University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2017
Assignment 4

Due 11am Thursday, October 26, 2017

1. **Non-Abelian currents.** In previous homework, we studied a complex scalar field. Now, we make a big leap to *two* complex scalar fields, $\Phi_{\alpha=1,2}$, with

$$S[\Phi_\alpha] = \int d^d x dt \left(\frac{1}{2} \partial_\mu \Phi^\star_\alpha \partial^\mu \Phi_\alpha - V\left(\Phi^\star_\alpha \Phi_\alpha\right) \right)$$

Consider the objects

$$Q^i \equiv \frac{1}{2} \int d^d x \mathbf{i} \left(\Pi^\dagger_\alpha \sigma^i_{\alpha\beta} \Phi^\dagger_\beta \right) + h.c.$$

where $\sigma^{i=1,2,3}$ are the three Pauli matrices.

   (a) What symmetries do these charges generate (*i.e.* how do the fields transform)? Show that they are symmetries of S.

   (b) If you want to, show that $[Q^i, H] = 0$, where H is the Hamiltonian.

   (c) Evaluate $[Q^i, Q^j]$. Hence, non-Abelian.

   (d) To complete the circle, find the Noether currents J^i_μ associated to the symmetry transformations you found in part 1a.

   (e) Generalize to the case of N scalar fields.

2. **More about 0+0d field theory.** Here we will study a bit more some field theories with no dimensions at all, that is, integrals.

   Consider the case where we put a label on the field: $q \to q_a, a = 1..N$. So we are studying

$$Z = \int \int_{-\infty}^{\infty} \prod_a dq_a \; e^{-S(q)}.$$

   Let

$$S(q) = \frac{1}{2} q_a K_{ab} q_b + T_{abcd} q_a q_b q_c q_d$$

   where T_{abcd} is a collection of couplings. Assume K_{ab} is a real symmetric matrix.

(a) Show that the propagator has the form:

$$a - - - - - - - b \equiv \langle q_a q_b \rangle_{T=0} = \left(K^{-1}\right)_{ab} = \sum_k \phi_a(k)^\star \frac{1}{k} \phi_b(k)$$

where $\{k\}$ are the eigenvalues of the matrix K and $\phi_a(k)$ are the eigenvectors in the a-basis.

(b) Show that in a diagram with a loop, we must sum over the eigenvalue label k. (For definiteness, consider the order-g correction to the propagator.)

(c) Consider the case where $K_{ab} = t\,(\delta_{a,b+1} + \delta_{a+1,b})$, with periodic boundary conditions: $a + N \equiv a$. Find the eigenvalues. Show that in this case if

$$T_{abcd} q_a q_b q_c q_d = \sum_a g q_a^4$$

the k-label is conserved at vertices, *i.e.* the vertex is accompanied by a delta function on the sum of the incoming eigenvalues.

(d) (Bonus question) What is the more general condition on T_{abcd} in order that the k-label is conserved at vertices?

(e) (Bonus question) Study the physics of the model described in 2c.

Back to the case without labels.

(f) By a change of integration variable show that

$$Z = \int_{-\infty}^{\infty} dq\ e^{-S(q)}$$

with $S(q) = \frac{1}{2} m^2 q^2 + g q^4$ is of the form

$$Z = \frac{1}{\sqrt{m^2}} \mathcal{Z}\left(\frac{g}{m^4}\right).$$

This means you can make your life easier by setting $m = 1$, without loss of generality.

(g) Convince yourself (*e.g.* with Mathematica) that the integral really is expressible as a Bessel function.

(h) It would be nice to find a better understanding for why the partition function of $(0+0)$-dimensional ϕ^4 theory is a Bessel function. Then find a Schwinger-Dyson equation for this system which has the form of Bessel's equation for

$$K(y) \equiv \frac{1}{\sqrt{y}} e^{-a/y} \mathcal{Z}(1/y)$$

for some constant a. (Hint: I found it more convenient to set $g = 1$ for this part and use $\xi \equiv m^2$ as the argument. If you get stuck I can tell you what function to choose for the 'anything' in the S-D equation.)

(i) Make a plot of the perturbative approximations to the 'Green function' $G \equiv \langle q^2 \rangle$ as a function of g, truncated at orders 1 through 6 or so. Plot them against the exact answer.

(j) (Bonus problem) Show that $c_{n+1} \sim -\frac{2}{3} n c_n$ at large n (by brute force or by cleverness).

3. **Combinatorics from 0-dimensional QFT.** [This is a bonus problem.]

Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$ arise as the answer to many combinatorics problems (beware: there is some disagreement about whether this is C_n or C_{n+1}). One such problem is: count random walks on a 1d chain with $2n$ steps which start at 0 and end at 0 without crossing 0 in between.

Another such problem is: in how many ways can $2n$ (distinguishable) points on a circle be connected by chords which do not intersect within the circle.

Consider a zero-dimensional QFT with the following Feynman rules:

- There are two fields h and l.
- There is an $\sqrt{t} h^2 l$ vertex in terms of a coupling t.
- The bare l propagator is 1.
- The bare h propagator is 1.
- All diagrams can be drawn on a piece of paper without crossing.[1]
- There are no loops of h.

The last two rules can be realized from a lagrangian by introducing a large N (below).

(a) Show that the full two-point green's function for h is

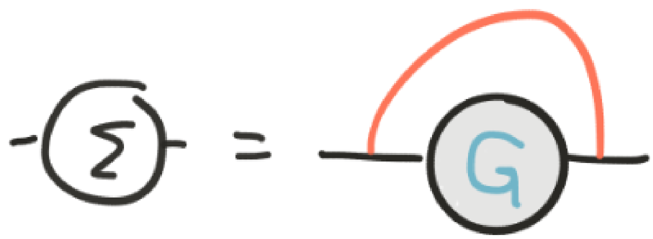
$$G(t) = \sum_n t^n C_n$$

[1] An annoying extra rule: All the l propagators must be on one side of the h propagators. You'll see in part 3f how to justify this.

the generating function of Catalan numbers.

(b) Let $\Sigma(t)$ be the sum of diagrams with two h lines sticking out which may not be divided into two parts by cutting a single intermediate line. (This property is called 1PI (one-particle irreducible), and Σ is called the "1PI self-energy of h". We'll use this manipulation all the time later on.) Show that $G(t) = \frac{1}{1-\Sigma(t)}$.

(c) Argue by diagrams for the equation (sometimes this is also called a Schwinger-Dyson equation)



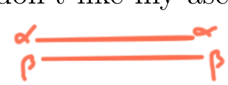
where Σ is the 1PI self-energy of h.

(d) Solve this equation for the generating function $G(t)$.

(e) If you are feeling ambitious, add another coupling N^{-1} which counts the crossings of the l propagators. The resulting numbers can be called Touchard-Riordan numbers.

(f) How to realize the no-crossings rule? Consider

$$L = \frac{\sqrt{t}}{\sqrt{N}} l_{\alpha\beta} h_\alpha h_\beta + \sum_{\alpha,\beta} l_{\alpha\beta}^2 + \sum_\alpha h_\alpha^2$$

where $\alpha, \beta = 1 \cdots N$. By counting index loops, show that the dominant diagrams at large N are the ones we kept above. Hint: to keep track of the index loops, introduce ('t Hooft's) double-line notation: since l is a matrix, it's propagator looks like: $\begin{matrix} \alpha - - - - - - - - \alpha \\ \beta - - - - - - - - \beta \end{matrix}$, while the h propagator is just one index line α______α, and the vertex is ___!!___. If you don't like my ascii diagrams, here are the respective pictures: $\langle l_{\alpha\beta} l_{\alpha\beta} \rangle =$  ,

$\langle h_\alpha h_\alpha \rangle =$  and the hhl vertex is:  .

(g) Use properties of Catalan numbers to estimate the size of non-perturbative effects in this field theory.

(h) There are many other examples like this. Another similar one is the relationship between symmetric functions and homogeneous products. A more different one is the enumeration of planar graphs. For that, see BIPZ.

4. **Brain-warmer: the identity does nothing twice.** Check our relativistic state normalization by squaring the expression for the identity in the 1-particle sector:

$$\mathbb{1}_1^2 \stackrel{!}{=} \mathbb{1}_1 = \int \frac{đ^d p}{2\omega_{\vec{p}}} |\vec{p}\rangle \langle \vec{p}| \, .$$