which can be regarded as related to the length of the bonds.

The eigenvalue equation can be rewritten as

$$v_i = \frac{t}{\epsilon} (v_{i-1} + v_{i+1}). \quad (10.10)$$

Notice that if i is odd, then the entries on the RHS only involve even sites. So this equation eliminates v_i at the odd sites in terms of the values at the even sites. Plugging this back into the equation for an even site gives

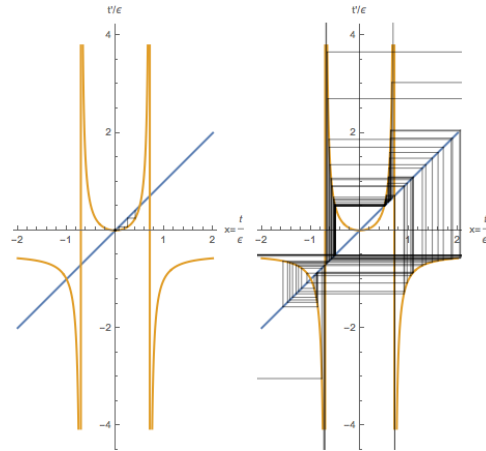
$$\begin{aligned} \epsilon v_{2l} &= t(v_{2l-1} + v_{2l+1}) = \frac{t^2}{\epsilon} (v_{2l-2} + v_{2l} + v_{2l} + v_{2l+2}) \\ \implies v_{2l} &= \underbrace{\frac{t^2}{\epsilon^2 - 2t^2}}_{\equiv t'/\epsilon} (v_{2l-2} + v_{2l+2}). \end{aligned}$$

This is the same equation as (10.10), but with half as many sites, *i.e.* the zoom factor is $b = 2$.

t' is a renormalized hopping amplitude:

$$\frac{t'}{\epsilon} = \frac{t^2}{\epsilon^2 - 2t^2} = \frac{(t/\epsilon)^2}{1 - 2(t/\epsilon)^2}.$$

This is a recursive map for the ratio $x = t/\epsilon$. Actually, it can be mapped to the logistic map $y \rightarrow ry(1 - y)$, with $r = 4$, by the change of variables $y = 4x^{-1} - 2$. A lot is known about this map.

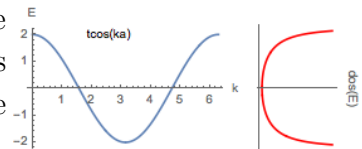


We can regard this recursion as a rule for growing the lattice (and all of its eigenvectors) starting from a small chunk of the stuff. How do we reconstruct the eigenvectors recursively? Suppose we start with a chain of 2^{n+1} sites and suppose we know an eigenstate v^{n, ϵ_n} for this case with $\epsilon_n \neq 0$. There is a solution on a lattice with twice as many sites with

$$v_{2j}^{n+1, \epsilon_{n+1}^\pm} = v_j^{n, \epsilon_n}, \quad v_{2j+1}^{n+1, \epsilon_{n+1}^\pm} = -\frac{v_j^{n, \epsilon_n} + v_{j-1}^{n, \epsilon_n}}{\epsilon_n^\pm}$$

where $\epsilon_{n+1}^\pm = \pm\sqrt{2 - \epsilon_n}$.

Let's cheat and remind ourselves of the known answer for the spectrum using translation invariance: $E(k) = 2t \cos ka$ ranges from $-2t$ to $2t$ as k varies over the BZ from 0 to $2\pi/a$. Let's use this to learn how to understand the iteration map.



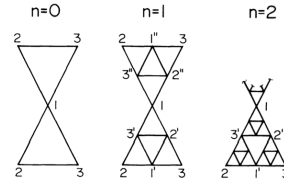
For the chain, the map has three fixed points, at $x = 0, \frac{1}{2}, -1$. Let's think of fixing E and varying the initial hopping rate. If $t_0 \in (-E/2, E/2)$ (that is, if $|E| > 2t$ is in the band gap) then $t_{n \rightarrow \infty} \rightarrow t^* = 0$ eventually reaches the fixed point at $x = 0$ (as in the left figure). More precisely, it goes like $t_n \stackrel{n \gg 1}{\sim} E e^{-2n\lambda}$ for some λ .

Such an orbit which asymptotes to $t \rightarrow 0$ can be described by decoupled clusters – the wavefunction is *localized*. I learned about this from [this paper](#).

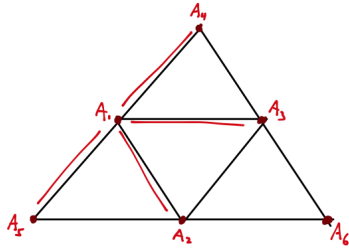
In contrast, one with finite or infinite asymptotic t is associated with an *extended* state. This happens if $|t_0| > E/2$ (so that $E \in (-2t, 2t)$ is in the band). Then $t_n > |E|/2$ for all n , and we have a nonzero effective hopping even between two sites that are arbitrarily far-separated.

The fixed point at $t_* = E/2$ is the state with $k = 0$, *i.e.* the uniform state.

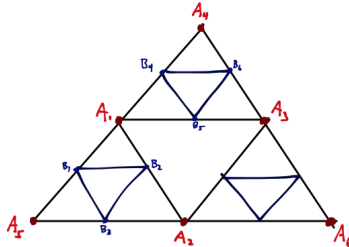
The procedure works for other examples, too, including some without translation invariance, where the spectrum can be quite different. Consider the Sierpinski triangle lattice.



[from Domany et al, linked above]



$$\epsilon A_1 = t(B_1 + B_2 + B_4 + B_5). \quad (10.11)$$



$$\begin{aligned} \epsilon B_1 &= t(A_1 + A_5 + B_2 + B_3), & \epsilon B_4 &= t(A_1 + A_4 + B_5 + B_6), \\ \epsilon B_2 &= t(A_1 + A_2 + B_1 + B_3), & \epsilon B_5 &= t(A_1 + A_3 + B_4 + B_6), \\ \epsilon B_3 &= t(A_2 + A_5 + B_1 + B_2), & \epsilon B_6 &= t(A_4 + A_3 + B_5 + B_4). \end{aligned}$$

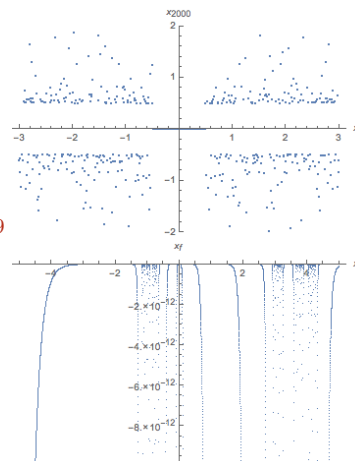
Eliminating the B sites by solving the previous six equations for them in terms of the A sites and plugging into (10.11) gives an equation of the same form on a coarser lattice

$$\epsilon A_1 = t'(A_2 + A_3 + A_4 + A_5), \quad t' = \frac{t^2}{\epsilon - 3t}.$$

Zoom factor is $b = \sqrt{2}$. In terms of the dimensionless ratio $x \equiv t/\epsilon$,

$$x \rightarrow \frac{x^2}{1 - 3x}$$

Here's a way to visualize the huge qualitative difference from this map relative to the result for the chain. Plot, as a function of some initial $x = t/\epsilon$, the value of the n th iterate, for some large value of n (here $10^5 \times 2$). For the chain (shown at the top), every x which starts in the band stays in the band ³⁹ ($x_n > 1/2$ if $x_0 > 1/2$), and vice versa. For the Sierpinski case, we get this Cantor-like set of localized states. Here the spacing on the x -axis is 10^{-2} ; if we scan more closely, we'll find more structure.



10.3 Spectral dimension

Here's one more notion of dimension, for a graph embedded in \mathbb{R}^d , following [Toulouse et al.](#) Think of the graph as a Debye solid, that is, put springs on the links of the graph, each with natural frequency $\omega_0^2 = K/m$. The normal modes of this collection of springs have frequencies ω with ω_n^2/ω_0^2 which are eigenvalues of the adjacency matrix.

The density of states of such modes for small ω is an ingredient in the heat capacity of the resulting model solid. Denote by $\rho(\omega)d\omega$ the number of modes with frequency in the interval $(\omega, \omega + d\omega)$.

For a translation-invariant system in d dimensions, the modes can be labelled by wavenumber and $\rho(\omega)d\omega = \mathfrak{d}^d k$ which at $\omega \rightarrow 0$ (in the thermodynamic limit) is governed by Goldstone's acoustic phonon with $\omega = v_s k$ and therefore $\rho(\omega) \propto \omega^{d-1}$. More generally, we define the *spectral dimension* d_s of the graph by the power law relation

$$\rho(\omega) \stackrel{N \rightarrow \infty}{\sim} \stackrel{\omega \rightarrow 0}{\sim} \omega^{d_s}.$$

Sometimes it's called the *diffusion dimension*. It is a useful idea! One cool application is to figuring out how many dimensions your average spacetime has when you do a simulation involving dynamical triangulations. (See §5.2 of [this paper](#).)

Now suppose that instead of translation-invariance, we have dilatation invariance, *i.e.* self-similarity. The number of sites for a graph Γ of linear size L scales as

$$N(L) \sim L^{D_\Gamma}$$

where D_Γ is the fractal dimension. This means that if we assemble a scaled up version whose linear size is scaled up by b , we have $N(bL) = b^{D_\Gamma} N(L)$ sites. And it means,

³⁹Thanks to Daniel Ben-Zion for help with these figures.

just by counting eigenvalues, that the density of states per unit cell must scale like

$$\rho_L(\omega) = b^{D_\Gamma} \rho_{L/b}(\omega). \quad (10.12)$$

Consider L finite so that the spectrum $\{\omega_n\}$ is discrete, and focus on the n th eigenvalue from the bottom, for some fixed n . If we knew that this eigenvalue scaled with system size like

$$\omega(L/b) = b^x \omega(L)$$

then

$$\begin{aligned} \rho_{L/b}(\omega) &= b^{-x} \rho_L(\omega b^{-x}) \\ \implies \rho_L(\omega) &\stackrel{(10.12), (10.13)}{=} b^{D_\Gamma - x} \rho_L(\omega b^{-x}) \stackrel{b = \omega^{1/x}}{\sim} \omega^{\frac{D_\Gamma - x}{x}}. \end{aligned} \quad (10.13)$$

Claim: The smooth part of the spectrum of the Sierpinski fractal solid does scale like b^x for some x which we can determine. A more earnest pursuit of the equations (10.11) implies that

$$\begin{aligned} \left(\frac{\omega^2}{\omega_0^2}\right) &\mapsto \left(\frac{(\omega')^2}{\omega_0^2}\right) = \left(\frac{\omega^2}{\omega_0^2}\right) \left(d + 3 - \left(\frac{\omega^2}{\omega_0^2}\right)\right) \\ \omega^2(L/2) &= \omega^2(L)(d + 3 + \mathcal{O}(\omega^2)) \equiv \omega^2(L)2^{2x} \\ \implies x &= \frac{\log(d + 3)}{2 \log 2}. \end{aligned}$$

(We used $b = 2$ since the number of sites per edge of the triangle is halved at each decimation step.) This means that the smooth part of the spectrum behaves as

$$\rho(\omega) \sim \omega^{\frac{D_\Gamma - x}{x}} = \omega^{d_s}.$$

10.4 Resistor networks

The resistor network on a Sierpinski d -gasket is studied [here](#). The scaling with size of the conductivity of stuff made from such a graph can be related to its spectral dimension.

Unlike the paragon of [nerd-sniping problems](#) (the resistor network on the square lattice), this problem cannot be solved by going to momentum space.

Consider sending a current I into one corner of a Sierpinski gasket. By symmetry, a current I/d must emerge from the other d corners.

Call $\rho(a)$ the resistance of one bond with lattice spacing a . Now we want to compute the effective, coarse-grained resistance $\rho(ba)$ for $b > 1$. The symmetry of the problem

forbids current from crossing the middle of the triangle, and this allows us to compute the voltage drop between the input corner and any of the others. Specifically, this voltage drop is preserved if

$$\rho(ba) = \frac{d+3}{d+1} \rho(a) \equiv b^\zeta \rho(a)|_{b=2}$$

$$\zeta = \frac{\log \frac{d+3}{d+1}}{\log 2}.$$

Now if we iterate this map ℓ times so that $b^\ell = \frac{L}{a}$ for some macroscopic L , then the resistance of the whole chunk of stuff is

$$\rho(L) \sim L^\zeta$$

and the conductivity of the stuff (in $\vec{J} = \sigma \vec{E}$, an intensive quantity) is

$$\sigma(L) = \frac{L^{2-d}}{\rho(L)} \sim L^{-t}$$

with scaling exponent $t = d - 2 + \zeta$.

Exercise: relate ζ to the spectral dimension d_s .

11 RG sampler platter

I didn't get this far in lecture, but the goal of this section is to convey some more the huge range of applications of the renormalization group perspective.

11.1 Disorder

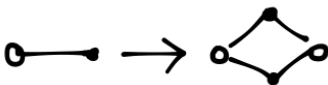
[McComb p.60; Creswick, chapter 3.]

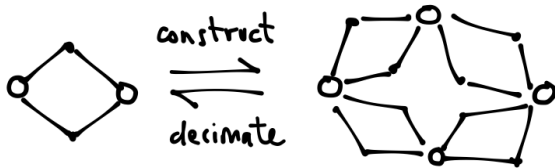
I want to emphasize that RG of couplings is a subset of RG of probability distributions.

So far in this course, we've been studying *clean* systems, ones whose couplings are the same at each location. It is often important and interesting to consider the case where the couplings are only uniform *on average*. Just encoding the values of the couplings in one realization of the system is then quite a job, never mind computing the resulting free energy. But for large systems, we can often appeal yet again to the central idea of statistical physics and choose the couplings from some probability distribution. (A physical quantity for which this assumption works is said to be 'self-averaging'.)

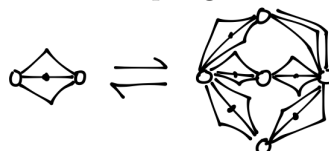
This probability distribution will then itself evolve under the RG.

Let's consider a case where we can study this in detail, namely the nearest-neighbor Ising ferromagnet on *hierarchical graphs*. Such a graph can be constructed by a sprouting rule: at each step of the construction, replace each link with some given motif.

For example, the sprouting rule  produces the *diamond hierarchical lattice*. I denote the new sites in black. The beauty of this construction for our purposes is that decimating the black sites precisely undoes the construction step:

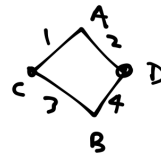


The generalization which replaces each link with q segments is called the Berker

lattice, I think. For $q = 3$, this looks like: 

Let $v_{\langle ij \rangle} \equiv \tanh \beta J_{\langle ij \rangle}$. Consider tracing over the black sites A and B in the figure at right. Using the high-temperature-expansion formula, this isn't hard:

$$\begin{aligned}
e^{-\Delta H_{\text{eff}}(s_C, s_D)} &= \sum_{s_A, s_B = \pm 1} e^{-H(s)} = \sum_{s_A, s_B \text{ links}, \langle ij \rangle} \prod (1 + v_{\langle ij \rangle} s_i s_j) \\
&= 2^2 (1 + v_1 v_2 s_C s_C) (1 + v_3 v_4 s_C s_D) \\
&= 2^2 ((1 + v_1 v_2 v_3 v_4) + (v_1 v_2 + v_3 v_4) s_C s_D) \\
&= 2^2 (1 + v_1 v_2 v_3 v_4) (1 + v' s_C s_D)
\end{aligned}$$



with

$$v'(v_1..v_4) = \frac{v_1 v_2 + v_3 v_4}{1 + v_1 v_2 v_3 v_4}. \quad (11.1)$$

In the clean limit where all couplings are the same, this is

$$v' = \frac{2v^2}{1 + v^4}.$$

This has fixed points at

$$v^* = 0, 1, 0.0437. \quad (11.2)$$

Just as we did for the Ising chain in §3, we can study the behavior near the nontrivial fixed point and find (here $b = \sqrt{2}$) that $\nu \simeq 1.338$. Redoing this analysis to include also a magnetic field, we would find $y_h = 1.758$ for the magnetization exponent.

But now suppose that the couplings are chosen from some initial product distribution, independently and identically distributed. Some examples for the bond distribution to consider are:

$$\begin{aligned}
\text{Random bond dilution:} & \quad P(J) = x\delta(J - J_0) + (1 - x)\delta(J) \\
\text{Source of frustration:} & \quad P(J) = x\delta(J - J_0) + (1 - x)\delta(J + J_0) \\
\text{Edwards-Anderson spin glass:} & \quad P(J) \propto \exp -\frac{J^2}{2J_0^2}
\end{aligned}$$

After the decimation step, the distribution for any link evolves according to the usual formula for changing variables in a probability distribution, using the RG relation (11.1):

$$P'(v') = \int dv_1 dv_2 dv_3 dv_4 \delta(v' - v'(v_1..v_4)) P(v_1) \cdots P(v_4).$$

The preceding relation is then an RG recursion equation for the *distribution* of couplings $P(v) \mapsto (\mathcal{R}(P))(v)$. As usual when confronted with such a recursion, we should ask about its fixed points, this case fixed distributions:

$$P_\star(v) \stackrel{!}{=} \int dv_1 dv_2 dv_3 dv_4 \delta(v - v'(v_1..v_4)) P_\star(v_1) \cdots P_\star(v_4). \quad (11.3)$$

We know some solutions of this equation. One is

$$P_\star(v) = 1\delta(v - v_\star)$$

with v_\star given by one of the solutions in (11.2).

Another set of fixed points is associated with bond percolation. Here's what I mean about percolation. Consider the $T \rightarrow 0$ (or $J \rightarrow \infty$) limit of a ferromagnetic NN Ising model on a graph, with one catch: For each pair of neighbors, we randomly decide whether or not to place a link. This is a model of *bond percolation*. This is realized in our system here by a distribution of the form

$$P_x(v) = x\delta(v - 1) + (1 - x)\delta(v).$$

Plugging this ansatz into the map gives

$$P'_x(v) = (x^4 + 4x^3(1 - x) + 2x^2(1 - x)^2) \delta(v-1) + (4x^2(1 - x^2) + 4x(1 - x)^3 + (1 - x)^4) \delta(v)$$

where the terms come from enumerating which of the four bonds is zero. So: the distribution is self-similar, but the bond-placing probability x evolves according to

$$x \mapsto x' = 2x^2 - x^4.$$

So each fixed point of this map gives a solution the fixed-distribution equation (11.3). They occur at

$$x^\star = \begin{cases} 0, & \text{nobody's home} \\ 1, & \text{everybody's home} \\ \frac{\sqrt{5}-1}{2}, & \text{percolation threshold on the DHL.} \end{cases}$$

We can study (a subset of) flows between these fixed points if we make the more general ansatz

$$p(v) = x\delta(v - v_0) + (1 - x)\delta(v)$$

with two parameters v_0, x . Then we get a 2d map, much more manageable, if the evolution preserves the form. It almost does. The evolution rule can be estimated by

$$\begin{aligned} x' &= -x^4 + 2x \\ x'v'_0 &= \langle v \rangle_{P'} \equiv \int dv v P'(v). \end{aligned} \tag{11.4}$$

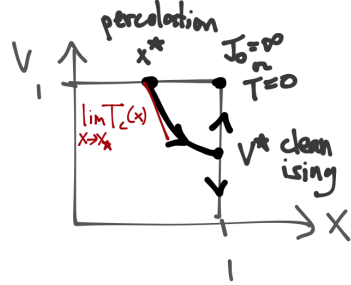
The result is

$$x'v'_0 = \int dv \int \prod_{i=1}^4 dv_i \delta(v - v'(v_1..v_4)) v \prod_i p(v_i)$$

$$\begin{aligned}
&= \int \prod_i dv_i \frac{v_1 v_2 + v_3 v_4}{1 + v_1 v_2 v_3 v_4} \prod_i p(v_i) \\
&= x^4 \frac{2v_0^2}{1 + v_0^4} + 4x^3(1-x)v_0^2 + 2x^2(1-x)^2v_0^2.
\end{aligned}$$

In the second line we used the delta function to do the v integral.

Indicated by the thick black link is the critical surface of the clean Ising fixed point at $x = 1, v = v^*$. The percolation fixed point at $x = x_*, v = 1$ is unstable to the clean Ising fixed point. Besides the structure of the phase diagram, we can infer the angle at which the Ising critical surface approaches $x = x_*$: $T_c(x) \xrightarrow{x \rightarrow x_*} -\frac{1}{\log(x-x_*)}$.



Strong-disorder RG. There is a simplifying limit where the distribution of the couplings is very broad. Such a distribution is sometimes an attractive fixed point of the RG, called a strong-disorder or even infinite-disorder fixed point depending on the extremity. This limit is simplifying because then we can order the RG analysis by looking at the largest coupling $t_i \equiv \Omega$ first, and we can use $\frac{t_{j \neq i}}{\Omega} \ll 1$ as a small parameter. A useful reference is [this paper by Altman and Refael](#). A more detailed but scarier discussion is [this review](#).

Let's analyze the example of an adjacency matrix of a graph with random values of the hopping parameter for each link. For simplicity let's think about the case where the graph is a chain.

So we want to solve

$$\begin{pmatrix} \ddots & & & & \\ & 0 & t_i & & \\ & t_i & 0 & t_{i+1} & \\ & & t_{i+1} & 0 & t_{i+2} \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ v_{i-1} \\ v_i \\ v_{i+1} \\ \vdots \end{pmatrix} = \epsilon \begin{pmatrix} \vdots \\ v_{i-1} \\ v_i \\ v_{i+1} \\ \vdots \end{pmatrix}$$

With some random t_i chosen from some broad distribution, so that individual t_i will be very different from each other. Consider the largest $t_i \equiv \mathcal{T}$, and assume that it is much bigger than all the others, including its neighbors. Then we can eliminate the two sites connected by the strong bond by solving the 2×2 problem

$$\begin{pmatrix} 0 & \mathcal{T} \\ \mathcal{T} & 0 \end{pmatrix} \begin{pmatrix} v_{i-1} \\ v_i \end{pmatrix} \simeq \epsilon \begin{pmatrix} v_{i-1} \\ v_i \end{pmatrix}.$$

More precisely, we can eliminate these two sites v_{i-1}, v_i in terms of their neighbors

using their two rows of the eigenvalue equation:

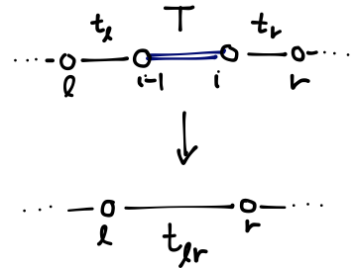
$$\begin{pmatrix} \ddots & & & & & & \\ & 0 & t_\ell & & & & \\ & t_\ell & 0 & \mathcal{T} & & & \\ & & \mathcal{T} & 0 & t_r & & \\ & & & t_r & 0 & & \\ & & & & & \ddots & \end{pmatrix} \begin{pmatrix} \vdots \\ v_\ell \\ v_{i-1} \\ v_i \\ v_r \\ \vdots \end{pmatrix} = \epsilon \begin{pmatrix} \vdots \\ v_\ell \\ v_{i-1} \\ v_i \\ v_r \\ \vdots \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} v_{i-1} \\ v_i \end{pmatrix} = \begin{pmatrix} -\epsilon & \mathcal{T} \\ \mathcal{T} & -\epsilon \end{pmatrix}^{-1} \begin{pmatrix} t_\ell \\ t_r \end{pmatrix}$$

The result of plugging this back into the neighboring rows of the equation is to make an effective hopping between ℓ and r of approximate strength

$$t_{\ell r} \sim \frac{t_\ell t_r}{\mathcal{T}}. \quad (11.5)$$

(I approximated $\epsilon \ll T$.)



This RG rule (11.5) (which we could name for Dasgupta and Ma in a slightly more fancy context) is very simple in terms of the logs of the couplings, $\zeta \equiv \log \mathcal{T}/t$:

$$\zeta' = \zeta_\ell + \zeta_r$$

– they just add. Here $\zeta \in (0, \infty)$ and $\zeta = 0$ is the strongest bond.

Let's make an RG for the probability distribution using this rule. The first step is to reduce the UV cutoff, by decimating the highest-energy or shortest distance degrees of freedom. Here the UV cutoff is just the largest hopping parameter, \mathcal{T} . Then the new effective bonds have the distribution (in the log)

$$P_{\text{new}}(\zeta) = \int_0^\infty d\zeta_\ell \int_0^\infty d\zeta_r P(\zeta_\ell) P(\zeta_r) \delta(\zeta - \zeta_\ell - \zeta_r).$$

Imagine we start the RG at some initial strongest bond \mathcal{T}_0 . Then $\Gamma = \log \mathcal{T}_0/\mathcal{T}$ says how much RGing we've done so far. The second rescaling step puts the distribution back in the original range, which requires shifting everyone

$$\zeta_i = \log \left(\frac{\mathcal{T}}{t_i} \right) \mapsto \log \left(\frac{\mathcal{T} - d\mathcal{T}}{t_i} \right) \simeq \zeta_i - \frac{d\mathcal{T}}{\mathcal{T}} = \zeta_i - d\Gamma$$

This moves the whole distribution to the left: $P(\zeta) \mapsto P(\zeta + d\Gamma) = P(\zeta) + d\Gamma P'(\zeta) + \mathcal{O}(d\Gamma^2)$, *i.e.*

$$d_{\text{rescale}} P(\zeta) = \frac{dP(\zeta)}{d\zeta} d\Gamma.$$

The change in the full distribution from adding in the new bonds is

$$d_{\text{new bonds}}P(\zeta) = \underbrace{d\Gamma P(0)}_{\text{strongest bond}} P_{\text{new}}(\zeta).$$

And the full evolution is

$$\frac{dP(\zeta)}{d\Gamma} = \frac{dP(\zeta)}{d\zeta} + P(0) \int_0^\infty d\zeta_\ell \int_0^\infty d\zeta_r P(\zeta_\ell) P(\zeta_r) \delta(\zeta - \zeta_\ell - \zeta_r).$$

This equation has a simple solution:

$$\begin{aligned} P_\Gamma(\zeta) = f(\Gamma)e^{-f(\Gamma)\zeta} &\implies \partial_\Gamma f = -f^2 \implies f(\Gamma) = \frac{1}{\Gamma}. \\ &\implies P_\Gamma(\zeta) = \frac{1}{\Gamma} e^{-\zeta/\Gamma}. \end{aligned}$$

In terms of the hoppings, this is

$$P_T(t) \sim \left(\frac{1}{t}\right)^{1-\frac{1}{\Gamma}} \xrightarrow{\Gamma \rightarrow \infty} \frac{1}{t}$$

– as we zoom out we approach a non-normalizable distribution. This is an infinite-randomness fixed point.

What is this analysis good for? For one thing, we can estimate the fraction of undecimated sites at each step. Each time we decimate a link, we remove two sites. Therefore, the number of undecimated sites evolves by

$$dN = -2P_\Gamma(0)N d\Gamma.$$

Approaching the fixed point, $P_\Gamma(0) = f(\Gamma) = \frac{1}{\Gamma}$, so the solution is

$$N(\Gamma) = \frac{N_0}{\Gamma^2} \sim \frac{N_0}{\log^2(\mathcal{T}_0/t)}.$$

The average distance between surviving sites is $L(\Gamma) \sim a \frac{N_0}{N(\Gamma)} \sim a\Gamma^2 \sim a \log^2(\mathcal{T}_0/t)$.

Here's an attempt at a physical application. Let's go back to using the spectrum of the (random-entry) adjacency matrix to determine the heat capacity of a Debye-Einstein solid. Let's add in a diagonal (Einstein) spring constant:

$$H = \sum_i \left(\frac{p_i^2}{2} + \omega_0^2 x_i^2 \right) + \omega_0^2 \sum_{ij} A_{ij} (x_i - x_j)^2.$$

So the spectrum of normal modes is

$$\omega_n^2 = \omega_0^2 (1 + \epsilon_n)$$

where ϵ_n are the eigenvalues of A . And we take $A_{ij} = (\delta_{i,i+1} + \delta_{i,i-1}) t_i$ and choose t_i from the strong-disorder distribution found above.

To find the heat capacity at temperature $1/\beta$, we should run the RG from some initial UV cutoff \mathcal{T}_0 down to the \mathcal{T} associated with temperature T , which is of order $\frac{T}{\omega_0}$. Because of the breadth of the distribution, the bonds with $t < T^2$ are likely to have $t \ll T^2$ and we can ignore them. Any site not participating in a bond produces a simple equipartition contribution $\Delta E = k_B T$ (i.e. it adds a constant to C_V) as long as $1/\beta > \Omega_0$. Sites participating in a bond have $\omega \ll T$ and are frozen out. So the heat capacity is

$$C_V(\beta) = N(T)$$

where $N(T)$ is the number of undecimated sites when the temperature is T , which means here that the RG scale is $\mathcal{T} \sim T^2$. So this model produces a *crazy* dependence on the temperature,

$$C_V \sim \frac{1}{\log^2(T^2)}.$$

11.2 RG viewpoint on matched asymptotic expansions

[Goldenfeld chapter 10]

It is possible to get anomalous dimensions even from systems without any stochastic element (*i.e.* thermal fluctuations or quantum fluctuations). It is even possible to get them from *linear* differential equations. The latter is demonstrated, for example, by [this analysis](#). (I apologize that I have not found the time to figure out how to explain this without all the surrounding complications.)

Goldenfeld gives an extended discussion of a diffusion equation, perturbed by a nonlinear, singular term, called the Barenblatt equation.

11.3 RG approach to the period doubling approach to chaos

[Creswick ch 2, Strogatz, Dan Arovav's 200B notes!]