University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2016
Assignment 9 ('Final Exam')

Due 11am Thursday, December 8, 2016

1. **The magnetic moment of a Dirac fermion.** [From L. Hall]

   In this problem we consider the hamiltonian density

   $$\mathfrak{h}_I = q\bar{\Psi}\gamma^\mu \Psi A_\mu \ .$$

   As we discussed, this describes a local, Lorentz invariant, and gauge invariant interaction between a Dirac fermion field Ψ and a vector potential A_μ. In this problem, we will treat the vector potential, representing the electromagnetic field, as a fixed, classical background field.

   Define single-particle states of the Dirac field by $\langle 0| \Psi(x) |\vec{p}, s\rangle = e^{-\mathbf{i}px} u^s(p)$. We wish to show that these particles have a magnetic dipole moment, in the sense that in their rest frame, their (single-particle) hamiltonian has a term $h_{NR} \ni \mu_B \vec{S}\cdot\vec{B}$ where $\vec{S} = \frac{1}{2}\vec{\sigma}$ is the particle's spin operator.

   (a) q is a real number. What is required of A_μ for $H_I = \int d^3x \mathfrak{h}_I$ to be hermitian?

   (b) How must A_μ transform under partiy P and charge conjugation C in order for H_I to be invariant? How to electric and magnetic fields transform? Show that this allows for a magnetic dipole moment but not an electric dipole moment.

   (c) Show that in the non-relativistic limit

   $$\bar{u}(p')\gamma^{\mu\nu} u'(p) F_{\mu\nu} = a\xi^\dagger \sigma \cdot \vec{B}\xi'$$

   for some constant a (find a) where u, u' are positive-energy solutions of the Dirac equation with mass m and $u \overset{NR}{\to} \sqrt{m}(\xi,\xi), u' \overset{NR}{\to} \sqrt{m}(\xi',\xi')$ in the non-relativistic limit.

   (d) Suppose that A_μ describes a magnetic field $\vec{B}$ which is uniform in space and time.

   Show that in the non-relativistic limit

   $$\langle \vec{p}', s'| H_I |\vec{p}, s\rangle = \not{\delta}^3 (\vec{p} - \vec{p}')\, h(\xi, \xi', \vec{B})$$

and find the function $h(\xi, \xi', \vec{B})$. You may wish to use the Gordon identity. Rewrite the result in terms of single-particle states with non-relativistic normalization (*i.e.* $\langle \vec{p}|\vec{p}'\rangle_{NR} = \not{\delta}^3(p-p')$). Interpret h as a non-relativistic hamiltonian term saying that the gyromagnetic ratio of the electron is $-g\frac{|q|}{2m}$ with $g = 2$.

(e) [optional] How does the result change if we add the term

$$\Delta H = \frac{c}{M}\bar{\Psi}F_{\mu\nu}[\gamma^\mu, \gamma^\nu]\Psi \quad ?$$

2. **Electron-positron scattering.**

   Draw and evaluate the two diagrams which contribute to $e^+e^- \to e^+e^-$ (Bhabha) scattering at tree level in QED. Be careful about the relative sign of their contributions.

   Compare with the case of $e^+e^- \to \mu^+\mu^-$ and with $e^-e^- \to e^-e^-$.

Do a subset of the following problems.

3. **Non-relativistic interactions from QFT.**

   (a) **Yukawa theory.**

   Consider $\Psi\bar{\Psi} \to \Psi\bar{\Psi}$ scattering in the Yukawa theory. Draw and evaluate the relevant Feynman diagrams.

   Show that the force arising in the non-relativistic limit is attractive.

   (b) **Yukawa theory with distinguishable fermions.** Consider two species of Dirac fermions, interacting with the same scalar field via Yukawa interactions

   $$V = -g_1\bar{\Psi}_1\Psi_1\phi - g_2\bar{\Psi}_2\Psi_2\phi.$$

   Show that if $g_1 g_2 < 0$, the force between a Ψ_1 particle and a Ψ_2 particle is repulsive.

   (c) **Coulomb potential.**

   Derive from QED that the force between electrons is a repulsive $1/r^2$ force law!

   (d) **Pseudoscalar Yukawa theory.**

   Consider the theory of a massive Dirac fermion Ψ and a massive pseudoscalar φ interacting via the term

   $$V_5 \equiv g_5\bar{\Psi}\gamma^5\Psi\varphi.$$

Convince yourself that this theory is parity invariant.

List the Feynman rules.

Draw and evaluate the diagrams contributing to $\Psi\Psi \to \Psi\Psi$ scattering at leading order in g_5.

Consider the non-relativistic limit, $m \gg |\vec{p}|$ and find the effective interaction hamiltonian. If you happen to find zero for the leading term, then it's not the leading term.

4. **Meson scattering.**

   Consider the Yukawa theory again. Draw the Feynman diagram which gives the leading contribution to the process $\phi\phi \to \phi\phi$.

   Derive the correct sign of the amplitude by considering the relevant matrix elements of powers of the interaction hamiltonian.

   Evaluate the diagram in terms of a spinor trace and a momentum integral. Do not do the momentum integral. Suppose that the integral is cutoff at large k by some cutoff Λ. Estimate the dependence on Λ.

5. **Electron-photon scattering at low energy.**

   Consider the process $e\gamma \to e\gamma$ at leading order.

   Draw and evaluate the two diagrams.

   Find $\frac{1}{4}\sum_{\text{spins/polarizations}} |\mathcal{M}|^2$.

   Construct the two-body final-state phase space measure in the limit where the photon frequency is $\omega \ll m$ (the electron mass), in the rest frame of the electron. I suggest the following kinematical variables:

   $$p_1 = (\omega, 0, 0, \omega), p_2 = (m, 0, 0, 0), p_4 = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta), p_3 = p_1 + p_2 - p_4 = (E', p')$$

   for the incoming photon, incoming electron, outgoing photon and outgoing electron respectively.

   Find the differential cross section $\frac{d\sigma}{d\cos\theta}$ as a function of ω, θ, m. (The expression can be prettified by using the on-shell condition $p_3^2 = m^2$ to relate ω' to ω, θ. It is named after Klein and Nishina.)

   Show that the limit $E \ll m$ gives the (Thomson) scattering cross section for classical electromagnetic radiation from a free electron.