University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 215A QFT Fall 2017
# Assignment 3

**Due 11am Thursday, October 19, 2017**

1. **Momentum.** Starting from the expression for the stress energy tensor we derived for free scalar field theory, find the generator of spatial translations $\vec{\mathbf{P}}$. Show that a state of $n$ particles of definite wavenumber

$$\mathbf{a}^\dagger_{\vec{k}_1} \cdots \mathbf{a}^\dagger_{\vec{k}_n} \left|0\right\rangle$$

is an eigenstate of $\vec{\mathbf{P}}$ with a reasonable answer for the eigenvalue. (This shows that it was OK to refer to the wavenumber $k$ as the momentum of the particle created by $\mathbf{a}^\dagger_k$.)

2. **Vacuum energy from the propagator.**

Consider a scalar field with

$$\mathbf{H} = \int d^d x \; \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right).$$

Reproduce the formal expression for the vacuum energy

$$\langle 0|\mathbf{H}|0\rangle = V \int đ^d k \frac{1}{2}\hbar\omega_{\vec{k}}$$

using the two point function

$$\langle 0|\,\phi(x)^2\,|0\rangle = \langle 0|\phi(0)\phi(0)|0\rangle = \lim_{\vec{x},t\to 0} \langle 0|\,\phi(x)\phi(0)\,|0\rangle$$

and its derivatives. ($V$ is the volume of space.)

Later we will learn to draw this process as a Feynman diagram which is a circle (a line connecting a point to itself).

3. **The propagator is a Green's function.**

   (a) Consider the *retarded* propagator for a real, free, massive scalar field:

$$D_R(x-y) \equiv \theta(x^0 - y^0)\,\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle\,.$$

Show that it is a Green's function for the Klein-Gordon operator, in the sense that

$$\left(\Box_x^2 + m^2\right) D_R(x-y) = a\delta^{d+1}(x-y), \qquad \Box_x \equiv \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}.$$

Find $a$.

(b) Use the previous result to generalize the mode expansion of the scalar field to the situation with an external *source*, *i.e.* when we add to the lagrangian density $\phi(x)j(x)$ for some fixed c-number function $j(x)$.

(c) Show that the *time-ordered* propagator for a real, free, massive scalar field

$$D(x-y) \equiv \langle 0| \mathcal{T}\phi(x)\phi(y) |0\rangle \equiv \theta(x^0-y^0) \langle 0| \phi(x)\phi(y) |0\rangle + \theta(y^0-x^0) \langle 0| \phi(y)\phi(x) |0\rangle$$

is *also* a Green's function for the Klein-Gordon operator. That is, consider what happens when you act with the wave operator $+\Box_x + m^2$ on the time-ordered two-point function.

[Hints: Use the canonical equal-time commutation relations:

$$[\phi(\vec{x}), \phi(\vec{y})] = 0, \quad [\partial_{x^0}\phi(\vec{x}), \phi(\vec{y})] = -\mathbf{i}\delta^{D-1}(\vec{x}-\vec{y}).$$

Do not neglect the fact that $\partial_t\theta(t) = \delta(t)$: the time derivatives act on the time-ordering symbol!]

(d) This is an example of something which is easier to understand from the path integral. In the next problem, we'll understand why the correlation functions of $\phi$ should (most of the time) solve the equations of motion.

4. **Schwinger-Dyson equations.**

Consider the path integral

$$\int [D\phi] e^{\mathbf{i}S[\phi]}.$$

Using the fact that the integration measure is independent of the choice of field variable, we have

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} \text{(anything)}$$

(as long as 'anything' doesn't grow at large $\phi$). So this equation says that we can integrate by parts in the functional integral.

(Why is this true? As always when questions about functional calculus arise, you should think of spacetime as discrete and hence the path integral measure as simply the product of integrals of the field value at each spacetime point, $\int [D\phi] \equiv \int \prod_x d\phi(x)$, this is just the statement that

$$0 = \int d\phi_x \frac{\partial}{\partial\phi_x} \text{(anything)}$$

with $\phi_x \equiv \phi(x)$, *i.e.* that we can integrate by parts in an ordinary integral if there is no boundary of the integration region.)

This trivial-seeming set of equations (we get to pick the 'anything') can be quite useful and they are called Schwinger-Dyson equations (or sometimes Ward identities). Unlike many of the other things we'll discuss, they are true non-perturbatively, *i.e.* are really true, even at finite coupling. They provide a quantum implementation of the equations of motion.

(a) Evaluate the RHS of

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} \left(\phi(y) e^{\mathbf{i}S[\phi]}\right)$$

to conclude that

$$\left\langle \mathcal{T} \frac{\delta S}{\delta\phi(x)} \phi(y) \right\rangle = +\mathbf{i}\delta(x-y). \tag{1}$$

(b) These Schwinger-Dyson equations are true in interacting field theories; to get some practice with them we consider here a free theory. Evaluate (1) for the case of a free massive real scalar field to show that the (two-point) time-ordered correlation functions of $\phi$ satisfy the equations of motion, most of the time. That is: the equations of motion are satisfied away from other operator insertions:

$$\left\langle \mathcal{T} \left(+\Box_x + m^2\right) \phi(x)\phi(y) \right\rangle = -\mathbf{i}\delta(x-y), \tag{2}$$

with $\Box_x \equiv \partial_{x^\mu}\partial^{x^\mu}$.

(c) Find the generalization of (2) satisfied by (time-ordered) three-point functions of the free field $\phi$.