University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2016
Assignment 7

Due 11am Thursday, November 17, 2016

1. **Charge conjugation in complex scalar field theory.**

   Consider again a free complex Klein-Gordon field Φ. Define a discrete symmetry operation (charge conjugation) C, by

   $$\Phi(x) \mapsto C\Phi(x)C^{-1} = \eta_c \Phi^\dagger(x)$$

   where C is a unitary operator, and η_c is an arbitrary phase factor. Assume that the vacuum is invariant under charge conjugation: $C|0\rangle = |0\rangle$.

   (a) Show that the free lagrangian is invariant under C , but the particle number current j^μ changes sign.

   (b) Show that the annihilation operators satisfy

   $$C\mathbf{a}_k C^{-1} = \eta_c \mathbf{b}_k, \quad C\mathbf{b}_k C^{-1} = \eta_c^\star \mathbf{a}_k$$

   and hence show that C interchanges particle and antiparticle states, up to a phase.

2. $\mathbb{Z}_2$ **symmetry of real scalar field theory.**

   What does the operator

   $$U \equiv e^{\mathbf{i}\pi \sum_k \mathbf{a}_k^\dagger \mathbf{a}_k}$$

   do to the real scalar field

   $$\phi(x) \to \phi'(x) = U\phi(x)U^\dagger$$

   whose ladder operators are $\mathbf{a}, \mathbf{a}^\dagger$?

   For which Lagrangians is this a symmetry?

3. **Parity symmetry of real scalar field theory.**

   Under the parity transformation

   $$\vec{x} \mapsto \vec{x}' = -\vec{x}$$

   a real Klein-Gordon transforms as

   $$\phi(t,\vec{x}) \mapsto P\phi(t,x)P^{-1} = \eta_p \phi(t,-\vec{x}) \tag{1}$$

   where P is unitary and $\eta_P = \pm 1$ is the *intrinsic parity* of the field ϕ. Again assume $P|0\rangle = |0\rangle$.

(a) Show that the parity transformation preserves the free Lagrangian, for both values of η_P.

(b) Show that an arbitrary n-particle state transforms as

$$P\left|\vec{k}_1, \cdots \vec{k}_n\right\rangle = \eta_P^n \left|-\vec{k}_1, \cdots, -\vec{k}_n\right\rangle.$$

(c) Here we give an explicit realization of the parity operator. Let

$$P_1 \equiv e^{-\mathbf{i}\frac{\pi}{2}\sum_k \mathbf{a}_k^\dagger \mathbf{a}_k}, \quad P_2 \equiv e^{\mathbf{i}\eta_p \frac{\pi}{2}\sum_k \mathbf{a}_k^\dagger \mathbf{a}_{-k}}.$$

Show that

$$P_1 \mathbf{a}_k P_1^{-1} = \mathbf{i}\mathbf{a}_k, \quad P_2 \mathbf{a}_k P_2^{-1} = -\mathbf{i}\eta_p \mathbf{a}_{-k}.$$

Hint: Use the following version of the Campbell-Baker-Hausdorff formula

$$e^{\mathbf{i}\alpha A} B e^{-\mathbf{i}\alpha A} = \sum_{n=0}^{\infty} \frac{(\mathbf{i}\alpha)^n}{n!} B_n$$

where $B_0 \equiv B$ and $B_n = [A, B_{n-1}]$ for $n = 1, 2.....$

Show that $P \equiv P_1 P_2$ is unitary, and satisfies (1).

4. **Action on the current.** [This problem is optional, since I added it late.]

Consider a complex scalar field. Using the results from problems 1 and 3, find the action of parity on the particle current $j^\mu \mapsto P j^\mu P^{-1}$ and $j^\mu \mapsto C j^\mu C^{-1}$. (You'll have to extend the action of P from the case of a real field to the complex case.)