Physics 215A: Particles and Fields
Fall 2016

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0.1 Introductory remarks

Quantum field theory (QFT) is the quantum mechanics of *extensive degrees of freedom*. What I mean by this is that at each point of space, there's some stuff that can wiggle.

It's not surprising that QFT is so useful, since this situation happens all over the place. Some examples of 'stuff' are: the atoms in a solid, or the electrons in those atoms, or the spins of those electrons. A less obvious, but more visible, example is the electromagnetic field, even in vacuum. More examples are provided by other excitations of the vacuum, and it will be our job here to understand those very electrons and atoms that make up a solid in these terms. The vacuum has other less-long-lasting excitations which are described by the Standard Model of particle physics.

Some examples of QFT are Lorentz invariant ('relativistic'). That's a nice simplification when it happens. Indeed this seems to happen in particle physics. We're going to focus on this case for most of this quarter. Still I would like to emphasize: though some of the most successful applications of QFT are in the domain of high energy particle physics, this is not a class on that subject, and I will look for opportunities to emphasize the universality of the subject.

A consequence of relativity is that the number of particles isn't fixed. That is: there are processes where the number of particles changes in time. This is a crucial point of departure for QFT, worth emphasizing, so let me stop and emphasize it. (Later on we'll understand in what sense it's a necessary consequence of Lorentz symmetry. The converse is false: particle production can happen even without relativity.)

Single-particle QM. In classes with the title 'Quantum Mechanics', we generally study quantum systems where the Hilbert space $\mathcal{H}_1$ holds states of a single particle (or sometimes a fixed small number of them).

The observables of such a system are represented by hermitian operators acting on $\mathcal{H}_1$. For example, the particle has a position $\vec{\mathbf{x}}$ and a momentum $\vec{\mathbf{p}}$ each of which is a d-vector of operators (for a particle in d space dimensions). The particle could be an electron (in which case it also has an inherent two-valuedness called spin) or a photon (in which case it also has an inherent two-valuedness called polarization).

Time evolution is generated by a Hamiltonian $\mathbf{H}$ which is made from the position and momentum (and whatever internal degrees of freedom it has), $\mathbf{i}\hbar\partial_t |\psi\rangle = \mathbf{H} |\psi\rangle$. Finally, the fourth (most ersatz) axiom regards measurement: when measuring an observable $\mathbf{A}$ in a state $|\psi\rangle \in \mathcal{H}$, we should decompose the state the eigenbasis $\mathbf{A}|a\rangle = a|a\rangle$, $|\psi\rangle = \sum_a \langle a|\psi\rangle |a\rangle$; the probability to get the answer a is $|\langle a|\psi\rangle|^2$.

By the way: The components of the state vector in the position basis $\langle \vec{x}|\psi\rangle = \psi(\vec{x})$ is a function of space, the wavefunction. This looks like a field. It is not what we mean by a field in QFT. Meaningless phrases like 'second quantization' may conspire to try to confuse you about this.

Now suppose you want to describe quantumly the emission of a photon from an excited electron in an atom. Surely this is something for which we need QM. How do you do it?

In the first section of this course we'll follow an organic route to discovering an answer to this question. This will have the advantage of making it manifest that the four axioms of QM just reviewed are still true in QFT. It will de-emphasize the role of Lorentz symmetry; in fact it will explicitly break it. It will emerge on its own!

'Divergences'. Another intrinsic and famous feature of QFT discernible from the definition I gave above is its flirtation with infinity. I said that there is 'stuff at each point of space'; how much stuff is that? Well, there are two senses in which 'the number of points of space' is infinite: space can go on forever (the infrared (IR)), and, in the continuum, in between any two points of space are more points (the ultraviolet (UV)). The former may be familiar from statistical mechanics, where it is associated with the thermodynamic limit, which is where interesting things happen. For our own safety, we'll begin our discussion in a padded room, protected on both sides from the terrors of the infinite.

Sources and acknowledgement. The material in these notes is collected from many places, among which I should mention in particular the following:

Peskin and Schroeder, *An introduction to quantum field theory* (Wiley)

Zee, *Quantum Field Theory* (Princeton, 2d Edition)

Le Bellac, *Quantum Physics* (Cambridge)

Schwartz, *Quantum field theory and the standard model* (Cambridge)

David Tong's lecture notes

Many other bits of wisdom come from the Berkeley QFT course of Prof. Lawrence Hall.

0.2 Conventions

Following most QFT books, I am going to use the $+---$ signature convention for the Minkowski metric. I am used to the other convention, where time is the weird one, so I'll need your help checking my signs. More explicitly, denoting a small spacetime displacement as $dx^\mu \equiv (dt, d\vec{x})^\mu$, the Lorentz-invariant distance is:

$$ds^2 = +dt^2 - d\vec{x}\cdot d\vec{x} = \eta_{\mu\nu}dx^\mu dx^\nu \quad \text{with} \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} .$$

(spacelike is negative). We will also write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\partial_t, \vec{\nabla}_x\right)^\mu$, and $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$. I'll use μ, ν... for Lorentz indices, and $i, k, ...$ for spatial indices.

The convention that repeated indices are summed is always in effect unless otherwise indicated.

A consequence of the fact that english and math are written from left to right is that time goes to the left.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is

$$đk \equiv \frac{\mathrm{d}k}{2\pi}.$$

I will also write $\not{\delta}^d(q) \equiv (2\pi)^d \delta^{(d)}(q)$. I will try to be consistent about writing Fourier transforms as

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ikx}\tilde{f}(k) \equiv \int đ^d k \; e^{ikx}\tilde{f}(k) \equiv f(x).$$

IFF $\equiv$ if and only if.
RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.
IBP $\equiv$ integration by parts. WLOG $\equiv$ without loss of generality.
$+\mathcal{O}(x^n) \equiv$ plus terms which go like x^n (and higher powers) when x is small.
$+h.c. \equiv$ plus hermitian conjugate.

We work in units where $\hbar$ and the speed of light, c, are equal to one unless otherwise noted. When I say 'Peskin' I mean 'Peskin & Schroeder'.

Please tell me if you find typos or errors or violations of the rules above.

1 From particles to fields to particles again

Here is a way to discover QFT starting with some prosaic ingredients. Besides the advantages mentioned above, it will allows us to check that we are on the right track with simple experiments.

1.1 Quantum sound: Phonons

Let's think about a crystalline solid. The specific heat of solids (how much do you have to heat it up to change its internal energy by a given amount) was a mystery before QM. The first decent (QM) model was due to Einstein, where he supposed that each atom is a (independent) quantum harmonic oscillator with frequency ω. This correctly predicts that the specific heat decreases as the temperature is lowered, but is very crude. Obviously the atoms interact: that's why they make a nice crystal pattern, and that's why there are sound waves, as we will see. By treating the elasticity of the solid quantum mechanically, we are going to discover quantum field theory. One immediate benefit of this will be a framework for quantum mechanics where particles can be created and annihilated.

As a more accurate toy model of a one-dimensional crystalline solid, let's consider a linear chain of particles of mass m, each connected to its neighbors by springs with spring constant κ. When in equilibrium, the masses form a regular one-dimensional crystal lattice (equally spaced mass points). Now let q_n denote the displacement of the nth mass from its equilibrium position x_n and let p_n be the corresponding momentum. Assume there are N masses and (for simplicity) impose periodic boundary conditions: $q_{n+N} = q_n$. The equilibrium positions themselves are

$$x_n = na, n = 1, 2...N$$

where a is the lattice spacing. The Hamiltonian for the collection of particles is:

$$\mathbf{H} = \sum_{n=1}^{N} \left(\frac{\mathbf{p}_n^2}{2m} + \frac{1}{2}\kappa \left(\mathbf{q}_n - \mathbf{q}_{n-1}\right)^2 \right) + \lambda \mathbf{q}^4. \tag{1.1}$$

Notice that this system is an ordinary QM system, made of *particles*. In particular, the whole story below will take place within the fixed Hilbert space of the positions of the N particles.

I've included a token anharmonic term $\lambda \mathbf{q}^4$ to remind us that we are leaving stuff out; for example we might worry whether we could use this model to describe *melting*. Now set $\lambda = 0$. (It will be a little while before we turn back on the *interactions* resulting from nonzero λ; bear with me.) This hamiltonian above describes a collection of coupled harmonic oscillators[1], with a matrix of spring constants $V = k_{ab}\mathbf{q}_a\mathbf{q}_b$. If we diagonalize the matrix of spring constants, we will have a description in terms of decoupled oscillators, called *normal modes*.

Since our system has (discrete) translation invariance, these modes are labelled by a wavenumber k[2]:

$$q_k = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{\mathbf{i}kx_n} q_n, \quad p_k = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{\mathbf{i}kx_n} p_n,$$

(Notice that in the previous expressions I didn't use boldface; that's because this step is really just classical physics. Note the awkward (but in field theory, inevitable) fact that

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[1] In case you are rusty, or forget the numerical factors like I do, here is a review of the operator solution of the SHO:

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{q}^2 = \frac{\hbar\omega}{2}\left(\mathbf{P}^2 + \mathbf{Q}^2\right) = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

with

$$\mathbf{a} \equiv \frac{1}{\sqrt{2}}\left(\mathbf{Q} + \mathbf{i}\mathbf{P}\right), \quad \mathbf{a}^\dagger \equiv \frac{1}{\sqrt{2}}\left(\mathbf{Q} - \mathbf{i}\mathbf{P}\right).$$

Here I've defined these new operators to hide the annoying factors:

$$\mathbf{Q} \equiv \left(\frac{m\omega}{\hbar}\right)^{1/2}\mathbf{q}, \quad \mathbf{P} \equiv \left(\frac{1}{m\hbar\omega}\right)^{1/2}\mathbf{p}.$$

$$[\mathbf{q}, \mathbf{p}] = \mathbf{i}\hbar\mathbb{1} \qquad \Longrightarrow \qquad [\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{1}.$$

The *number operator* $\mathbf{N} \equiv \mathbf{a}^\dagger\mathbf{a}$ satisfies

$$[\mathbf{N}, \mathbf{a}] = -\mathbf{a}, \quad [\mathbf{N}, \mathbf{a}^\dagger] = +\mathbf{a}^\dagger \quad .$$

So $\mathbf{a}$ and $\mathbf{a}^\dagger$ are lowering and raising operators for the number operator. The eigenvalues of the number operator have to be positive, since

$$0 \leq \|\, \mathbf{a}\left|n\right\rangle \|^2 = \left\langle n\right|\mathbf{a}^\dagger\mathbf{a}\left|n\right\rangle = \left\langle n\right|\mathbf{N}\left|n\right\rangle = n\left\langle n|n\right\rangle$$

which means that for $n = 0$ we have $\mathbf{a}\left|n = 0\right\rangle = 0$. If it isn't zero (*i.e.* if $n \geq 1$), $\mathbf{a}\left|n\right\rangle$ is also an eigenvector of $\mathbf{N}$ with eigenvalue $n - 1$. It has to stop somewhere! So the eigenstates of $\mathbf{N}$ (and hence of $\mathbf{H} = \hbar\omega\left(\mathbf{N} + \frac{1}{2}\right)$ are

$$\left|0\right\rangle, \quad \left|1\right\rangle \equiv \mathbf{a}^\dagger\left|0\right\rangle, \quad ..., \left|n\right\rangle = c_n\left(\mathbf{a}^\dagger\right)^n\left|0\right\rangle ...$$

where we must choose c_n to normalize these states. The answer which gives $\left\langle n|n\right\rangle = 1$ is $c_n = \frac{1}{\sqrt{n!}}$.

[2] The inverse transformation is:

$$q_n = \frac{1}{\sqrt{N}}\sum_{k>0}^{2\pi/a} e^{-\mathbf{i}kx_n} q_k, \quad p_n = \frac{1}{\sqrt{N}}\sum_{k>0}^{2\pi/a} e^{-\mathbf{i}kx_n} p_k. \tag{1.2}$$

we'll have (field) momentum operators $\mathbf{p}_k$ labelled by a wavenumber aka momentum.) The nice thing about the fourier kernel is that it diagonalizes the translation operator:

$$\mathbf{T}e^{\mathbf{i}kx} \equiv e^{\mathbf{i}k(x+a)} = e^{\mathbf{i}ka}e^{\mathbf{i}kx}.$$

Regulators: Because N is finite, k takes discrete values ($1 = e^{\mathbf{i}kNa}$); this is a long-wavelength "IR" property. Because of the lattice structure, k is periodic (only $e^{\mathbf{i}kan}, n \in \mathbb{Z}$ appears): $k \equiv k + 2\pi/a$; this is a short-distance "UV" property. The range of k can be taken to be

$$0 \leq k \leq \frac{2\pi(N-1)}{Na} .$$

Because of the periodicity in k, we can equivalently label the set of wavenumbers by[3]:

$$0 < k \leq \frac{2\pi}{a} \quad \text{or} \quad -\frac{\pi}{a} < k \leq \frac{\pi}{a}. \tag{1.3}$$

Summary: Because the system is in a box (periodic), k-space is discrete. Because the system is on a lattice, k-space is periodic. There are N oscillator modes altogether.

The whole hamiltonian is a bunch of decoupled oscillators, labelled by these funny wave numbers:

$$\mathbf{H} = \sum_k \left(\frac{\mathbf{p}_k\mathbf{p}_{-k}}{2m} + \frac{1}{2}m\omega_k^2\mathbf{q}_k\mathbf{q}_{-k} \right) \tag{1.4}$$

where the frequency of the mode labelled k is

$$\omega_k \equiv 2\sqrt{\frac{\kappa}{m}} \sin\frac{|k|a}{2}. \tag{1.5}$$

Why might we care about this frequency? For one thing, consider the Heisenberg equation of motion for the deviation of one spring:

$$\mathbf{i}\partial_t\mathbf{q}_n = [\mathbf{q}_n, \mathbf{H}] = \frac{\mathbf{p}_n}{m}, \quad \mathbf{i}\partial_t\mathbf{p}_n = [\mathbf{p}_n, \mathbf{H}]$$

Combining these gives:

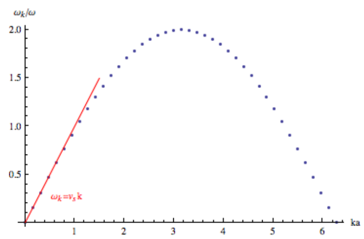
$$m\ddot{q}_n = -\kappa\left((q_n - q_{n-1}) - (q_n - q_{n+1})\right) = -\kappa\left(2q_n - q_{n-1} - q_{n+1}\right).$$

In terms of the fourier-mode operators:

$$m\ddot{\mathbf{q}}_k = -\kappa\left(2 - 2\cos ka\right)\mathbf{q}_k \ .$$

[3]This range of independent values of the wavenumber in a lattice model is called the *Brillouin zone*. There is some convention for choosing a fundamental domain which prefers the last one but I haven't found a reason to care about this.

Plugging in a fourier ansatz in time $q_k(t) = \sum_\omega e^{-\mathbf{i}\omega t} q_{k,\omega}$ turns this into an algebraic equation which says $\omega^2 = \omega_k^2 = \left(\frac{2\kappa}{m}\right) \sin^2 \frac{|k|a}{2}$ for the allowed modes. We see that (the classical version of) this system describes *waves*:



$$0 = \left(\omega^2 - \omega_k^2\right) q_{k,\omega} \overset{k \ll 1/a}{\simeq} \left(\omega^2 - v_s^2 k^2\right) q_{k,\omega}.$$

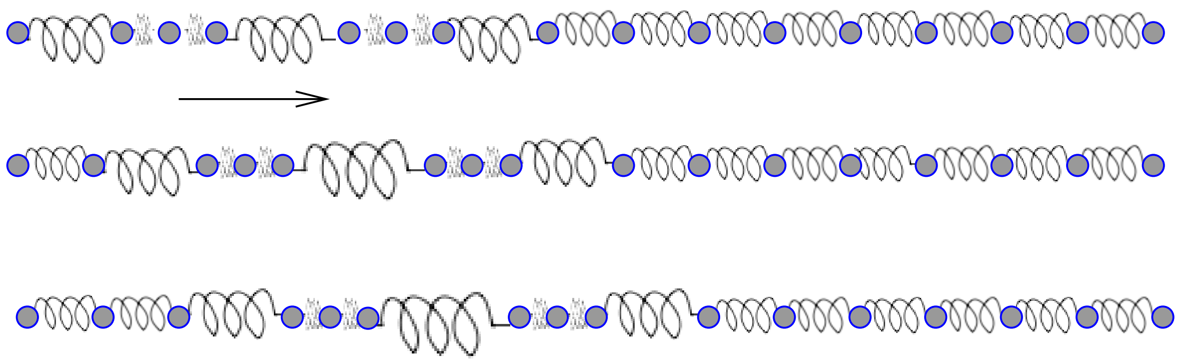
The result for small k is the fourier transform of the wave equation:

$$\left(\partial_t^2 - v_s^2 \partial_x^2\right) q(x,t) = 0 \ . \tag{1.6}$$

v_s is the speed of propagation of the waves, in this case the speed of sound. Comparing to the dispersion relation (1.5), we have found

$$v_s = \frac{\partial \omega_k}{\partial k}|_{k\to 0} = a\sqrt{\frac{\kappa}{m}}.$$

The wave looks something like this:



So the story I am telling is a quantization of sound waves. Soon we will quantize electromagnetic (EM) waves, too.

So far the fact that quantumly $[\mathbf{q}_n, \mathbf{p}_{n'}] = \mathbf{i}\hbar \delta_{nn'} \mathbb{1}$ hasn't really mattered in our analysis (go back and check – we could have derived the wave equation classically). For the Fourier modes, this implies the commutator

$$[\mathbf{q}_k, \mathbf{p}_{k'}] = \sum_{n,n'} \mathbf{U}_{kn} \mathbf{U}_{k'n'} [\mathbf{q}_n, \mathbf{p}_{n'}] = \mathbf{i}\hbar \mathbb{1} \sum_n \mathbf{U}_{kn} \mathbf{U}_{k'n} = \mathbf{i}\hbar \delta_{k,-k'} \mathbb{1}.$$

(In the previous expression I called $\mathbf{U}_{kn} = \frac{1}{\sqrt{N}} e^{\mathbf{i}kx_n}$ the unitary matrix realizing the discrete Fourier kernel.)

To make the final step to decouple the modes with k and $-k$, introduce the annihilation and creation operators[4]

$$\textbf{For } k \neq 0\textbf{:} \qquad \mathbf{q}_k = \sqrt{\frac{\hbar}{2m\omega_k}}\left(\mathbf{a}_k + \mathbf{a}_{-k}^\dagger\right), \quad \mathbf{p}_k = \frac{1}{\mathbf{i}}\sqrt{\frac{\hbar m\omega_k}{2}}\left(\mathbf{a}_k - \mathbf{a}_{-k}^\dagger\right).$$

They satisfy

$$[\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = \delta_{kk'}\mathbb{1}.$$

In terms of these, the hamiltonian is

$$\mathbf{H}_0 = \sum_k \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2}\right) + \frac{p_0^2}{2m}$$

– it is a sum of decoupled oscillators, and a free particle describing the center-of-mass.

The discovery of Fock space (aka particle phononemology[5]). The ground state satisfies $\mathbf{a}_k |0\rangle = 0$ for all k (and has eigenvalue zero for the center-of-mass momentum, $p_0 = 0$). The first excitation above the ground state

$$\mathbf{a}_k^\dagger |0\rangle \propto |\text{one phonon with momentum } \hbar k\rangle \tag{1.7}$$

has energy $\hbar\omega_k$. It is called a *phonon* with momentum $\hbar k$.[6] This is what in undergrad QM we would have called "$|k\rangle$"; we can make a state with one phonon in a position eigenstate by taking superpositions:

$$|\text{one phonon at position } x\rangle = \sum_k e^{\mathbf{i}kx} |\text{one phonon with momentum } \hbar k\rangle \sim \sum_k e^{\mathbf{i}kx}\mathbf{a}_k^\dagger |0\rangle .$$

The number operator (of the SHO with label k) $\mathbf{N}_k \equiv \mathbf{a}_k^\dagger \mathbf{a}_k$ counts the number of phonons with momentum k. The ground state is the state with no phonons. We can also make a state with two phonons:

$$|k, k'\rangle = \mathbf{a}_k^\dagger \mathbf{a}_{k'}^\dagger |0\rangle$$

whose energy is $E = \omega_k + \omega_{k'}$. Note that all these states have non-negative energy.

So this construction allows us to describe situations where the number of particles $\mathbf{N} = \sum_k \mathbf{N}_k$ can vary! That is, we can now describe dynamical processes in which the

[4]You might notice that when $k = 0, \omega_k = 0$. This last step applies to the modes with $\omega_k \neq 0$, hence $k \neq 0$. The 'zero-mode' must be treated specially. It is neglected in many treatments of this topic but actually as a lot of physics in it. If you are curious see this discussion, page 11.

[5]This little gem is due to Sami Ortoleva.

[6]I put 'proportional to' rather than 'equal' in (1.7) because there can be a k-dependent normalization factor. We'll see soon that Lorentz symmetry prefers a particular normalization here which we will adopt.

number of particles changes. Let me emphasize: In QM, we would describe the hilbert space of two (distinguishable) particles as a tensor product of the hilbert space of each. How can we act with an operator which enlarges the hilbert space?? We just figured out how to do it.

We can specify basis states for this Hilbert space

$$\left(\mathbf{a}^\dagger_{k_1}\right)^{n_{k_1}} \left(\mathbf{a}^\dagger_{k_2}\right)^{n_{k_2}} \cdots |0\rangle = |\{n_{k_1}, n_{k_2}, ...\}\rangle$$

by a collection of *occupation numbers* n_k, eigenvalues of the number operator for each normal mode (and the center-of-mass momentum p_0).

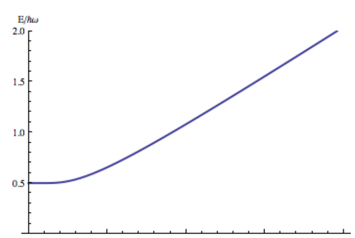
Notice that in this description it is manifest that a phonon has no identity. We only keep track of how many of them there are and what is their momentum. They cannot be distinguished. Also notice that we can have as many we want in the same mode – n_k can be any non-negative integer. These are an example of *bosons*.

[End of Lecture 1]

Notice that there are some energies where there aren't any phonon states. In particular, the function (1.5) has a maximum. More generally, in a system with discrete translation invariance, there are *bands* of allowed energies. In the continuum limit, to which we devolve soon, this maximum goes off to the sky.

Heat capacity of (insulating) solids: phonons are real. The simplest demonstration that phonons are real is the dramatic decrease at low temperatures of the heat capacity of insulating solids. At high temperatures, the equipartition theorem of classical thermodynamics correctly predicts that the energy of the solid from the lattice vibrations should be T times the number of atoms, so the pacity, $C_V = \partial_T E$ should be independent of T.

At low temperatures $T < \Theta_D$, this is wrong. Θ_D is the temperature scale associated with the frequencies of the lattice vibrations (say the maximum of the curve ω_k above). The resolution lies in the thermal energy of a quantum harmonic oscillator for $T < \omega$, the energy goes to a constant $\frac{1}{2}\hbar\omega$:



So the heat capacity (the slope of this curve) goes to zero as $T \to 0$.

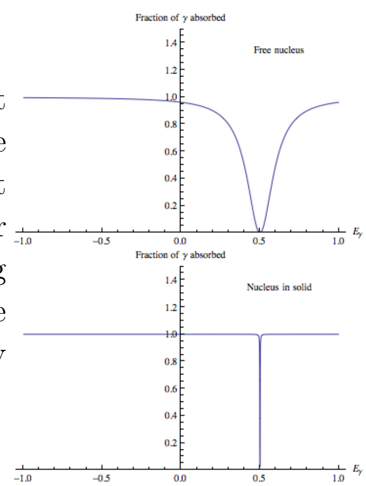
The Mössbauer effect: phonons are real. Here is another dramatic consequence of the quantization of the lattice vibrations of solids, known as the *Mössbauer effect*, first described in words. The nuclei of the atoms in a solid have various energy levels; when hit with a γ-ray photon, these nuclei can experience transitions from the groundstate to some excited energy level. If an excited nucleus somewhere in the lattice gets hit by a very energetic photon (a γ-ray) of some very specific energy

$E_\gamma = \Delta E \equiv E_{\text{excited}} - E_0$, the nucleus can absorb and re-emit that photon. The resulting sharp resonant absorption lines at $E_\gamma = \Delta E$ are indeed observed.

This sounds simple, but here is a mystery about this: Consider a nucleus alone in space in the excited state, after it gets hit by a photon. The photon carried a momentum $p_\gamma = E_\gamma/c$. Momentum is conserved, and it must be made up by some *recoil* of the absorbing nucleus. When it emits a photon again, it needn't do so in the same direction. This means that the nucleus remains in motion with momentum $\Delta\vec{p} = \vec{p}_1 - \vec{p}_2$. But if some of its energy $\Delta E = E_{\text{excited}} - E_0$ goes to kinetic energy of recoil, not all of that energy can go to the final photon, and the emitted photon energy will be *less* than E_γ by $E_{\text{recoil}} = \frac{\Delta p^2}{2M}$. This can be as big as $E^{\max}_{\text{recoil}} = \frac{(2\vec{p})^2}{2M} = \frac{(2E\gamma/c)^2}{2M}$ (in the case of scattering by angle π). So instead of a sharp absorption line, it seems that we should see a broad bump of width $\frac{(E\gamma/c)^2}{M}$. But we *do* see a sharp line!

The solution of the puzzle is phonons: for a nucleus in a lattice, its recoil means that the springs are stretched – it must excite a lattice vibration, it must create some phonons. But there is a nonzero probability for it to create *zero* phonons. In this case, the momentum conservation is made up by an acceleration of *the whole solid*, which is very massive, and therefore does not recoil very much at all (it loses only energy $\frac{p_\gamma^2}{2NM}$).

This allows for very sharp resonance lines. In turn, this effect has allowed for some very high-precision measurements. The different widths in these cartoon absorption spectra don't do justice to the relative factor of N. An essentially similar effect makes it possible to get precise peaks from scattering of X-rays off of a solid (Bragg scattering) – there is a finite amplitude for the scattering to occur without exciting any phonons.



This is actually a remarkable thing: although solids seem ordinary to us because we encounter them frequently, the rigidity of solids is a quantum mechanical emergent phenomenon. You can *elastically* scatter photons off of a solid only because the atoms making up the solid participate in this collective behavior wherein the whole solid acts like a single quantum object!

Towards scalar field theory. It is worthwhile to put together the final relation between the 'position operator' and the phonon annihilation and creation operators:

$$\mathbf{q}_n = \sqrt{\frac{\hbar}{2\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{\mathbf{i}kx_n} \mathbf{a}_k + e^{-\mathbf{i}kx_n} \mathbf{a}_k^\dagger \right) + \frac{1}{\sqrt{N}} \mathbf{q}_0 \tag{1.8}$$

and the corresponding relation for its canonical conjugate momentum

$$\mathbf{p}_n = \frac{1}{\mathbf{i}}\sqrt{\frac{\hbar\mu}{2}}\sum_k \sqrt{\omega_k}\left(e^{\mathbf{i}kx_n}\mathbf{a}_k - e^{-\mathbf{i}kx_n}\mathbf{a}_k^\dagger\right) + \frac{1}{\sqrt{N}}\mathbf{p}_0.$$

The items in red are the ways in which p and q differ; they can all be understood from the relation $p = \mu\dot{q}$ as you will see on the homework. Notice that these expressions are formally identical to the formulae in a QFT textbook expressing a scalar field in terms of creation and annihilation operators (such as Peskin eqns. (2.25) and (2.26)). The stray factors of μ arise because we didn't 'canonically normalize' our fields and absorb the ms into the field, *e.g.* defining $\phi \equiv \sqrt{m}q$ would get rid of them. The other difference is because we still have an IR regulator in place.

Path integral reminder in a box. At this point I will use the path-integral description. Let's remind ourselves how this works for a particle in one dimension with $\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q})$. The basic statement is the following formula for the propagator

$$\langle q| e^{-\mathbf{i}\mathbf{H}t} |q_0\rangle = \int_{q(0)=q_0}^{q(t)=q} [dq] e^{\mathbf{i}\int_0^t \mathrm{d}t\ \left(\frac{1}{2}\dot{q}^2 - V(q)\right)} .$$

Here $[dq] \equiv \mathcal{N}\prod_{l=1}^{M_\tau} dq(t_l)$ – the path integral measure is defined by a limiting procedure, and $\mathcal{N}$ is a normalization factor that always drops out of physical quantities so I don't need to tell you what it is.

Recall that the key step in the derivation of this statement is the evaluation of the propagator for an infinitesimal time:

$$\langle q_2| e^{-\mathbf{i}\mathbf{H}\Delta t} |q_1\rangle = \langle q_2| e^{-\mathbf{i}\Delta t\frac{\mathbf{p}^2}{2m}} e^{-\mathbf{i}\Delta t V(\mathbf{q})} |q_2\rangle + \mathcal{O}(\Delta t^2) .$$

An integral expression for this can be obtained by inserting resolutions of the identity

$$\mathbb{1} = \mathbb{1}^2 = \left(\int \mathrm{d}p\, |p\rangle \langle p|\right)\left(\int \mathrm{d}q\, |q\rangle \langle q|\right)$$

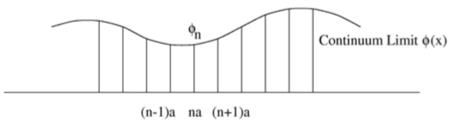
in between the two exponentials. For a more extensive reminder, please see §2.4 of this document.

1.2 Scalar field theory

Scalar field theory in one dimension. Notice that if we use the path integral description, some of these things (in particular the continuum, sound-wave limit) are more obvious-seeming. The path integral for our collection of oscillators is

$$Z = \int [dq_1 \cdots dq_N] e^{\mathbf{i}S[q]}$$

with $S[q] = \int dt \left(\sum_n \frac{1}{2} m_n \dot{q}_n^2 - V(\{q\})\right) \equiv \int dt L(q, \dot{q})$. The potential is $V(\{q\}) = \sum_n \frac{1}{2}\kappa \left(q_{n+1} - q_n\right)^2$. Now let's try to take the continuum limit $a \to 0, N \to \infty$ (now N is the number of points in space, not in time like in the last chapter). Basically the only thing we need is to think of $q_n = q(x = na)$ as defining a smooth function:



[Note that the continuum field is often called $\phi(x)$ instead of $q(x)$ for some reason. At least the letters $q(x)$ and $\phi(x)$ look similar.]

We now have

$$\left(q_n - q_{n-1}\right)^2 \simeq a^2 \left(\partial_x q\right)^2 |_{x=na}$$

Now the path integral becomes:

$$Z = \int [Dq] e^{\mathbf{i} S[q]}$$

with Dq now representing an integral over all configurations $q(t, x)$ (*defined* by this limit) and

$$S[q] = \int dt \int dx \frac{1}{2} \left(\mu \left(\partial_t q\right)^2 - \mu v_s^2 \left(\partial_x q\right)^2 - r q^2 - u q^4 - ...\right) \equiv \int dt \int dx \mathcal{L}$$

where I've introduced some parameters μ, v_s, r, u determined from m, κ... in some ways that we needn't worry about. $\mathcal{L}$ is the Lagrangian *density* whose integral over space is the Lagrangian $L = \int dx \mathcal{L}$.

The equation of motion is the stationary phase condition,

$$0 = \frac{\delta S}{\delta q(x,t)} = -\mu \ddot{q} - \mu v_s^2 \partial_x^2 q - r q - 2 u q^3 - ...$$

In this expression I have written a functional derivative; with our lattice regulator, it is simply a(n extremely useful) shorthand notation for the collection of partial derivatives $\frac{\partial}{\partial q_n}$. [7]

[7] Functional derivatives will be very useful to us. The definition is

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta(x - y) \tag{1.9}$$

plus the Liebniz properties (linearity, product rule). More prosaically, they are just partial derivatives, if we define a collection of values of the independent variable $\{x_n\}$ to regard as grid points, and let

$$\phi_n \equiv \phi(x_n)$$

so that (1.9) is just

$$\frac{\partial \phi_n}{\partial \phi_m} = \delta_{nm}.$$

If you are not yet comfortable with the machinery of functional derivatives, please work through pages 2-28 through 2-30 of this document now.

From the phonon problem, we automatically found $r = u = 0$, and the equation of motion is just the wave equation (1.6). This happened because of the symmetry $q_n \to q_n + \epsilon$. This is the operation that *translates* the whole crystal, It guarantees low-energy phonons near $k = 0$ because it means $q(x)$ can only appear in S via its derivatives. (This is a general property of *Goldstone modes*; more on this later.)

The following will be quite useful for our subsequent discussion of quantum light. We can construct a hamiltonian from this action by defining a canonical field-momentum $\pi(x) = \frac{\partial \mathcal{L}}{\partial_t q} = \mu \partial_t q$ and doing the Legendre transformation:

$$H = \sum_n (p_n \dot{q}_n - L_n) = \int dx\, (\pi(x)\dot{q}(x) - \mathcal{L}) = \int dx \left(\frac{\pi(x)^2}{2\mu} + \mu v_s^2 \left(\partial_x q(x)\right)^2 + r q^2 + u q^4 + ... \right). \tag{1.10}$$

Note that I suppress the dependence of all the fields on t just so it doesn't get ugly, not because it isn't there. Also, I emphasize that the position along the chain x here is just a *label* on the fields, not a degree of freedom or a quantum operator.

The field q is called a *scalar field* because it doesn't have any indices decorating it. This is to be distinguished from the Maxwell field, which is a vector field, and which is our next subject. (Note that vibrations of a crystal in three dimensions actually do involve vector indices! We will omit this complication from our discussion.)

The lattice spacing a and the size of the box Na in the discussion above are playing very specific roles in *regularizing* our 1-dimensional scalar field theory. The lattice spacing a implies a maximum wavenumber or shortest wavelength and so is called an "ultraviolet (UV) cutoff", because the UV is the short-wavelength end of the visible light spectrum. The size of the box Na implies a maximum wavelength mode which fits in the box and so is called an "infrared (IR) cutoff".

If we also take the infinite volume limit, then the sums over k become integrals. In this limit we can make the replacement

$$\frac{1}{L^d}\sum_k \rightsquigarrow \int đ^d k, \quad L^d \delta_{kk'} \rightsquigarrow (2\pi)^d \delta^{(d)}(k - k').$$

A check of the normalization factors comes from combining these two rules $1 = \sum_k \delta_{k,k'} = \int đ^d k (2\pi)^d \delta^{(d)}(k - k')$.

Continuum (free) scalar field theory in $d+1$ dimensions. Notice that these continuum expressions are easy to generalize to scalar field theory in any number of dimensions. Let's do them directly in infinite volume and set $\mu = 1$ by rescaling fields. The action is

$$S[\phi] = \int d^d x dt \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2} v_s^2 \vec{\nabla}\phi \cdot \vec{\nabla}\phi - V(\phi) \right). \tag{1.11}$$

This is what we would have found for the long-wavelength ($ka \ll 1$) description of a d-dimensional lattice of masses on springs, like a mattress. The equation of motion is

$$0 = \frac{\delta S[\phi]}{\delta \phi(x)} = -\partial_t^2 \phi + v_s^2 \nabla^2 \phi - V'(\phi). \tag{1.12}$$

For the harmonic case $V(\phi) = \frac{1}{2} m^2 \phi^2$ we know what we're doing, and (1.12) is called the Klein-Gordon equation,

$$0 = \left(\partial_\mu \partial^\mu + m^2\right) \phi. \tag{1.13}$$

(Notice that I've set $v_s = c = 1$ here, and this is where we have committed to a choice of signature convention; take a look at the conventions page §0.2.). In relativistic notation, the Lagrangian density is just $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right)$. This describes free continuum real massive relativistic scalar quantum field theory. (Match the adjectives to the associated features of the lagrangian; collect them all!)

The canonical momentum is $\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi$ and the Hamiltonian (which we can instantly promote to a quantum operator by using boldface symbols) is then

$$\mathbf{H} = \int d^d x \left(\frac{\boldsymbol{\pi}(x)^2}{2} + \frac{1}{2} v_s^2 \left(\vec\nabla \boldsymbol{\phi} \cdot \vec\nabla \boldsymbol{\phi} \right) + \frac{1}{2} m^2 \boldsymbol{\phi}^2 \right).$$

Note that all these terms are positive. [End of Lecture 2]

A translation invariant problem is solved by Fourier transforms[8]: $\boldsymbol{\phi}(x) = \int đ^d k \, e^{-\mathbf{i}\vec k \cdot \vec x} \boldsymbol{\phi}_k$, and $\boldsymbol{\pi}(x) = \int đ^d k \, e^{-\mathbf{i}\vec k \cdot \vec x} \boldsymbol{\pi}_k$, this is

$$\mathbf{H} = \int đ^d k \left(\frac{1}{2} \boldsymbol{\pi}_k \boldsymbol{\pi}_{-k} + \frac{1}{2}\left(v_s^2 k^2 + m^2\right) \boldsymbol{\phi}_k \boldsymbol{\phi}_{-k} \right)$$

where $k^2 = (-\mathbf{i}\vec k) \cdot (\mathbf{i}\vec k) = \vec k \cdot \vec k$. Just as in (1.4), this is merely a sum of decoupled oscillators, except for the coupling between wavenumbers k and $-k$. Comparing with (1.4), we can read off the normal mode frequencies, aka the dispersion relation:

$$\omega_k^2 = v_s^2 k^2 + m^2.$$

Notice that this is also the condition for a fourier mode $e^{\mathbf{i}\vec k \cdot \vec x - \mathbf{i}\omega t}$ to solve the Klein-Gordon equation (1.13).

We can decouple the modes with wavenumber k and $-k$ as above by introducing the ladder operators

$$\boldsymbol{\phi}_k \equiv \sqrt{\frac{\hbar}{2\omega_k}} \left(\mathbf{a}_k + \mathbf{a}_{-k}^\dagger \right), \quad \boldsymbol{\pi}_k \equiv \frac{1}{\mathbf{i}} \sqrt{\frac{\hbar \omega_k}{2}} \left(\mathbf{a}_k - \mathbf{a}_{-k}^\dagger \right), \qquad [\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = (2\pi)^d \delta^{(d)}(k - k').$$

[8]Beware that the mode operators $\mathbf{a}_k$ defined here differ by powers of $2\pi/L$ from the finite-volume objects in the previous discussion. These agree with Peskin's conventions.

Their commutator follows from $[\phi(x), \pi(y)] = \mathbf{i}\delta^{(d)}(x-y)$. In terms of the ladder operators,

$$\mathbf{H} = \int đ^d k\ \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} \right).$$

The field operators

$$\begin{aligned} \boldsymbol{\phi}(\vec{x}) &= \int đ^d k\ \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}} \mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x}} \mathbf{a}_k^\dagger \right), \\ \boldsymbol{\pi}(\vec{x}) &= \frac{1}{\mathbf{i}} \int đ^d k\ \sqrt{\frac{\hbar\omega_k}{2}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}} \mathbf{a}_k - e^{-\mathbf{i}\vec{k}\cdot\vec{x}} \mathbf{a}_k^\dagger \right), \end{aligned} \tag{1.14}$$

satisfy the canonical commutation relation

$$[\boldsymbol{\phi}(\vec{x}), \boldsymbol{\pi}(\vec{x}')] = \mathbf{i}\hbar \mathbb{1} \delta^d(\vec{x} - \vec{x}').$$

I emphasize that this is really the same equation as our starting point for each ball on springs:

$$[\mathbf{q}_n, \mathbf{p}_{n'}] = \mathbf{i}\hbar \mathbb{1} \delta_{nn'}.$$

The mode expansions (1.14) contain a great deal of information. First notice that $\boldsymbol{\phi}$ is manifestly hermitian. Next, notice that from $\boldsymbol{\phi}(\vec{x}) \equiv \boldsymbol{\phi}(\vec{x}, 0)$ by itself we cannot disentangle $\mathbf{a}_k$ and $\mathbf{a}_k^\dagger$, since only the combination $\mathbf{a}_k + \mathbf{a}_{-k}^\dagger$ multiplies $e^{\mathbf{i}\vec{k}\cdot\vec{x}}$. The momentum $\boldsymbol{\pi}$ contains the other linear combination. However, if we evolve the field operator in time using the Heisenberg equation (as you did on the HW), we find

$$\phi(\vec{x}, t) \equiv e^{\mathbf{i}\mathbf{H}t} \phi(\vec{x}) e^{-\mathbf{i}\mathbf{H}t} = \int đ^d k \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x} - \mathbf{i}\omega_{\vec{k}} t} \mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x} + \mathbf{i}\omega_{\vec{k}} t} \mathbf{a}_k^\dagger \right). \tag{1.15}$$

Indeed we can check that the relation $\boldsymbol{\pi} = \dot{\boldsymbol{\phi}}$ holds.

Notice that the dependence on spacetime is via a sum of terms of the form:

$$e^{\mathbf{i}\vec{k}\cdot\vec{x} - \mathbf{i}\omega_{\vec{k}} t} = e^{\mathbf{i}k_\mu x^\mu}|_{k^0 = \omega_{\vec{k}}}$$

and their complex conjugates. These are precisely all the solutions to the wave equation (1.13). For each $\vec{k}$, there are two solutions, one with positive frequency and one with negative frequency. You might have worried that solutions with both signs of the frequency mean that the world might explode or something (like it would if we tried to replace the Schrödinger equation for the wavefunction with a Klein-Gordon equation). This danger is evaded in a beautiful way: the coefficient of the positive frequency solution with wavenumber $\vec{k}$ is the destruction operator for the mode; the associated negative frequency term comes with the creation operator for the same mode, as a consequence of reality of the field. (Some words about antimatter would be appropriate here, but it will be clearer later when we talk about an example of particles which are not their own antiparticles.)

Finally, in a relativistic system, anything we can say about time should also be true of space, up to some signs. So the fact that we were able to generate the time dependence by conjugation with the unitary operator $e^{\mathbf{iH}t}$ (as in (1.15)) says that we should be able to generate the space dependence by conjugating by a unitary operator of the form $e^{-\mathbf{i}\vec{\mathbf{P}}\cdot\vec{x}}$. Here $\vec{\mathbf{P}}$ is the last in a long list of object with a claim to the name 'momentum'. It is the conserved charge associated with spatial translation symmetry, the generator of spatial translations. Its form in terms of the fields will be revealed below when we speak about Noether's theorem. For now, let me emphasize that is distinct from the objects $p_n, \pi(x)$ (which were 'momenta' in the sense of canonical momenta of various excitations) and also from the wavenumbers $\vec{k}$ of various modes, which (when multiplied by $\hbar$) are indeed spatial momenta of single particles. (This statement gives us an expectation for what is the total momentum of a state of a collection of particles which we will check below in §1.4.) In terms of the momentum operator, then, we can write

$$\boldsymbol{\phi}(x^\mu) = e^{\mathbf{iP}_\mu x^\mu}\boldsymbol{\phi}(0)e^{-\mathbf{iP}_\mu x^\mu}$$

with $\mathbf{P}_\mu \equiv (\mathbf{H}, \vec{\mathbf{P}})_\mu$.

1.3 Quantum light: Photons

The quantization of the Maxwell field is logically very similar to the preceding discussion. There are just a few complications from its several polarizations, and from the fact that quantum mechanics means that the vector potential is real and necessary (whereas classically it is just a convenience). This is a quick-and-dirty version of the story. I mention it here to emphasize that the machinery we are developing applies to a system you have already thought a lot about!

Maxwell's equations are:

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\partial_t\vec{B}, \tag{1.16}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho, \quad \nabla\times\vec{B} = \partial_t\vec{E} + \frac{4\pi}{c}\vec{j}. \tag{1.17}$$

The first two equations (1.16) are constraints on $\vec{E}$ and $\vec{B}$ which mean that their components are not independent. This is annoying for trying to treat them quantumly. To get around this we introduce *potentials* which determine the fields by taking derivatives and which automatically solve the constraints (1.16):

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$

Potentials related by a *gauge transformation*

$$\vec{A} \to \vec{A}^{\lambda} = \vec{A} - \vec{\nabla}\lambda, \quad \Phi \to \Phi^{\lambda} = \Phi + \partial_t \lambda$$

for any function $\lambda(\vec{r}, t)$, give the same $\vec{E}, \vec{B}$. The Bohm-Aharonov effect is proof that (some of the information in) the potential is real and useful, despite this redundancy. We can partially remove this redundancy be choosing our potentials to satisfy *Coulomb gauge*

$$\vec{\nabla} \cdot \vec{A} = 0 \ .$$

In the absence of sources $\rho = 0 = \vec{j}$, we can also set $\Phi = 0$. In this gauge, Ampere's law becomes

$$c^2 \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = c^2 \vec{\nabla} \cdot \left(\vec{\nabla} \cdot \vec{A} \right) - c^2 \nabla^2 \vec{A} = -\partial_t^2 \vec{A} \quad \text{i.e.} \ \boxed{\partial_t^2 \vec{A} - c^2 \nabla^2 \vec{A} = 0}.$$

This wave equation is different from our scalar wave equation (1.6) in three ways:

- we're in three spatial dimensions,
- the speed of sound v_s has been replaced by the speed of light c,
- the field $\vec{A}$ is a vector field obeying the constraint $\vec{\nabla} \cdot \vec{A} = 0$. In fourier space $\vec{A}(x) = \sum_k e^{i\vec{k}\cdot\vec{x}} \vec{A}(k)$ this condition is

  $$0 = \vec{k} \cdot \vec{A}(k)$$

  – the vector field is *transverse*.

Recall that the energy density of a configuration of Maxwell fields is $u = \frac{\epsilon_0}{2} \left(\vec{E}^2 + \vec{B}^2 \right)$. So the quantum Hamlitonian is

$$\mathbf{H} = \frac{\epsilon_0}{2} \int d^3 r \left(\vec{E}^2 + c^2 \vec{B}^2 \right) \quad . \tag{1.18}$$

Here $\vec{E} = -\partial_t \vec{A}$ plays the role of field momentum $\pi(x)$ in (1.10), and $\vec{B} = \vec{\nabla} \times \vec{A}$ plays the role of the spatial derivative $\partial_x q$. We immediately see that we can quantize this system just like for the scalar case, with the canonical commutator

$$[\phi(x), \pi(x')] = \mathbf{i}\hbar\delta(x - x') \qquad \rightsquigarrow \qquad [\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = -\mathbf{i}\hbar\delta^3(\vec{r} - \vec{r}')\delta_{ij}$$

where $i, j = 1..3$ are spatial indices. [9] So we can immediately write down an expression for the quantum Maxwell field in terms of independent creation and annihilation operators:

$$\vec{\mathbf{A}}(\vec{r}) = \sum_{\vec{k}} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k L^3}} \sum_{s=1,2} \left(\mathbf{a}_{\vec{k},s} \vec{e}_s(\hat{k}) e^{\mathbf{i}\vec{k}\cdot\vec{r}} + \mathbf{a}^\dagger_{\vec{k},s} \vec{e}^\star_s(\hat{k}) e^{-\mathbf{i}\vec{k}\cdot\vec{r}} \right)$$

The field momentum is $\vec{\mathbf{E}} = -\partial_t \vec{A}$:

$$\vec{\mathbf{E}}(\vec{r}) = \mathbf{i} \sum_{\vec{k}} \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 L^3}} \sum_{s=1,2} \left(\mathbf{a}_{\vec{k},s} \vec{e}_s(\hat{k}) e^{\mathbf{i}\vec{k}\cdot\vec{r}} - \mathbf{a}^\dagger_{\vec{k},s} \vec{e}^\star_s(\hat{k}) e^{-\mathbf{i}\vec{k}\cdot\vec{r}} \right)$$

[10] Also, the magnetic field operator is

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}} = \sum_{\vec{k}} \sum_{s} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k L^3}} \mathbf{i}\vec{k} \times \left(\mathbf{a}_{\vec{k},s} \vec{e}_s(\hat{k}) e^{\mathbf{i}\vec{k}\cdot\vec{r}} - \mathbf{a}^\dagger_{\vec{k},s} \vec{e}^\star_s(\hat{k}) e^{-\mathbf{i}\vec{k}\cdot\vec{r}} \right) \ ;$$

the magnetic field is analogous to $\vec{\nabla}\phi$ in the scalar field theory. Plugging these expressions into the Hamiltonian (1.18), we can write it in terms of these oscillator modes (which create and annihilate *photons*). As for the scalar field, the definitions of these modes were designed to make this simple: It is:

$$\mathbf{H} = \sum_{\vec{k},s} \hbar\omega_k \left(\mathbf{a}^\dagger_{\vec{k},s} \mathbf{a}_{\vec{k},s} + \frac{1}{2} \right) .$$

Notice that the vacuum energy is

$$E_0 = \frac{1}{2} \sum_{\vec{k},s} \hbar\omega_k = \frac{L^3}{(2\pi)^3} \int d^3k \hbar c k.$$

[9] As a check, note that using this Hamiltonian and the canonical commutator, we can reproduce Maxwell's equations using Ehrenfest's theorem:

$$\left\langle \partial_t^2 \vec{A} \right\rangle = \partial_t \left\langle \vec{E} \right\rangle = -\frac{\mathbf{i}}{\hbar} \left\langle [\mathbf{H}, \vec{E}] \right\rangle = \left\langle c^2 \vec{\nabla}^2 \vec{A} \right\rangle .$$

[10] I am short-changing you a little bit here on an explanation of the *polarization vectors*, $\vec{e}_s$. They conspire to make it so that there are only two independent states for each $\vec{k}$ and they are transverse $\vec{k} \cdot \vec{e}_s(\hat{k}) = 0$, so $s = 1, 2$. The bit that I'm leaving out is the completeness relation satisfied by the polarization vectors of a given k:

$$\sum_s e_{si}(\hat{k}) e^\star_{sj}(\hat{k}) = \delta_{ij} - \hat{k}_i \hat{k}_j \ . \tag{1.19}$$

This says that they span the plane perpendicular to $\hat{k}$.

Notice that in this case we began our story in the continuum, rather than with microscopic particles connected by springs. (However, if you read Maxwell's papers you'll see that he had in mind a particular UV completion involving gears and cogs. I actually don't understand it; if you do please explain it to me.)

The fact that $\sum_k$ is no longer a finite sum might be something to worry about. We will see below in §1.5 that this vacuum energy has physical consequences.

Consolidation of understanding

So far in this chapter, we have studied systems of increasing complexity: the simple harmonic oscillator, a non-interacting scalar field, and the EM field in vacuum (*i.e.* in the absence of charge). All these free field theories have the same structure, in the following sense.

In the following, Here $\mathrm{Re}\,\mathbf{A} \equiv \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^\dagger\right)$ as usual. The normalization constant is $\mathcal{N} = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}$.

$$\mathbf{H}_{SHO} = \frac{1}{2m}\mathbf{p}^2 + \frac{1}{2}m\omega^2\mathbf{q}^2 = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

$$[\mathbf{q}, \mathbf{p}] = \mathbf{i}\hbar \implies [\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{1}.$$

$$\mathbf{q} = \mathrm{Re}\mathcal{N}\mathbf{a}, \quad \mathbf{p} = m\mathrm{Im}\,\omega\mathcal{N}\mathbf{a}.$$

$$\mathbf{H}_{scalar} = \int dx \left(\frac{1}{2\mu}\boldsymbol{\pi}^2 + \frac{1}{2}\mu c^2\left(\partial_x\boldsymbol{\phi}\right)^2\right) = \sum_k \hbar\omega_k\left(\mathbf{a}_k^\dagger\mathbf{a}_k + \frac{1}{2}\right)$$

$$[\boldsymbol{\phi}(x), \boldsymbol{\pi}(x')] = \mathbf{i}\hbar\delta(x - x') \implies [\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = \mathbf{i}\hbar\delta_{kk'}.$$

$$\boldsymbol{\phi}(x) = \mathrm{Re}\left(\sum_k \mathcal{N}_k e^{\mathbf{i}kx}\mathbf{a}_k\right), \quad \boldsymbol{\pi}(x) = \mu\mathrm{Im}\left(\sum_k \omega_k\mathcal{N}_k e^{\mathbf{i}kx}\mathbf{a}_k\right).$$

$$\mathbf{H}_{EM} = \int d^3x \left(\frac{\epsilon_0}{2}\vec{\mathbf{E}}^2 + \frac{\epsilon_0 c^2}{2}\vec{\mathbf{B}}^2\right) = \sum_{k,s=1,2} \hbar\omega_k\left(\mathbf{a}_{ks}^\dagger\mathbf{a}_{ks} + \frac{1}{2}\right)$$

$$[\mathbf{A}_i(x), \mathbf{E}_j(x')] = \mathbf{i}\hbar\delta^3(x - x')\delta_{ij} \implies [\mathbf{a}_{ks}, \mathbf{a}_{k's'}^\dagger] = \hbar\delta_{kk'}\delta_{ss'}.$$

$$\vec{\mathbf{A}}(x) = \mathrm{Re}\left(\sum_k \mathcal{N}_k e^{\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_{ks}\vec{e}_s(\hat{k})\right), \quad \vec{\mathbf{E}}(x) = \mu\mathrm{Im}\left(\sum_k \omega_k\mathcal{N}_k e^{\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_{ks}\vec{e}_s(\hat{k})\right).$$

Note that $\vec{\mathbf{E}}$ is the canonical momentum of $\vec{\mathbf{A}}$ since (in Coulomb gauge) $\vec{E} = -\partial_t\vec{A}$.

1.4 Lagrangian field theory

[Here we fill in the bits of Peskin §2.2 that we missed above.] Let's consider a classical field theory in the Lagrangian description. This means that the degrees of freedom are a set of fields $\phi_r(x)$, where r is a discrete index (for maybe spin or polarization or flavor), and we specify the dynamics by the classical action. If the world is kind to us (in this class we assume this), the action is an integral over space and time of a Lagrangian density

$$S[\phi] \equiv \int d^{d+1}x \mathcal{L}(\phi, \partial^\mu \phi).$$

This important assumption is an implementation of *locality*.

This central object encodes the field equations, the canonical structure on the phase space, the Hamiltonian, the symmetries of the theory.

I've sneakily implied that we are going to assume Lorentz invariance, so that $\mathcal{L}$ depends on the 4-vector $\partial^\mu \phi$, and not its components separately.

I am also going to assume that the action S is real.

We've seen basically two examples so far

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

and (you get to do the Legendre transformation from the Hamiltonian of §1.3 on the homework)

$$\mathcal{L}_{EM} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4e^2}\left(E^2 - B^2\right)$$

with A_μ regarded as the independent degrees of freedom.

A word about units: in units with $\hbar = c = 1$, everything has units of mass to some power, called its mass dimension. Energy and momentum $p_\mu = \hbar k_\mu$ have mass dimension $+1$. The space and time coordinates x^μ have mass dimension -1. The action goes in the exponential of the path integral measure $\int [D\phi] e^{\frac{\mathbf{i}S}{\hbar}}$ and so must be dimensionless. So the Lagrangian density has mass dimension $d+1$. This means that the KG field has mass dimension $\frac{d-1}{2}$ (and the mass m has mass dimension 1 (yay!)). In $d+1 = 3+1$ dimensions, $E \sim \dot{A}, B \sim \vec{\nabla} A$ have mass dimension 2 and A has mass dimension one (and e is dimensionless). This is nice because then the covariant derivative $\partial_\mu + A_\mu$ has mass dimension one. Notice that $E^2 + B^2$ has dimension 4 which is good for an energy per unit volume.

object	mass dim.
p_μ	1
x^μ	-1
S	0
$\mathcal{L}$	$d+1$
ϕ	$\frac{d-1}{2}$
A_μ	1

[End of Lecture 3]

The equation of motion is

$$0 = \frac{\delta S}{\delta \phi_r(x)}.$$

Note the functional derivative. You can check that in the case when $\mathcal{L}$ depends only on ϕ and $\partial_\mu \phi$, this is the same as the Lagrange EOM

$$0 = \frac{\partial \mathcal{L}}{\partial \phi_r} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)}$$

(for each r) which I can't remember.

By redefining the field $\phi \equiv \frac{1}{D}(\chi - B/C)$, we can make the KG theory uglier

$$\mathcal{L} = A + B\chi + \frac{1}{2}C\chi^2 + \frac{1}{2}D\partial^\mu \chi \partial_\mu \chi + ...$$

From the path integral point of view, the field is just an integration variable. Sometimes, its normalization is meaningful, like in the phonon example where it began its life as the displacement of the atoms from their equilibrium. So you see that we are not losing generality except in our neglect of interactions, and in our neglect of terms with more derivatives. The former neglect we will repair little by little in this course, by doing perturbation theory. The latter is justified well by the renormalization group philosophy, which is a subject for Physics 215C, *i.e.* the Spring Quarter.

Canonical field momentum and Hamiltonian. The Hamiltonian viewpoint in field theory has the great virtue of bringing out the physical degrees of freedom. It has the great shortcoming that it picks out the time coordinate as special and obscures Lorentz symmetry.

The canonical field momentum is defined to be

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi(x))}.$$

Notice that this expression assumes a local Lagrangian density. π is actually a 'field momentum density' in the sense that the literal canonical momentum is $\frac{\partial}{\partial \dot{\phi}(x)} L = d^d x \pi(x)$ (as opposed to $\mathcal{L}$). I will often forget to say 'density' here.

The hamiltonian is then

$$H = \sum_n p_n q_n - L = \int d^d x \left(\pi(x)\dot{\phi}(x) - \mathcal{L} \right) \equiv \int d^d x \ \mathfrak{h}.$$

Noether's theorem and the Noether method. Yay, symmetries. Why do physicists love symmetries so much? One reason is that they offer possible resting places along our never-ending chains of 'why?' questions. For example, one answer (certainly the one given in Weinberg's text, but just as certainly not the only one) to the question "Why QFT?" is: quantum mechanics plus Poincaré symmetry.

They are also helpful for solving physical systems: Continuous symmetries are associated with conserved currents. Suppose the action is invariant under a continuous transformation of the fields ϕ, $\phi(x) \mapsto \phi'(x)$. (The invariance of the action is what makes the transformation a symmetry.) 'continuous' here means we can do the transformation just a little bit, so that $\phi(x) \mapsto \phi(x) + \epsilon\Delta\phi(x)$ where ϵ is an infinitesimal parameter. If the transformation with constant ϵ (independent of space and time) is a symmetry, then the variation of the action with $\epsilon = \epsilon(x,t)$ must be proportional to $\partial_\mu\epsilon$ (at least assuming some smoothness properties of the action), and so that it vanishes $\forall\phi$ when ϵ is constant:

$$S[\phi + \epsilon(x)] - S[\phi] = \int d^d x dt \partial_\mu \epsilon(x) j^\mu \stackrel{\text{IBP}}{=} -\int d^d x dt \epsilon(x) \partial_\mu j^\mu \ .$$

But if the equations of motion are obeyed, then the action is invariant under *any* variation, including this one, for arbitrary $\epsilon(x)$. But this means that $\partial_\mu j^\mu = 0$, the current is conserved. These words are an accurate description of the equation because they mean that the charge

$$Q_R \equiv \int_R d^d x\ j^0$$

in some region of space R can only change by leaving the region (assume the definition of R is independent of time):

$$\partial_t Q_R = \int_R d^d x\ \partial_t j^0 = -\int_R d^d x\ \vec{\nabla} \cdot \vec{j} = \int_{\partial R} d^{d-1} x \hat{n} \cdot \vec{j}$$

where in the last step we used Stokes' theorem.

This trick with pretending the parameter depends on space is called the *Noether method*. More prosaically, the condition that the action is invariant means that the Lagrangian density changes by a total derivative (we assume boundary terms in the action can be ignored):

$$\mathcal{L}(\phi', \partial_\mu \phi') \stackrel{\text{symmetry}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \partial_\mu \mathcal{J}^\mu$$

but on the other hand, by Taylor expansion,

$$\begin{aligned}
\mathcal{L}(\phi', \partial_\mu \phi') &\stackrel{\text{calculus}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \left(\frac{\partial \mathcal{L}}{\partial \phi} \Delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\mu \Delta\phi \right) \\
&\stackrel{\text{IBP}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \left(\underbrace{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}}_{eom} \right) \Delta\phi + \epsilon \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \Delta\phi \right) \ .
\end{aligned}$$

By combining the previous two equations for $\mathcal{L}(\phi')$, we see that on configurations which satisfy the EOM, $0 = \partial_\mu j^\mu$ with

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_r\right)} \Delta\phi_r - \mathcal{J}^\mu. \tag{1.20}$$

Notice that I stuck the index back in at the last step.

There is a converse to the Noether theorem, which is easier to discuss directly in quantum mechanics. Given a conserved charge Q, that is, a hermitian operator with $[H, Q] = 0$, we can make a symmetry transformation of the fields ϕ by

$$\delta\phi \equiv \mathbf{i}\epsilon[Q, \phi]. \tag{1.21}$$

We'll say that Q *generates* the symmetry, for the following reason. (1.21) is the infinitesimal version of the finite transformation

$$\phi \to \phi' \equiv e^{\mathbf{i}\epsilon Q}\phi e^{-\mathbf{i}\epsilon Q}.$$

The object $\mathbf{U} \equiv e^{\mathbf{i}\epsilon Q}$ is a unitary operator (since Q is hermitian) which represents the action of the symmetry on the Hilbert space of the QFT. It is a symmetry in the sense that its action commutes with the time evolution operator $e^{\mathbf{i}Ht}$.

Some examples will be useful:

- For example, suppose $S[\phi]$ only depends on ϕ through its derivatives, for example, $S[\phi] = \int \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. Then there is a shift symmetry $\phi \to \phi' \equiv \phi + \epsilon$. Letting ϵ depend on spacetime, the variation of the action is $S[\phi + \epsilon(x)] = \int \epsilon\partial_\mu\partial^\mu\phi$, so the current is $j_\mu = \partial_\mu\phi$. Let's check the converse: Indeed, the charge $Q = \int_{\text{space}} j_0$ generates the symmetry in the sense that for small ϵ, the variation in the field is

$$\delta\phi \equiv \phi' - \phi = \epsilon = \mathbf{i}\epsilon[Q, \phi]$$

  (if we were doing classical mechanics, we should replace $\mathbf{i}[Q, \phi]$ with the Poisson bracket). Using our expression for the current this is

$$\delta\phi = \mathbf{i}\epsilon\Big[\int d^dy\, \underbrace{\dot{\phi}(y)}_{=\pi(y)}, \phi(x)\Big] = \epsilon$$

  which is indeed true. In this case the finite transformation is again $\phi \to \phi + \epsilon$.

- On the homework you're studying a complex scalar Φ, with $S[\Phi, \Phi^\star]$ is invariant under $\Phi \to e^{\mathbf{i}\epsilon}\Phi = \Phi + \mathbf{i}\epsilon\Phi + \mathcal{O}(\epsilon^2)$. This $\mathsf{U}(1)$ phase transformation can be rewritten in terms of the real and imaginary parts as an $\mathsf{SO}(2)$ rotation. You'll find that the charge can be written as

$$Q = \int d^d x j^0 = \int đ^d p \left(\mathbf{a}_p^\dagger \mathbf{a}_p - \mathbf{b}_p^\dagger \mathbf{b}_p \right)$$

where the two sets of creation and annihilation operators are associated with excitations of Φ and $\Phi^\dagger$ respectively. (That is, quantize $\phi_{1,2}$ as we did for a single real scalar field, in terms of mode operators $\mathbf{a}_{1,2}$ respectively. Then let $\mathbf{a} \equiv \mathbf{a}_1 + \mathbf{i}\mathbf{a}_2, \mathbf{b} \equiv \mathbf{a}_1 - \mathbf{i}\mathbf{a}_2$, up to numerical prefactors which I tried to get right in the posted solutions.) So the particles created by $\mathbf{a}$ and $\mathbf{b}$ have opposite charge (this is essential given the mode expansion $\Phi_k \sim \mathbf{a}_k + \mathbf{b}_{-k}^\dagger$) and can be interpreted as each others' antiparticles: there can be symmetry-respecting processes where an $\mathbf{a}$ particle and $\mathbf{b}$ particle take each other out.

- Consider spatial translations, $x^\mu \to x^\mu - a^\mu$. We can think of this as a transformation of the fields by

$$\phi(x) \mapsto \phi(x+a) = \phi(x) + a^\nu \underbrace{\partial_\nu \phi}_{\equiv \Delta_\nu \phi} + \mathcal{O}(a^2).$$

Our transformation parameter is now itself a four-vector, so we'll get a four-vector of currents T^μ_ν. This will be a symmetry as long as the lagrangian doesn't depend explicitly on space and time (so $\partial_\nu \mathcal{L} = 0$) but rather depends on space and time only via the fields (so $0 \neq \frac{d}{dx^\nu}\mathcal{L} \overset{\text{chain rule}}{=} \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$). Let's use the prosaic method for this one: the shift in the Lagrangian density also can be found by Taylor expansion

$$\mathcal{L} \mapsto \mathcal{L} + a^\mu \frac{d}{dx^\mu}\mathcal{L} = \mathcal{L} + a^\nu \partial_\mu \left(\delta^\mu_\nu \mathcal{L} \right).$$

So the formula (1.20) gives

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \underbrace{\partial_\nu \phi}_{\Delta_\nu \phi} - \mathcal{L}\delta^\mu_\nu.$$

For the time translation, the conserved charge T^0_0 gives back the hamiltonian density $\mathfrak{h} = \pi\dot{\phi} - \mathcal{L}$ obtained by Legendre transformation. The conserved quantity from spatial translations is the momentum carried by the field, which for the KG field is

$$\mathbf{P}_i = \int d^d x \; T^0_i = - \int d^d x \; \pi \partial_i \phi.$$

For the Maxwell field, this gives the Poynting vector.

There is some ambiguity in the definition of the stress tensor which you'll study on the homework.

Let's check that expression above for the conserved momentum agrees with our expectations. In particular, in free field theory the total momentum of the state $\left|\vec{k}_1, \cdots \vec{k}_n\right\rangle$ should be just the sum of the momenta of the particles, $\vec{P} = \sum_{\ell=1}^{n} \hbar \vec{k}_\ell$ (with interactions the story can be more complicated). Indeed

$$\mathbf{P}_i = -\int d^d x \; \pi \partial_i \phi = \int đ^d k k_i \mathbf{a}^\dagger_{\vec{k}} \mathbf{a}_{\vec{k}}$$

agrees with this. (Notice that I used rotation invariance of the vacuum to not worry about a possible constant term.)

- We'll say more about the rest of the Poincaré group, *i.e.* rotations and boosts, later on.

1.5 Casimir effect: vacuum energy is real

[A. Zee, *Quantum Field Theory in a Nutshell*, §I.9] This subsection has two purposes. One is to show that the $\frac{1}{2}\hbar\omega$ energy of the vacuum of the quantum harmonic oscillator is real. Sometimes we can get rid of it by choosing the zero of energy (which doesn't matter unless we are studying dynamical gravity). But it is meaningful if we can *vary* ω (or the collection of ωs if we have many oscillators as for the radiation field) and compare the difference.

The other purpose is to give an object lesson in asking the right questions. In physics, the right question is often a question which can be answered by an experiment, at least in principle. The answers to such questions are less sensitive to our silly theoretical prejudices, *e.g.* about what happens to physics at very short distances.

In the context of the bunch of oscillators making up the radiation field, we can change the spectrum of frequencies of these oscillators $\{\omega_k\}$ by putting it in a box and varying the size of the box. In particular, two parallel conducting plates separated by some distance d experience an attractive force from the change in the vacuum energy of the EM field resulting from their presence. The plates put boundary conditions on the field, and therefore on which normal modes are present.

To avoid some complications of E&M which are not essential for our point here, we're going to make two simplifications:

- we're going to solve the problem in 1+1 dimensions
- and we're going to solve it for a scalar field.

To avoid the problem of changing the boundary conditions *outside* the plates we use the following device with *three* plates:

$$| \leftarrow d \rightarrow | \longleftarrow \qquad L-d \qquad \longrightarrow |$$

(We will consider $L \gg d$, so we don't really care about the far right plate.) The 'perfectly conducting' plates impose the boundary condition that our scalar field $q(x)$ vanishes there. The normal modes of the scalar field $q(x)$ in the left cavity are then

$$q_j = \sin\left(j\pi x/d\right), \quad j = 1, 2, ...$$

with frequencies $\omega_j = \frac{\pi|j|}{d}c$. There is a similar expression for the modes in the right cavity which we won't need. We're going to add up all the $\frac{1}{2}\hbar\omega$s for all the modes in both cavities to get the vacuum energy $E_0(d)$; the force on the middle plate is then $-\partial_d E_0$.

The vacuum energy in the whole region of interest between the outer plates is the sum of the vacuum energies of the two cavities

$$E_0(d) = f(d) + f(L-d)$$

where

$$f(d) = \frac{1}{2}\hbar\sum_{j=1}^{\infty}\omega_j = \hbar c\frac{\pi}{2d}\sum_{j=1}^{\infty} j \overset{?!?!!}{=} \infty.$$

We have done something wrong. What? Cliffhanger! [End of Lecture 4]

Our crime is hubris: we assumed that we knew what the modes of arbitrarily large mode number k (arbitrarily short wavelength, arbitrarily high frequency) are doing, and in particular we assumed that they cared about our silly plates. In fact, no metal in existence can put boundary conditions on the modes of large enough frequency – those modes don't care about d. The reason a conductor puts boundary conditions on the EM field is that the electrons move around to compensate for an applied field, but there is a limit on how fast the electrons can move (*e.g.* the speed of light). The resulting cutoff frequency is called the *plasma frequency* but we don't actually need to know about all these details. To parametrize our ignorance of what the high-frequency modes do, we must *cut off* (or *regularize*) the contribution of the high-frequency modes. Let's call modes with $\omega_j \gg \pi/a$ high frequency where a is some short time[11]. Replace

$$f(d) \rightsquigarrow f(a,d) \;=\; \hbar\frac{\pi}{2d}\sum_{j=1}^{\infty} e^{-a\omega_j/\pi} j$$

[11] You can think of a as the time it takes the waves to move by one lattice spacing. If we work in units where the velocity is $c = 1$, this is just the lattice spacing. I will do so for the rest of this discussion.

$$
\begin{aligned}
&= -\frac{\pi\hbar}{2}\partial_a \underbrace{\left(\sum_{j=1}^{\infty} e^{-aj/d}\right)}_{=\frac{1}{1-e^{-a/d}}-1} \\
&= +\frac{\pi\hbar}{2d}\frac{e^{a/d}}{\left(e^{a/d}-1\right)^2} \\
&\overset{a\ll d}{\simeq} \hbar\left(\underbrace{\frac{\pi d}{2a^2}}_{\to\infty \text{ as } a\to 0} - \frac{\pi}{24d} + \frac{\pi a^2}{480 d^3} + \ldots\right)
\end{aligned}
\tag{1.22}
$$

Answers which don't depend on a have a chance of being meaningful. The thing we can measure is the force:

$$
\begin{aligned}
F &= -\partial_d E_0 = -\left(f'(d) - f'(L-d)\right) \\
&= -\hbar\left(\left(\frac{\pi}{2a^2} + \frac{\pi}{24d^2} + \mathcal{O}(a^2)\right) - \left(\frac{\pi}{2a^2} + \frac{\pi}{24\left(L-d\right)^2} + \mathcal{O}(a^2)\right)\right) \\
&\overset{a\to 0}{=} -\frac{\pi\hbar}{24}\left(\frac{1}{d^2} - \frac{1}{(L-d)^2}\right) \\
&\overset{L\gg d}{=} -\frac{\pi\hbar c}{24 d^2}\left(1 + \mathcal{O}(d/L)\right) \ .
\end{aligned}
\tag{1.23}
$$

This is an attractive force between the plates. (I put the c back in the last line.)

The analogous force between real conducting plates, caused by the change of boundary conditions on the electromagnetic field, has been measured.

The string theorists will tell you that $\sum_{j=1}^{\infty} j = -\frac{1}{12}$, and our calculation above agrees with them in some sense. But what this foolishness means is that if we compute something which is not dependent on the cutoff we have to get the same answer no matter what cutoff we use. Notice that it is crucial to ask the right questions.

An important question is to what extent could we have picked a different cutoff function (instead of $e^{-\pi\omega/a}$) and gotten the same answer for the physics. This interesting question is answered affirmatively in Zee's wonderful book, 2d edition, section I.9 (available electronically here!).

A comment about possible physical applications of the calculation we actually did: you could ask me whether there is such a thing as a Casimir force due to the vacuum fluctuations of *phonons*. Certainly it's true that the boundary of a chunk of solid puts boundary conditions on the phonon modes, which change when we change the size of the solid. The problem with the idea that this might produce a measurable force (which would lead the solid to want to shrink) is that it is hard to distinguish the 'phonon vacuum energy' from the rest of the *energy of formation* of the solid, that is, the energy difference between the crystalline configuration of the atoms and the configuration when they are all infinitely separated. Certainly the latter is not well-described in the harmonic approximation ($\lambda = 0$ in (1.1)).

A few comments about the 3+1 dimensional case of E&M. Assume the size of the plates is much larger than their separation L. Dimensional analysis shows that the force per unit area from vacuum fluctuations must be of the form

$$P = A\frac{\hbar c}{L^4}$$

where A is a numerical number. A is not zero!

Use periodic boundary conditions in the xy planes (along the plates). The allowed wave vectors are then

$$\vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}\right)$$

with n_x, n_y integers.

We have to do a bit of E&M here. Assume the plates are perfect conductors (this where the hubris about the high-frequency modes enters). This means that the transverse component of the electric field must vanish at the surface. Instead of plane waves in z, we get standing waves: $\phi(z) \propto \sin\left(n\pi z/L\right)$.

The frequencies of the associated standing waves are then

$$\omega_n(\vec{k}) = c\sqrt{\frac{\pi^2 n^2}{L^2} + \vec{k}^2}, n = 0, 1, 2$$

Also, there is only one polarization state for $n = 0$.

So the zero-point energy is

$$E_0(L) = \frac{\hbar}{2}\left(2{\sum_{n,\vec{k}}}' \omega_n(\vec{k})\right)$$

where it's useful to define

$${\sum_{n,\vec{k}}}' \equiv \frac{1}{2}\sum_{n=0,\vec{k}} + \sum_{n\geq 1,\vec{k}}$$

Now you can imagine introducing a regulator like the one we used above, and replacing

$${\sum_{n,\vec{k}}}' \cdot \rightsquigarrow {\sum_{n,\vec{k}}}' e^{-a\omega_n(\vec{k})/\pi}.$$

and doing the sums and integrals and extracting the small-a behavior.

1.6 Lessons

Starting from a collection of particles, we chained them together, and made a field; treating this system quantumly, we found a new set of particles. The new particles (the normal modes) are *collective* excitations: their properties can be very different from those of the constituent particles. (For example, the constituent particles are distinguishable by their locations, but phonons are indistinguishable from each other.)

Some lessons from all this hard work:

Identical particles. Every photon is the same as every other photon, except for their position (or momentum) and polarization state. For photons this is an immediate consequence of how we discovered them by quantizing the Maxwell field: the state with n photons of the same momentum and polarization is

$$\left| n \text{ photons with } \vec{k}, \alpha \right\rangle = \frac{\left(\mathbf{a}^\dagger_{\vec{k},\alpha}\right)^n}{\sqrt{n!}} \left|0\right\rangle .$$

The same is true of all the other kinds of particles we know about, including electrons (for which we haven't seen a similar classical field description).

This means that we can write the state of N such *indistinguishable particles* merely by specifying a collection of positions and of spin states – we don't need to say which is which (and in fact, we cannot).

A (momentum-space) wavefunction for N such particles is of the form

$$\Psi(k_1, \alpha_1; ...; k_N, \alpha_N) \equiv \langle k_1\alpha_1; \cdots ; k_N, \alpha_N|\Psi\rangle = \langle 0| \mathbf{a}_{k_1\alpha_1}\mathbf{a}_{k_2\alpha_2}\cdots \mathbf{a}_{k_N\alpha_N}\left|\Psi\right\rangle .$$

But the same state is described if we switch the labels of any two of the particles:

$$\Psi(k_2, \alpha_2; k_1, \alpha_1; ....) = a\Psi(k_1, \alpha_1; k_2, \alpha_2; ....)$$

where a is some phase (recall: multiplying the whole wavefunction by a phase does not change the state). Switching them back gives back the first state:

$$\Psi(k_1, \alpha_1; k_2, \alpha_2; ....) = a^2\Psi(k_1, \alpha_1; k_2, \alpha_2; ....)$$

so $a^2 = 1$. There are two solutions: $a = +1$ and $a = -1$ and the two classes of particles associated with these two choices are called respectively *bosons* and *fermions*.

Note that the Hilbert space of N indistinguishable particles (bosons or fermions) is therefore not quite a tensor product of the Hilbert spaces of the individual particles.

An immediate consequence of the minus sign under exchange of fermion labels is the *Pauli exclusion principle*:

$$\Psi_{\text{Fermions}}(k_1, \alpha_1; k_1, \alpha_1; ...) = 0.$$

No two fermions can occupy the same single-particle state. The ground state of a collection of (non-interacting) fermions is therefore quite interesting, since we must find a different single-particle state in which to put each of our fermions. This has many dramatic consequences, including the periodic table of elements, and the distinction between metals and insulators.

A puzzle to consider: the bosonic statistics of phonons and photons was an immediate consequence of the ladder operator algebra $[\mathbf{a}, \mathbf{a}^\dagger] = 1$. How could we possibly get a different answer?

Lorentz invariance can emerge. The dispersion relation for the sound mode and for the light mode was $\omega^2 = v^2 \vec{k}^2$. This is the fourier transform of $\partial_\mu \partial^\mu \phi(x) = 0$, a wave-equation which has Lorentz symmetry (if v is the speed appearing in the Minkowski metric). In the case of sound, we had to ignore the $\mathcal{O}(a^4 k^4)$ terms in the long-wavelength expansion of the dispersion relation, $\cos(ka)$. The lattice breaks Lorentz symmetry, but its effects go away for $ka \ll 1$. This point might make you think that the Lorentz symmetry which is so precious to us in particle physics could emerge in a similar way, but with a much smaller a than the lattice spacing in solids. There are strong constraints on how small this can be (*e.g.* this well-appreciated paper) so it is very useful to treat it as a fundamental principle.

2 Lorentz invariance and causality

[Peskin §2.2, 2.3, 2.4] Now we take Lorentz invariance to be an exact symmetry and see what its consequences are for QFT.

Relativistic normalization of 1-particle states. Fock space is spanned by the states $|\vec{p}_1, \cdots \vec{p}_n\rangle \propto \mathbf{a}^\dagger_{p_1} \cdots \mathbf{a}^\dagger_{p_n} |0\rangle$ where $\mathbf{a}_p |0\rangle = 0$. Now it is time to turn that $\propto$ into an $=$. Fock space is a direct sum of sectors labelled by the number of particles: $\sum_k N_k = 0, 1, 2....$ (Without interactions, the hamiltonian is block diagonal in this decomposition.) In the no-particle sector, it is clear what we should do: $\langle 0|0\rangle = 1$.

A one-particle state is $|p\rangle \equiv c_p \mathbf{a}^\dagger_p |0\rangle$. How best to choose c_p?

(This discussion is shaded because it contains equations which will not be true in the normalization we'll use below. In this regard, beware the section of Peskin called "how not to quantize the Dirac equation".) Suppose we choose $c_p \stackrel{?}{=} 1$. Then

$$\left\langle \vec{k} | \vec{p} \right\rangle \stackrel{?}{=} \langle 0 | \mathbf{a}_k \mathbf{a}_p^\dagger | 0 \rangle \stackrel{\mathbf{a}_k|0\rangle=0}{=} \langle 0 | [\mathbf{a}_k, \mathbf{a}_p^\dagger] | 0 \rangle = (2\pi)^d \delta^{(d)}(\vec{k}-\vec{p}) \equiv \not{\delta}(\vec{k}-\vec{p}).$$

Suppose the previous equation is true in my rest frame F. Since $1 = \int đ^d p \not{\delta}(p-k)$, we see that $đ^d p \not{\delta}(p-k)$ is Lorentz invariant. More precisely,

$$\delta(f(x)) = \sum_{\text{zeros } x_0 \text{ of } f} \frac{\delta(x-x_0)}{|f'(x_0)|}.$$

If another F' is obtained by a boost in the x direction, $p'_\mu = \Lambda^\mu_\nu p_\nu$,

$$\begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} \implies \begin{aligned} \frac{dp'_x}{dp_x} &= \gamma\left(1 - \beta\frac{dE}{dp_x}\right) \\ &= \gamma\left(1 - \beta\frac{p_x}{E}\right) \\ &= \frac{\gamma}{E}(E - \beta p_x) = \frac{E'}{E} \end{aligned}$$

where we used $E^2 = \vec{p}^2 + m^2 = p_x^2 + p_\perp^2 + m^2$ and $2E\frac{dE}{dp_x} = 2p_x$ and $\frac{dE}{dp_x} = \frac{p_x}{E}$.

So

$$\delta^{(d)}(\vec{p}-\vec{k}) = \frac{đ^d p'}{đ^d p}\delta^{(d)}(\vec{p}'-\vec{k}') = \frac{dp'_x}{dp_x}\delta^{(d)}(\vec{p}'-\vec{k}') = \frac{E'}{E}\delta^{(d)}(\vec{p}'-\vec{k}').$$

Which means that in F' we would have

$$\left\langle \vec{k}' | \vec{p}' \right\rangle \stackrel{?}{=} \frac{E'}{E}\not{\delta}^{(d)}(\vec{p}'-\vec{k}').$$

There is a special frame, it's no good.

There is an easy fix:

$$|\vec{p}\rangle \equiv \sqrt{2\omega_{\vec{p}}}\, a^\dagger_{\vec{p}} |0\rangle .$$

In that case the calculation in the shaded text is replaced by

$$\left\langle \vec{k} | \vec{p} \right\rangle = \sqrt{4\omega_k\omega_p}\not{\delta}^{(d)}(k-p) = 2\omega_p \not{\delta}^{(d)}(k-p)$$

while

$$\left\langle \vec{k}' | \vec{p}' \right\rangle = 2\omega_p \frac{\omega_{p'}}{\omega_p}\not{\delta}^{(d)}(k'-p').$$

So the expression is the same in any frame, yay.

Now you can ask why the factor of $\sqrt{2}$. We'd like to use these states to resolve the identity in the 1-particle sector, $\mathbb{1}_1 \equiv \sum_i \sum_i |i\rangle\langle i|$. I claim that the following expression does this and makes Lorentz symmetry manifest:

$$\mathbb{1}_1 \stackrel{?}{=} \int đ^{d+1}k\, \underbrace{\theta(k^0)}_{E>0} 2\pi\delta(k^2 - m^2)\, |k\rangle\langle k|$$
$$= \int đ^d k \int dk_0 \theta(k^0) \left(\frac{\delta\left(k^0 - \sqrt{\vec{k}^2 - m^2}\right)}{2k^0} \right) |k\rangle\langle k| = \int \frac{đ^d k}{2\omega_{\vec{k}}} \left|\omega_{\vec{k}}, \vec{k}\right\rangle \left\langle \omega_{\vec{k}}, \vec{k}\right| .$$

We used the general fact $\delta(f(x)) = \sum_{x_0 | f(x_0)=0} \frac{1}{|f'(x_0)|}\delta(x - x_0)$.

So in retrospect, a quick way to check the normalization is to notice that the following combination is Lorentz invariant:

$$\frac{đ^d k}{2\omega_k} = \int dk_0 \theta(k_0)\, đ^d k \delta(k^2 - m^2) = \frac{đ^d k'}{2\omega_{k'}}.$$

Actually, this statement has a hidden assumption, that $m^2 > 0$. In that case, the 4-vector k^μ satisfying $k^2 = m^2 > 0$ is timelike, and no Lorentz transformation connected to the identity can change the sign of k_0, it can only move it around within the lightcone. So the $\theta(k_0)$ is Lorentz invariant.

Notice that our convenient choice of normalization doesn't show that our Hamiltonian description of scalar field theory is actually Lorentz invariant. For example, we have

$$[\phi(\vec{x}), \pi(\vec{y})]_{ETCR} = \mathbf{i}\delta^{(d)}(\vec{x} - \vec{y})$$

at equal times, in one frame. What about other frames?

A second reason to study commutators is ...

2.1 Causality and antiparticles

Causality: This is the very reasonable condition on our description of physics that *events should not precede their causes.*

It will be worthwhile to think about how to implement this condition in a QFT. (The following discussion is based on appendix D of this paper.) Suppose B wants to send a message to A. How does he do this? He applies an operator[12], call it $\mathbf{B}$, localized near B, to the shared state of their quantum many body system $|\psi\rangle_{ABE}$.

[12] 'Applying an operator' is more complicated than it seems. Actually it means 'changing the Hamiltonian so that the time evolution operator is $\mathbf{B}$'.

(Here E is for 'environment', the rest of the world besides A and B.) Then A measures some observable $\mathbf{A}$; let's assume $1 = \mathrm{tr}\mathbf{A} = \sum(\text{eigenvalues of } \mathbf{A})$. To send a different message, he should apply a different operator, say B'.

Under the circumstances just stated, the expectation for A's measurement of $\mathbf{A}$ is

$$\langle \mathbf{A} \rangle_B = \langle\psi| \mathbf{B}^\dagger \underbrace{e^{\mathbf{iH}t} \mathbf{A} e^{-\mathbf{iH}t}}_{=\mathbf{A}(t)} \mathbf{B} |\psi\rangle = \langle \mathbf{A}(t)\rangle + \left\langle \mathbf{B}^\dagger [\mathbf{B}, \mathbf{A}(t)] \right\rangle .$$

Therefore, if $[\mathbf{B}, \mathbf{A}(t)] = 0$, the expectation doesn't depend on what B did. In fact, replacing $\mathbf{A}$ with $\mathbf{A}^\eta$ for any η and using ($[\mathbf{B}, \mathbf{A}(t)] = 0 \implies [\mathbf{B}, \mathbf{A}(t)^\eta] = 0$) shows that all the moments of the distribution for A's measurement will also be independent of what B did, so no message gets through[13].

Causality in relativistic QFT. In a Lorentz invariant system, 'precede' is sometimes a frame-dependent notion. If A is in the future lightcone of B, *i.e.* $0 < (x_A - x_B)^2 = (t_A - t_B)^2 - (\vec{x}_A - \vec{x}_B)^2$ and $t_A > t_B$, then everyone agrees that A is after B. This is the easy case. But if A and B are spacelike separated, $0 > (x_A - x_B)^2 = (t_A - t_B)^2 - (\vec{x}_A - \vec{x}_B)^2$, then there is a frame where they occur at the same time, and frames where they occur in either order. This is the dangerous case.

[End of Lecture 5]

So: causality will follow if $[\mathbf{A}(x_A), \mathbf{B}(x_B)] = 0$ whenever x_A and x_B are spacelike separated, $0 > (x_A - x_B)^2$. Recall that spacelike separated means that there is a Lorentz frame where A and B are at the same time.

A general operator in a scalar QFT can be made from ϕs and $\partial_\mu\phi$s, so the general

[13] The loophole-seeking reader (ideally, this is everyone) will worry that a distribution is not in general determined by its moments. (For example, there are perfectly normalizable distributions with finite averages but where the higher moments are all infinite, such as $p(x) = \frac{\sqrt{2}a^3/\pi}{x^4+a^4}$ on the real line: $\left\langle x^2 \right\rangle = a^2$, but $\left\langle x^{2n} \right\rangle = \infty$ for $n > 1$.) What we would really like to show is that the conditional probability $p(a|B)$ is independent of B, in which case for sure A couldn't learn anything about what B did. That is

$$p(a|B) = \langle\psi| \mathbf{B}^\dagger \underbrace{e^{\mathbf{iH}t} |a\rangle \langle a| e^{-\mathbf{iH}t}}_{=\mathbf{P}_a(t)} \mathbf{B} |\psi\rangle = \langle \mathbf{P}_a(t)\rangle + \left\langle \mathbf{B}^\dagger [\mathbf{B}, \mathbf{P}_a(t)] \right\rangle .$$

Does $[\mathbf{A}, \mathbf{B}] = 0$ imply that the projector onto a particular eigenvalue of a commutes with $\mathbf{B}$? In a finite dimensional Hilbert space, it does for sure, since $0 = [\mathbf{A}^\eta, \mathbf{B}] = \sum_a a^\eta [\mathbf{P}_a, \mathbf{B}]$ is true for all η, which gives infinitely many equations for $[\mathbf{P}_a, \mathbf{B}]$. In the case of infinite dimensional $\mathcal{H}$ I think there is some room for functional analysis horror. On the other hand, any measurement has finite resolution. Thanks to Sami Ortoleva and Chuncheong Lam for help with this point.

statement will follow from considering commutators of

$$\phi(x) = \int \frac{đ^d p}{\sqrt{2\omega_{\vec p}}} \left(\mathbf{a}_{\vec p} e^{-\mathbf{i}p_\mu x^\mu} + \mathbf{a}^\dagger_{\vec p} e^{+\mathbf{i}p_\mu x^\mu} \right) |_{p^0=\omega_{\vec p}} \equiv \phi^{(+)} + \phi^{(-)} \ .$$

Here we have decomposed the field into positive- and negative-frequency parts. Notice that since $\phi^{(+)}$ ($\phi^{(-)}$)only involves annihilation (creation) operators, $[\phi^{(\pm)}(x), \phi^{(\pm)}(y)] = 0$ for any x, y. Using the ladder algebra,

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{đ^d p}{2\omega_{\vec p}} \left(e^{-\mathbf{i}p_\mu (x-y)^\mu} - e^{+\mathbf{i}p_\mu (x-y)^\mu} \right) \\ &= \underbrace{\int đ^{d+1} p \ 2\pi\delta(p^2 - m^2)\theta(p^0)}_{\text{Lorentz inv't}} \left(e^{-\mathbf{i}p_\mu (x-y)^\mu} - e^{+\mathbf{i}p_\mu (x-y)^\mu} \right) \end{aligned} \qquad (2.1)$$

Here comes the slippery stuff: Suppose $x - y$ is spacelike. Let's choose a frame where they are at the same time, and let Λ be the Lorentz matrix that gets us there: $\Lambda^\mu_\nu (x-y)^\nu = (0, \vec{\Delta x})^\mu \equiv \tilde{x}^\mu$. Then we can change integration variable to $\tilde{p}^\mu = \Lambda^\mu_\nu p^\nu$.

$$[\phi(x), \phi(y)] \overset{(x-y)^2<0}{=} \int đ^{d+1}\tilde{p} \ 2\pi\delta(\tilde{p}^2 - m^2)\theta(\tilde{p}^0) \underbrace{\left(e^{-\mathbf{i}\vec{\tilde{p}}\cdot\vec{\Delta}x} - e^{+\mathbf{i}\vec{\tilde{p}}\cdot\vec{\Delta}x} \right)}_{\text{odd under } \vec{\tilde{p}} \to -\vec{\tilde{p}}} = 0. \qquad (2.2)$$

We conclude that $[\phi(x), \phi(y)] = 0$ if $(x-y)^2 < 0$, *i.e.* for spacelike separation.

The same argument works for $[\phi, \pi]$ and $[\pi, \pi]$. For $[\phi, \pi]$, the strict inequality is important.

So vanishing ETCR (for nonzero separation) plus Lorentz symmetry implies causality.

Notice that the argument fails if $(x-y)^2 > 0$, since then we can't get rid of the time component of the exponents by a Lorentz transformation, and they don't cancel.

Now let's think more about the bit which is nonzero:

$$[\phi(x), \phi(y)] = \underbrace{[\phi^{(+)}(x), \phi^{(-)}(y)]}_{\equiv \hat{\Delta}^+(x-y)} + \underbrace{[\phi^{(-)}(x), \phi^{(+)}(y)]}_{\equiv \hat{\Delta}^-(x-y)} .$$

Because $[\mathbf{a}, \mathbf{a}^\dagger] \propto \mathbb{1}$, $\hat{\Delta}^\pm$ is a c-number, independent of what state it acts on. So, for any normalized state $|\psi\rangle$,

$$\begin{aligned} \Delta^+(x-y) = \langle\psi| \hat{\Delta}^+(x-y) |\psi\rangle &= \langle 0| \hat{\Delta}^+(x-y) |0\rangle \\ &= \langle 0| [\phi^{(+)}(x), \phi^{(-)}(y)] |0\rangle \\ &= \langle 0| \phi^{(+)}(x)\phi^{(-)}(y) |0\rangle - \langle 0| \phi^{(-)}(y) \underbrace{\phi^{(+)}(x) |0\rangle}_{=0: \phi^+ \ni \mathbf{a}, \mathbf{a}|0\rangle = 0} \end{aligned}$$

$$= \langle 0| \phi(x)\phi(y) |0\rangle \equiv D(x-y)$$

where in the last step we again used $\phi^+ |0\rangle = 0$. [14] This is the vacuum-to-vacuum amplitude, or *propagator*, in the sense that

$$\phi(y) |0\rangle = |\text{`particle created at } y\text{'}\rangle$$

$$\langle 0| \phi(x) = \langle \text{`particle destroyed at } x\text{'}|$$

That is,

$$\Delta^+(x-y) = \sum_p (x \cdot - - - \leftarrow - - - - \cdot y) = \int \frac{đ^d p}{2\omega_{\vec p}} e^{-\mathbf{i}p\cdot(x-y)}.$$

This integral can be done in terms of functions with names[15], but the most useful information is about its asymptotics in the very timelike ($t \equiv |x^0 - y^0| \gg |\vec{x} - \vec{y}|$) and very spacelike ($|x^0 - y^0| \ll |\vec{x} - \vec{y}| \equiv r$) limits. You can read about how to arrive at these expressions in Peskin (page 27):

$$\Delta^+(x-y) \simeq \begin{cases} e^{-\mathbf{i}mt}, & |x^0 - y^0| \gg |\vec{x} - \vec{y}|, (x-y)^2 \equiv t^2 \\ e^{-mr}, & |x^0 - y^0| \ll |\vec{x} - \vec{y}|, (x-y)^2 \equiv -r^2 \end{cases}$$

Notice that this quantity $\langle \phi(x)\phi(y)\rangle$ is *not* zero outside the lightcone. What gives?

Causality only requires the vanishing of commutators outside the lightcone, which we already showed in (2.2). I got mixed up in the book-keeping at this point in lecture. The purpose of the following discussion is to interpret the cancellation in (2.2) as destructive interference between particles and antiparticles. What happened here in terms of physics pictures? It's clearer for the complex scalar field, where

$$\Phi^{(+)} = \int \frac{đ^d p}{\sqrt{2\omega_p}} e^{-\mathbf{i}px} \mathbf{a}_p, \quad \Phi^{(-)} = \int \frac{đ^d p}{\sqrt{2\omega_p}} e^{+\mathbf{i}px} \mathbf{b}_p^\dagger$$

(with the expressions for the $+$ and $-$ frequency components for $\Phi^\star$ following by taking hermitian conjugates). So consider the analogous

$$D(x-y) \equiv \langle 0| \Phi(x)\Phi^\star(y) |0\rangle = \langle 0| [\Phi^+(x), \Phi^{\star -}(y)] |0\rangle = \underbrace{\Delta_{\mathbf{a}}^+(x-y)}_{\text{from } [\mathbf{a}, \mathbf{a}^\dagger]}$$

$$D^\star(y-x) \equiv \langle 0| \Phi^\star(y)\Phi(x) |0\rangle = \langle 0| [\Phi^{\star +}(y), \Phi^-(x)] |0\rangle = -\underbrace{\Delta_{\mathbf{b}}^-(y-x)}_{\text{from } [\mathbf{b}, \mathbf{b}^\dagger]}$$

[14] Note from the definition that $\Delta^\pm(x-y) = [\phi^{(\pm)}(x), \phi^{(\mp)}(y)] = -\Delta^\mp(y-x)$.

[15] Specifically, in four spacetime dimensions and spacelike separation, $(x-y)^2 \equiv -r^2$, $\Delta^+(x-y) = \frac{m}{2\pi^2 r} K_1(r)$.

So if we consider the commutator,

$$\begin{aligned}\langle 0|\,[\Phi(x),\Phi^\star(y)]\,|0\rangle &= D(x-y) - D^\star(y-x)\\ &= \sum_p \left(\underbrace{(x\cdot - - - \leftarrow p - - - \cdot y)}_{\text{particle}} \;-\; \underbrace{(x\cdot - - - p \rightarrow - - - - \cdot\, y)}_{\text{antiparticle!}}\right)\end{aligned}$$

then in the spacelike case, the antiparticle bit from the first term of the commutator cancels the particle bit of the second, as above in (2.2). Antimatter makes QFT causal.

2.2 Propagators, Green's functions, contour integrals

I claim that the propagator for a real free scalar field can be represented as :

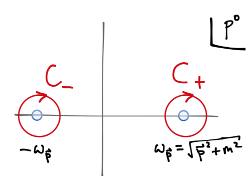
$$\Delta(x) = \underbrace{\int_C đ^{d+1}p}_{\equiv \int_C đp_0 \int đ^d p} \; e^{-\mathbf{i}p_\mu x^\mu} \frac{\mathbf{i}}{p^2 - m^2}.$$

To see that this is related to the object we discussed above, first note that the denominator is

$$p^2 - m^2 = (p_0 - \omega_{\vec p})(p_0 + \omega_{\vec p}) \;\;,\quad \omega_{\vec p} \equiv \sqrt{\vec p \cdot \vec p + m^2},$$

so there are two poles, which seem to be on the real axis; this means that our integral is ambiguous and we need more information, indeed some physical input.

We can specify the contour C by taking linear combinations of $C_\pm$ which are small circles going *clockwise* around the poles at $\pm\omega_{\vec p}$.



These basic integrals are[16]:

$$\int_{C_+} đ^{d+1}p\; e^{-\mathbf{i}p_\mu x^\mu}\frac{\mathbf{i}}{p^2-m^2} = \int đ^d p \frac{1}{2\omega_{\vec p}} e^{-\mathbf{i}px}|_{p_0=\omega_{\vec p}} = \Delta^+(x). \tag{2.3}$$

$$\int_{C_-} đ^{d+1}p\; e^{-\mathbf{i}p_\mu x^\mu}\frac{\mathbf{i}}{p^2-m^2} = \int đ^d p \frac{1}{-2\omega_{\vec p}} e^{-\mathbf{i}px}|_{p_0=-\omega_{\vec p}} = -\Delta^+(-x) \overset{\text{let } \vec q \equiv -\vec p}{=} \Delta^-(x).$$

If we add these up, we get the full propagator:

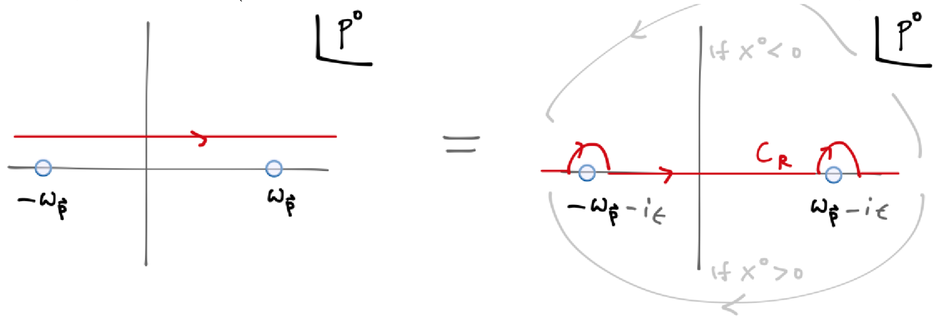
$$\Delta(x) = \Delta^+(x) + \Delta^-(x) = \int_{C=C_+ + C_-} đ^{d+1}p \frac{\mathbf{i}}{p^2-m^2} e^{-\mathbf{i}px}.$$

[16] We are going to use the Cauchy residue theorem $\oint_C dz = f(z) = 2\pi\mathbf{i}\sum_{z_j} \text{Res}_{z=z_j} f$ where z_j are the poles of f. To remember the sign, consider a small circle C_0 counterclockwise around the origin and $f(z) = 1/z$, so $\oint_{C_0}\frac{dz}{z} = \mathbf{i}\int_0^{2\pi} d\theta = 2\pi\mathbf{i}$.

This is one particular choice of contour, and others are also interesting. Consider the *retarded* propagator,

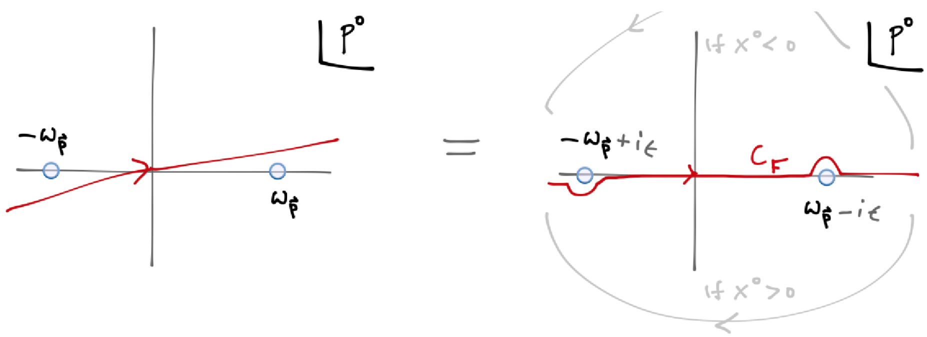
$$D_R(x-y) \equiv \theta(x^0-y^0)\,\langle 0|[\phi(x),\phi(y)]|0\rangle\,.$$

We can reproduce this by routing our contour to go above the poles in the complex p^0 plane: if $x^0-y^0>0$, then the factor $e^{-\mathbf{i}p^0(x^0-y^0)}$ decays when $\mathrm{Im}p^0<0$, so we can close the contour for free in the LHP, and we pick up both poles; if $x^0-y^0>0$, we must close in the UHP and we pick up no poles and get zero. Notice that we could get the same result by replacing $p^0\to p^0+\mathbf{i}\epsilon$ in the denominator, where ϵ is an *infinitesimal* (this means that $\epsilon^2=0$ and $\epsilon c=\epsilon$ for any positive quantity c).



Another interesting way to navigate the poles is by replacing p^2-m^2 with $p^2-m^2+\mathbf{i}\epsilon$. This shifts the poles to

$$p^0=\pm\omega_p\sqrt{1-\mathbf{i}\epsilon/\omega_p}=\pm\omega_p\,(1-\mathbf{i}\epsilon)\,.$$



This is called the *Feynman contour*, C_F, and it seems rather ad hoc; if we had built our understanding of field theory using the path integral, as Zee does, this would have popped out as an inevitable consequence of making the path integral well-defined (see page 20 of the 2d edition). To salvage some self-respect here, consider the *euclidean* propagator, where we get rid of the relative sign in the metric :

$$I_E(\vec{x}) \equiv \int đ^D p \frac{-\mathbf{i}}{\sum_{i=1}^D p_i^2+m^2} e^{-\mathbf{i}\sum_{i=1}^D p_i x_i}.$$

Its poles are at $p^D=\pm\mathbf{i}\sqrt{\vec{p}\cdot\vec{p}+m^2}$, far from the integration contour, so there is no problem defining it. Now consider smoothly rotating the contour by varying θ from

0 to $\pi/2 - \epsilon$ in $p^0 \equiv e^{\mathbf{i}\theta} p^D$. The Feynman contour is the analytic continuation of the euclidean contour, and the ϵ is the memory of this.

In position space, the Feynman propagator is

$$\Delta_F(x) \equiv \int_{C_F} đ^{d+1}p\, e^{-\mathbf{i}p_\mu x^\mu} \frac{\mathbf{i}}{p^2 - m^2} = \theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^+(-x) = \theta(x^0)\Delta^+(x) - \theta(-x^0)\Delta^-(x).$$

If $x^0 > 0$ (< 0), we must close the contour in the LHP (UHP) and get Δ^+ (Δ^-). Recalling that $\Delta^+(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$,

$$\Delta_F(x-y) = \langle 0| \left(\theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)\right)|0\rangle \equiv \langle 0| \mathcal{T}\left(\phi(x)\phi(y)\right)|0\rangle .$$

The $\mathcal{T}$ is the *time-ordering symbol*: the operators at the earliest times go on the right, so we can regard time as increasing to the left.

The propagator is a Green's function. So we've learned that

$$\frac{\mathbf{i}}{p^2 - m^2} \equiv \tilde{\Delta}(p)$$

is the Fourier transform of $\Delta(x)$, the momentum-space propagator (either retarded or Feynman). From this we can see that $\Delta(x)$ is a Green's function for the differential operator $\partial_\mu \partial^m + m^2$ in the sense that

$$\left(\partial_\mu \partial^\mu + m^2\right)\Delta(x) = -\mathbf{i}\delta(x)$$

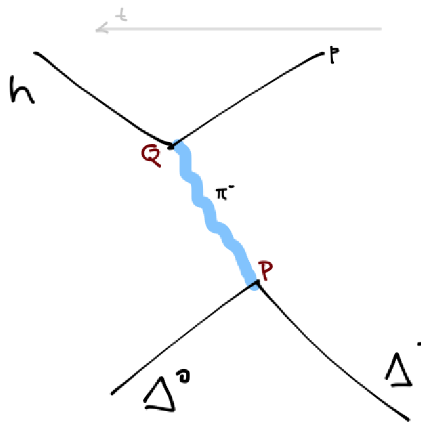
(by plugging in the Fourier expansion of Δ and of the delta function, $\delta^{(d+1)}(x) = \int đ^{d+1}p\; e^{-\mathbf{i}px}$, and differentiating under the integral). Notice that this did not depend on the choice of contour, so this equation in fact has several solutions, differing by the routes around the poles (hence by $\Delta^\pm$, which are solutions to the homogeneous equation, without the delta function). On the homework, you will show this directly from the position space definition.

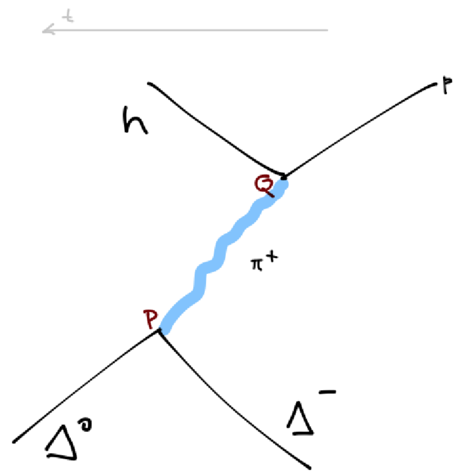
[End of Lecture 6]

Physics preview. Here is a preview of the physics of the Feynman propagator. Imagine we introduce some interactions, such as a cubic term in the Lagrangian, *e.g.*

$$\mathcal{L} \ni \phi_p(x)\phi_n(x)\phi_\pi(x) + h.c. \tag{2.4}$$

where the fields appearing here destroy or create particles with the names in the subscripts. Here are two stories we might tell about a collection of such particles.





In both pictures, time goes to the left. In the first picture, a Δ^- emits a π^-, becoming a Δ^0 at spacetime point P. This π^- propagates to Q where it is absorbed by a p, which turns into an n. In the second picture, a p emits a π^+ at Q, and becomes n; that π^+ is absorbed by a Δ^- which becomes a Δ^0.

But these two stories are related by a Lorentz boost which exchanges the relative times of the interaction events – they must be the same story. The Feynman propagator includes both automatically.

Antiparticles going backwards in time. This story is clearer if we discuss the complex scalar, where particles (created by $\mathbf{a}^\dagger$) and antiparticles (created by $\mathbf{b}^\dagger$) are distinct:

$$\Phi(x) = \underbrace{\int \frac{\text{đ}^d p}{\sqrt{2\omega_p}} \mathbf{a}_p e^{-\mathbf{i}px}}_{\equiv \Phi^+(x)} + \underbrace{\int \frac{\text{đ}^d p}{\sqrt{2\omega_p}} \mathbf{b}_p^\dagger e^{+\mathbf{i}px}}_{\equiv \Phi^-(x)} .$$

The commutator is

$$\begin{aligned}
[\Phi(x), \Phi^\star(y)] &= \underbrace{[\Phi^+(x), \Phi^{\star -}(y)]}_{\text{from } [\mathbf{a}, \mathbf{a}^\dagger], \text{ particles}} + \underbrace{[\Phi^-(x), \Phi^{\star +}(y)]}_{\text{from } [\mathbf{b}, \mathbf{b}^\dagger], \text{ antiparticles}} \\
&= \Delta^+_{\mathbf{a}}(x-y) + \underbrace{\Delta^-_{\mathbf{b}}(x-y)}_{= -\Delta^+_{\mathbf{b}}(y-x)} \\
&= \int_{C_+} \text{đ}^{d+1} e^{-\mathbf{i}p(x-y)} \frac{\mathbf{i}}{p^2 - m^2} + \int_{C_-} \text{đ}^{d+1} e^{-\mathbf{i}p(x-y)} \frac{\mathbf{i}}{p^2 - m^2} .
\end{aligned} \tag{2.5}$$

The propagator that we'll really need to compute S-matrix elements is

$$\begin{aligned}
\Delta_F(x-y) &\equiv \langle 0 | \mathcal{T}\left(\Phi(x)\Phi^\star(y)\right) |0\rangle \\
&= \theta(x^0 - y^0) \qquad \underbrace{\langle 0 | \Phi(x)\Phi^\star(y) |0\rangle}_{= \langle 0|\Phi^+(x)\Phi^{\star -}(y)|0\rangle = \langle 0|[\Phi^+(x), \Phi^{\star -}(y)]|0\rangle = \Delta^+_{\mathbf{a}}(x-y), \quad \text{particles}}
\end{aligned}$$

$$+\ \theta(y^0 - x^0) \qquad \underbrace{\langle 0|\,\Phi^\star(y)\Phi(x)\,|0\rangle}_{=-\langle 0|[\Phi^{\star +}(y),\Phi^-(x)]|0\rangle=-\Delta^+_{\mathbf{b}}(x-y),\quad \text{antiparticles}}$$

2.3 Interlude: where is 1-particle quantum mechanics?

[Tong, §2.8] Consider a relativistic complex free massive scalar field Φ, with mass m. The minimum energy of a single-particle state is $\omega_{p=0} = m$ (above the vacuum); in its rest frame, the wavefunction is $e^{-\mathbf{i}mt}$. Consider the change of variables:

$$\Phi(\vec{x}, t) =: e^{-\mathbf{i}mt}\Psi(\vec{x}, t)\frac{1}{\sqrt{2m}}. \tag{2.6}$$

The Klein-Gordon equation is

$$0 = \partial_t^2\Phi - \nabla^2\Phi + m^2\Phi = \sqrt{2m}e^{-\mathbf{i}mt}\left(\ddot{\Psi} - 2\mathbf{i}m\dot{\Psi} - \nabla^2\Psi\right),$$

where the terms with m^2 cancel; so far we've just changed variables. The non-relativistic limit is $|\vec{p}| \ll m^2$ which on-shell, $\omega_p = \sqrt{\vec{p}^2 + m^2} \simeq m + \frac{p^2}{2m} + \cdots$, implies that $|\ddot{\Psi}| \ll m|\dot{\Psi}|$ so we can ignore that term in the KG equation, leaving

$$\mathbf{i}\partial_t\Psi = -\frac{1}{2m}\nabla^2\Psi. \tag{2.7}$$

This looks like the Schrödinger equation for a particle in no potential, in position space, but that is a coincidence: Ψ is not a wavefunction. This equation (2.7) is the eom associated with the lagrange density

$$\mathcal{L} = \mathbf{i}\Psi^\star\dot{\Psi} - \frac{1}{2m}\vec{\nabla}\Psi^\star \cdot \vec{\nabla}\Psi$$

from which $\pi_\Psi = \mathbf{i}\Psi^\star, \pi_{\Psi^\star} = 0$ (which you can also get by plugging (2.6) into $\partial_\mu\Phi^\star\partial^\mu\Phi$). The ETCRs are then

$$[\Psi(\vec{x}), \Psi^\star(\vec{y})] = \delta^d(\vec{x} - \vec{y}), \quad [\Psi, \Psi] = 0 = [\Psi^\star, \Psi^\star]$$

and the Hamiltonian is

$$H = \int d^dx \frac{1}{2m}\vec{\nabla}\Psi^\star \cdot \vec{\nabla}\Psi.$$

The solution in terms of creation operators is then

$$\Psi(x) = \int đ^dp e^{\mathbf{i}\vec{p}\cdot\vec{x}}\mathbf{a}_p, \quad \Psi^\star(x) = \int đ^dp e^{-\mathbf{i}\vec{p}\cdot\vec{x}}\mathbf{a}_p^\dagger$$

with $[\mathbf{a}_p, \mathbf{a}_q^\dagger] = (2\pi)^d\delta^d(p - q)$ as before. The hamiltonian is then

$$H = \int đ^dp \frac{\vec{p}^2}{2m}\mathbf{a}_p^\dagger\mathbf{a}_p \tag{2.8}$$

(with no normal-ordering constant – the vacuum of this non-relativistic theory is extremely boring and has no dance of birth and death of the fluctuating particle-antiparticle pairs).

The crucial point here is that the antiparticles are gone, despite the fact that the field is complex. In the limit we've taken, we don't have enough energy to make any. The states are

$$\mathbf{a}^\dagger_{\vec{p}_1} \cdots \mathbf{a}^\dagger_{\vec{p}_n} |0\rangle \equiv |\{p\}\rangle , \quad \mathbf{a}_p |0\rangle = 0$$

and these are energy eigenstates with energy $H |\{p\}\rangle = \sum_i \frac{\vec{p}_i^2}{2m} |\{p\}\rangle$, a nice non-relativistic (NR) dispersion for each particle. Notice that the NR limit required a complex field (otherwise we can't multiply by a phase). The particle-number symmetry is still present $\Psi \to e^{-\mathbf{i}\alpha}\Psi$, but now the current is

$$j^\mu = \left(\Psi^\star \Psi, \frac{\mathbf{i}}{2m} \Psi^\star \vec{\nabla} \Psi + h.c. \right)^\mu = \left(\rho, \vec{j} \right)^\mu .$$

Now we can find the QM of a single particle which cannot go away (since we got rid of the antiparticles), with some position and momentum *operators*. In fact the momentum operator is just the charge associated with translation invariance, and takes the form (just like on the homework)

$$\vec{\mathbf{P}} = \int đ^d p \vec{p} \mathbf{a}^\dagger_p \mathbf{a}_p$$

and $\vec{\mathbf{P}} |\{p\}\rangle = \sum_a \vec{p}_a |\{p\}\rangle$. What's the position operator? A state with a particle at position $\vec{x}$ is

$$|\vec{x}\rangle = \Psi^\star(x) |0\rangle = \int đ^d p e^{-\mathbf{i}\vec{p}\cdot\vec{x}} \mathbf{a}^\dagger_p |0\rangle .$$

If we let

$$\vec{\mathbf{X}} \equiv \int d^d x \Psi^\star(x) \vec{x} \Psi(x)$$

then indeed $\vec{\mathbf{X}} |\vec{x}\rangle = \vec{x} |\vec{x}\rangle$. To see that the Heisenberg algebra $[\mathbf{X}, \mathbf{P}] = \mathbf{i}$ works out, consider the general 1-particle state

$$|\psi\rangle = \int d^d x \psi(x) |x\rangle .$$

The function $\psi(x)$ here is the usual position-basis Schrödinger wavefunction. You can check on the homework that

$$\mathbf{X}^i |\psi\rangle = \int d^d x x^i \psi(x) |x\rangle , \quad \mathbf{P}^i |\psi\rangle = \int d^d x \left(-\mathbf{i} \frac{\partial}{\partial x^i} \psi(x) \right) |x\rangle$$

which implies the Heisenberg commutator. Finally, the hamiltonian (2.8) gives the time evolution equation

$$\mathbf{i}\partial_t \psi = -\frac{\nabla^2}{2m}\psi$$

which really is the Schrödinger equation.

Many particles which one studies in NR QM are actually fermions $(e, p, n...)$ and therefore not described by a scalar field. But in the 1-particle sector, who can tell? No one. Later we'll see the NR limit of the Dirac equation, which is basically the same, but with some extra juicy information about spin.

Next we will speak about 'interactions'. This term is used in two ways. In NR QM, it is sometimes used to describe an external potential $V(x)$ appearing as an extra term in the Schrödinger equation

$$\mathbf{i}\partial_t \psi = -\frac{\nabla^2}{2m}\psi + V(x)\psi(x).$$

Such a term explicitly violates translation symmetry. It can be accomplished by adding to the action the *quadratic* term

$$\Delta S_V = -\int d^d x \Psi^\star(x)\Psi(x)V(x) = -\int d^d x \rho(x)V(x).$$

This says that nonzero density of particles at x costs energy $V(x)$. A second sense of 'interaction' which is how it will be used forever below is interaction *between* particles. With only one particle this cannot happen. NR QM theory does accommodate more than one particle, and we can consider an interaction between them like

$$\Delta S = -\int d^d x \int d^d y \Psi^\star(x)\Psi(x)V(x-y)\Psi^\star(y)\Psi(y).$$

If $V(x-y) = \delta^d(x-y)$, this interaction is local.

3 Interactions, not too strong

3.1 Where does the time dependence go?

[Peskin chapter 4.2] Now we must get straight where to put the time dependence. Different ways of doing the book-keeping are called different 'pictures'. At some reference time, say $t = 0$, they all agree: label states by $|\psi, 0\rangle$ and operators by $\mathcal{O}(0)$. At a later time, in picture P, these evolve to $|\psi, t\rangle_P, \mathcal{O}_P(t)$. Physics, such as any amplitude like

$$ {}_P \langle \psi, t| \mathcal{O}_P(t) |\psi, t\rangle_P \tag{3.1} $$

is independent of P. Let's assume the hamiltonian H has no explicit time dependence.

In Heisenberg picture ($P = H$), $|\psi, t\rangle_H \equiv |\psi, 0\rangle$ for all time, and the burden of the time dependence is all on the operators

$$ \mathcal{O}_H(t) = U_H^\dagger(t) \mathcal{O}(0) U_H(t). $$

The Heisenberg equations of motion are

$$ \mathbf{i}\dot{\mathcal{O}}_H = [\mathcal{O}_H, H_H] \tag{3.2} $$

so in particular $\dot{H}_H = 0$ so $H_H(t) = H_H(0) = H$. Then (3.2) is solved by $U_H(t) = e^{-\mathbf{i}tH}$. For example, another example of an operator is the free field:

$$ \Phi_H(\vec{x}, t) = \int \frac{đ^d p}{\sqrt{2\omega_p}} \left(\mathbf{a}_p e^{-\mathbf{i}px} + \mathbf{b}_p^\dagger e^{\mathbf{i}px} \right). $$

In fact, this equation is basically the whole story of free field theory. The field makes particles which don't care about each other.

In Schrödinger picture ($P = S$), $\frac{d}{dt}\mathcal{O}_S = \partial_t \mathcal{O}_S$ time dependence of operators comes only from explicit, external dependence in the definition of the operator (which will not happen here), so $\mathcal{O}_S(t) = \mathcal{O}(0)$, and (3.1) then requires

$$ |\psi, t\rangle_S = U_H(t) |\psi, 0\rangle. $$

And the unitary evolution operator is

$$ U_H(t) = e^{-\mathbf{i}H(0)t} = e^{-\mathbf{i}H_S t} = U_S = U $$

so does not require a picture label.

Interactions. So, in an interacting field theory, all we need to do is to find U to figure out what it does. For better or worse, this isn't a realistic goal in general. In this class we are going to focus on the special case where the interactions are weak, so that the hamiltonian takes the form

$$H = H_0 + V$$

where H_0 is quadratic in fields (linear terms are allowed but annoying) and we assume that the interaction term V is proportional to a small parameter. This by no means exhausts all interesting questions in field theory; on the other hand, a surprisingly enormous amount of physics can be done using this assumption.

Interaction picture. $(P = I)$ In this case, it is very convenient to use a hybrid picture where the time-dependence of the operators is as in the Heisenberg picture for the hamiltonian with $V \to 0$. This free field evolution is solvable:

$$\mathcal{O}_I(t) \equiv U_0^\dagger \mathcal{O}(0) U_0, \quad U_0(t) \equiv e^{-\mathbf{i}H_0 t}. \tag{3.3}$$

Note that in this picture, $H_0(t) = H_0(0) = H_0$. Equivalently, $\mathbf{i}\dot{\mathcal{O}}_I = [\mathcal{O}_I, H_0]$, where in this expression, crucially, $H_0 = H_0(\Phi_I)$ is made from interaction picture fields, whose evolution we know from above; for example, for a complex scalar,

$$\Phi_I(\vec{x}, t) = \int \frac{đ^d p}{\sqrt{2\omega_p}} \left(\mathbf{a}_p e^{-\mathbf{i}px} + \mathbf{b}_p^\dagger e^{\mathbf{i}px} \right).$$

[End of Lecture 7]

The catch is that the interaction-picture states are still time-dependent:

$$\underbrace{{}_H \langle \varphi, t | \mathcal{O}_H(t) | \psi, t \rangle_H}_{= \langle \psi, 0 | U_H^\dagger(t) \mathcal{O}(0) U_H(t) | \psi, 0 \rangle} \overset{(3.1)!}{=} {}_I \langle \varphi, t | \underbrace{\mathcal{O}_I(t)}_{= U_0^\dagger(t) \mathcal{O}(0) U_0(t)} | \psi, t \rangle_I$$

$\forall \varphi, \psi$ which says that

$$|\psi, t\rangle_I = U_0^\dagger(t) U_H(t) |\psi, 0\rangle \equiv U_I(t) |\psi, 0\rangle.$$

In the interaction picture, the interaction hamiltonian itself evolves according to

$$\mathbf{i}\frac{d}{dt} V_I = [V_I, H_0] \implies V_I(t) = U_0^\dagger V(0) U_0.$$

So for example, if $V(0) = \int d^d x g \phi^3(x, 0)$, then using $1 = U_0 U_0^\dagger$ repeatedly,

$$U_0^\dagger V(0) U_0 = g \int d^d x U_0^\dagger \phi^3(x,0) U_0 = g \int d^d x U_0^\dagger \phi(x,0) U_0 U_0^\dagger \phi(x,0) U_0 U_0^\dagger \phi(x,0) U_0 = g \int d^d x \left(\phi_I(x,t) \right)^3.$$

This trick wasn't special to ϕ^3 and works for any local interaction:

$$\left(V(t) \right)_I = V|_{t=0} \left(\phi_I(t) \right)$$

– just stick the interaction-picture-evolved fields into the form of the interaction at $t = 0$, easy.

How do the states evolve? Notice that $[U_0^\dagger, U_H] \neq 0$, if the interactions are interesting. So

$$\begin{aligned}
\partial_t |\psi, t\rangle_I = \partial_t \left(U_I(t)|\psi,0\rangle\right) &= \partial_t \left(U_0^\dagger(t) \underbrace{U_H(t)}_{=e^{-\mathbf{i}H(0)t}=e^{-\mathbf{i}(H_0+V)t}} |\psi, 0\rangle \right) \\
&= U_0^\dagger \left(\underbrace{\mathbf{i}H_0 - \mathbf{i}H}_{=-\mathbf{i}V(0)} \right) U_H \underbrace{|\psi, 0\rangle}_{=U_H^\dagger(t)U_0(t)|\psi,t\rangle_I} \\
&= -\mathbf{i}U_0^\dagger V(0) \underbrace{U_H U_H^\dagger}_{=1} U_0 |\psi, t\rangle_I = -\mathbf{i} \underbrace{U_0^\dagger V(0) U_0}_{=V(t)} |\psi, t\rangle_I .
\end{aligned} \tag{3.4}$$

That is

$$\boxed{\mathbf{i}\partial_t |\psi, t\rangle_I = V(t) |\psi, t\rangle_I}.$$

Alternatively, the interaction-picture evolution operator satisfies

$$\mathbf{i}\partial_t U_I(t) = V(t)U_I(t).$$

Notice how this differs from the Heisenberg evolution equation (3.2): although the full H is time-independent, $V_I(t)$ actually does depend on t, so $[V(t), V(t')] \neq 0$, and so the solution is not just a simple exponential. We'll find a nice packaging for the solution next in the form of Dyson's expansion.

Peskin's notation for this object is $U_I(t) = U(t, t_0)|_{t_0=0}$. We can figure out how to change the reference time from zero as follows:

$$|\psi, t\rangle_I = U_I(t) |\psi, 0\rangle , \quad |\psi, t'\rangle_I = U_I(t') |\psi, 0\rangle \quad \Longrightarrow \quad |\psi, 0\rangle = U_I^\dagger(t) |\psi, t'\rangle$$

$$|\psi, t\rangle_I = \underbrace{U_I(t)U_I^\dagger(t')}_{=U(t,t')} |\psi, t'\rangle_I$$

From which we infer that

$$U(t, t') = U_0^\dagger(t)U_H(t)U_H^\dagger(t')U_0(t') = e^{\mathbf{i}H_0 t}e^{-\mathbf{i}H(t-t')}e^{-\mathbf{i}H_0 t'}.$$

From now on we drop the $P = I$ subscripts: everything is I.

Definition of S-matrix. What are we going to do with the evolution operator? Here is a basic (only slightly naive) three-step framework for doing particle physics (not yet for making predictions).

- At time t_i, specify (*e.g.* measure) all the particle types, spins, momenta in the form of an initial state $|i\rangle$ in the QFT Hilbert space.
- Wait. At time t, the state is

$$U(t, t_i)|i\rangle = |\psi, t\rangle .$$

- At time t_f, measure all the particle data and get some state $|f\rangle$.

Quantum mechanics says that the probability for this outcome is

$$|\langle f| U(t_f, t_i)|i\rangle|^2 \quad .$$

(One way in which this is significantly naive is that the space of outcomes is continuous, so we must instead make probability distributions. More soon.) Because particle interactions are a fast-moving business, a useful idealization is to take $t_i \to -\infty$ and $t_f \to \infty$, and let

$$S_{fi} \equiv \langle f| \hat{S} |i\rangle , \quad \hat{S} \equiv U(\infty, -\infty)$$

the S-matrix ('S' is for 'scattering').

This has only three ingredients: initial state, final state, and time evolution operator. Let's focus on the last one:

Dyson expansion. We need to solve the equation

$$\partial_t |\psi, t\rangle = -\mathbf{i}V(t)|\psi, t\rangle , \quad \text{with initial condition} \quad |\psi, t_i\rangle = |i\rangle .$$

Here's a "solution":

$$|\psi, t\rangle = |i\rangle + (-\mathbf{i}) \int_{t_i}^{t} dt_1 V(t_1)|\psi, t_1\rangle \quad .$$

The only small problem is that we don't know $|\psi, t_1\rangle$. But we can use this expression for that too:

$$\begin{aligned} |\psi, t\rangle &= |i\rangle + (-\mathbf{i}) \int_{t_i}^{t} dt_1 V(t_1) \left(|i\rangle + (-\mathbf{i}) \int_{t_i}^{t_1} dt_2 V(t_2) \right) \\ &= |i\rangle + (-\mathbf{i}) \int_{t_i}^{t} dt_1 V(t_1)|i\rangle + (-\mathbf{i})^2 \int_{t_i}^{t} dt_1 \int_{t_i}^{t_1} dt_2 V(t_1)V(t_2)|\psi, t_2\rangle \end{aligned}$$

Why stop there? Two comments: (1) This is a good idea when $V \propto \lambda \ll 1$. (2) Notice the time-ordering: the range of integration restricts $t_1 \geq t_2$, and the earlier operator $V(t_2)$ is to the right. The ad absurdam limit is

$$|\psi, t\rangle = \sum_{n=0}^{\infty} (-\mathbf{i})^n \int_{t_i}^{t} dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n V(t_1)V(t_2)\cdots V(t_n)|i\rangle = U(t, t_i)|i\rangle \tag{3.5}$$

which since this is true for any $|i\rangle$ tells us a formula for U.

To review, the equation we are trying to solve is:

$$\underbrace{\mathbf{i}\partial_t |\psi, t\rangle}_{=V(t)|\psi,t\rangle} = \mathbf{i}\partial_t U |\psi, t_i\rangle = \mathbf{i}\partial_t U U^\dagger |\psi, t\rangle .$$

This is true for all $|\psi, t\rangle$, so it means $\mathbf{i}\partial_t U U^\dagger = V$. Multiplying the BHS on the right by U gives

$$\implies \partial_t U = -\mathbf{i} V U.$$

We might expect that an equation like this has a solution which is something like $U \overset{?}{\sim} e^{-\mathbf{i}Vt}$.

Now we must deal with what Lawrence Hall calls "the wretched $n!$". Starting from our series solution (3.5),

$$\begin{aligned} U(t, t_i) &= \sum_{n=0}^{\infty} (-\mathbf{i})^n \int_{t_i}^{t} dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n V(t_1)V(t_2)\cdots V(t_n) \\ &= \sum_{n=0}^{\infty} (-\mathbf{i})^n \int_{t_i}^{t} dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n \mathcal{T} \left(V(t_1)V(t_2)\cdots V(t_n)\right) \\ &= \sum_{n=0}^{\infty} (-\mathbf{i})^n \frac{1}{n!} \int_{t_i}^{t} dt_1 \int_{t_i}^{t} dt_2 \cdots \int_{t_i}^{t} dt_n \mathcal{T} \left(V(t_1)V(t_2)\cdots V(t_n)\right) \end{aligned} \quad (3.6)$$

In the first step I used the fact that the operators are already time ordered (this followed from the differential equation we are solving, since the V always acts from the left). In the second step we used the fact that the time-ordered integrand doesn't change if we permute the labels on the times. So we can just average over the $n!$ possible orderings of n times. If we pull out the time-ordering symbol, this is an exponential series:

$$U(t, t_i) = \mathcal{T}\left(e^{-\mathbf{i}\int_{t_i}^{t} dt' V(t')}\right).$$

The time-ordered exponential is defined by its Taylor expansion.

3.2 S-matrix

Taking the times to $\pm\infty$ in the previous equation gives an expression for the S-matrix:

$$\hat{S} = U(-\infty, \infty) = \mathcal{T}\left(e^{-\mathbf{i}\int_{-\infty}^{\infty} dt V(t)}\right). \quad (3.7)$$

The practical value of these expressions is that they give a (compact) recipe for evaluating the time evolution operator as a series in powers of the small parameter in front of $V(0)$: we know $V(t)$ in terms of things like $\mathbf{a}, \mathbf{a}^\dagger$, can pull them down term-by-term.

I should have called the previous expression the 'S-operator', since the thing we are after is really the S-matrix elements, $\langle f|\, \hat{S}\, |i\rangle$, for which we still need $\langle i|$ and $|f\rangle$. Here we encounter some small trouble. Can we just use the states like $\sqrt{2\omega_p}\mathbf{a}_p^\dagger |0\rangle$ (the eigenstates of the free hamiltonian) which we've grown to love? In fact, even the vacuum $|0\rangle$ is not an eigenstate of the actual $H_0 + V$ (since $[H_0, V] \neq 0$), so it will not stay where we put it. The vacuum of the interacting theory $|\Omega\rangle$ is itself an object of mystery (a boiling sea of virtual particles and antiparticles), and the stationary excited states are too (a particle carries with it its disturbance of the vacuum). We'll learn to deal with this in perturbation theory, but here's an expedient: pick a function $f(t)$ which is zero at one end, one in the middle, and then zero again at the far end. Now replace the interaction hamiltonian V with $f(t)V(t)$. Then, if we take $t_i <$ the time before which the interaction turns on, and $t_f >$ the time after which we turn it off, then we can use the free hamiltonian eigenstates. This is in fact wrong in detail, but it will get us started.

Example. Let's return to the 'scalar Yukawa theory' that we briefly encountered earlier in (2.4). Simplifying the notation a bit, the whole Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^\star \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^\star \Phi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + \mathcal{L}_I \tag{3.8}$$

$$\text{with} \qquad \mathcal{L}_I = -g \Phi^\star \Phi \phi.$$

The mode expansions are

$$\phi = \int \frac{d^d p}{\sqrt{2\omega_p}} \left(\mathbf{a}_p e^{-\mathbf{i}px} + \mathbf{a}_p^\dagger e^{\mathbf{i}px} \right) |_{p^0 = \omega_p}$$

$$\Phi = \int \frac{d^d p}{\sqrt{2E_p}} \left(\mathbf{b}_p e^{-\mathbf{i}px} + \mathbf{c}_p^\dagger e^{\mathbf{i}px} \right) |_{p^0 = E_p}$$

where I've written $\omega_p \equiv \sqrt{M^2 + p^2}, E_q \equiv \sqrt{m^2 + q^2}$. Notice that the $\Phi \to e^{-\mathbf{i}\alpha}\Phi$ symmetry is conserved; the charge is

$$q = N_c - N_b.$$

But the ϕ particles are not conserved.[17]

Artisanal meson decay. [Tong §3.2.1] Consider

$$|i\rangle = \sqrt{2\omega_p} a_p^\dagger |0\rangle\,, \quad |f\rangle = \sqrt{2E_{q_1} 2E_{q_2}} b_{q_1}^\dagger c_{q_2}^\dagger |0\rangle\,.$$

[17] You might notice a possible problem with this theory: what happens to the quadratic term for Φ when ϕ is very negative? Let's not take it too seriously.

The S-matrix element between these states is

$$\langle f| \hat{S} |i\rangle = \langle f| \left(1 + (-\mathbf{i}) \int d^{d+1}x g\phi_x \Phi^\star_x \Phi_x + \mathcal{O}(g^2)\right) |i\rangle$$

where all operators are in interaction picture. The first term dies because $\langle f|i\rangle = 0$. The $\phi \sim a + a^\dagger$ takes a one-particle state into a superposition of states with zero and two ϕ-particles. We need to end up with zero ϕ-particles. The leading-order nonzero term is

$$\begin{aligned}
&= -\mathbf{i}g \int d^{d+1}x \langle f| \Phi^\star_x \Phi_x \int \frac{đ^d k}{\sqrt{2\omega_k}} e^{-\mathbf{i}kx} \underbrace{\mathbf{a}_k \mathbf{a}^\dagger_p |0\rangle}_{=đ^d(k-p)|0\rangle} \sqrt{2\omega_k} \\
&= -\mathbf{i}g \int d^{d+1}x e^{-\mathbf{i}px} \langle 0| \mathbf{b}_{q_1} \mathbf{c}_{q_2} \sqrt{4E_{q_1}E_{q_2}} \int \frac{đ^d k_1}{\sqrt{2E_{k_1}}} e^{\mathbf{i}k_1 x} \mathbf{b}^\dagger_{k_1} \int \frac{đ^d k_2}{\sqrt{2E_{k_2}}} e^{\mathbf{i}k_2 x} \mathbf{c}^\dagger_{k_1} |0\rangle \\
&= -\mathbf{i}g \int d^{d+1}x e^{\mathbf{i}(q_1+q_2-p)x} = -\mathbf{i}g(2\pi)^{d+1}\delta^{d+1}(q_1+q_2-p)
\end{aligned}$$

This is a small victory. This delta function imposes conservation of energy and momentum on the transition amplitude. In the ϕ rest frame, $p^\mu = (M, 0)$ which says the amplitude is only nonzero when $\vec{q}_1 = -\vec{q}_2$ and when $M = 2\sqrt{|q_1|^2 + m^2}$. Notice that this can only happen if $M \geq 2m$.

How do we get from this amplitude to a probability? We have to square it:

$$P_{fi} \sim |S_{fi}|^2 = g^2 \left(\delta^{d+1}(p_f - p_i)\right)^2.$$

The square of a delta function is infinity. What did we do wrong?

[End of Lecture 8]

Not so much, we just asked a dumb question. Here is where it helps to be a physicist. Consider:

$$\left(\delta^{d+1}(p)\right)^2 = \delta^{d+1}(p)\delta^{d+1}(0) = \delta^{d+1}(p) \int d^{d+1}x e^{\mathbf{i}0x} = \delta^{d+1}(p) VT$$

where VT is the *volume of spacetime* – the size of the box times how long we're willing to wait. There is a nonzero probability per unit time per unit volume that a ϕ particle in a plane wave state will decay. We'll get its lifetime out momentarily.

For more complicated examples, it will help to streamline this process, which is the job of §3.3.

3.3 Time-ordered equals normal-ordered plus contractions

We have an expression for $\hat{S}$ in (3.7) involving only time-ordered products of operators. If we stick this between states with just a few particles, the annihilation operators in there very much want to move to the right so they can get at the vacuum and annihilate it, as is their wont. Wick's theorem tells us how to do this, along the following lines:

$$\text{Wick:} \qquad \underbrace{\mathcal{T}(\phi...\phi)}_{\text{have}} = \underbrace{:\phi...\phi:}_{\text{want}} \ + \ ?$$

In the previous schematic non-equation, I introduced a notation for a *normal-ordered product* $:\phi\cdots\phi:$ which means each term has all the annihilation operators to the right of all the creation operators, for example

$$:\phi(x)\phi(y):\equiv \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y). \tag{3.9}$$

This operation is by definition linear.

Normal-ordering difficulty. Actually, there is a trap here which was brought to my attention by Chuncheong Lam: If we want the normal-ordering operation to be linear, then we must have

$$:\left(\underbrace{\mathbf{a}^\dagger\mathbf{a}}_{=\mathbf{a}\mathbf{a}^\dagger-1}\right):=:\mathbf{a}\mathbf{a}^\dagger:-:1:.$$

But $\left\langle 0\right| :\mathbf{a}^\dagger\mathbf{a}: \left|0\right\rangle = \left\langle 0\right| :\mathbf{a}\mathbf{a}^\dagger: \left|0\right\rangle = 0$ so we must have $\langle 0| :1: |0\rangle = 0$, which means we must have the shocking-looking equation:

$$:1:=0$$

that is: the normal-ordered product of a c-number must be zero. This definition (which, beware, differs from Peskin's) has the advantage that my statement below that the vacuum expectation value (VEV) of any normal ordered product is zero (with no exceptions for c-numbers). The price is that we cannot put the normal-ordering symbol around the c-number bits, as Peskin does.

More generally, let A, B, C be the positive- and negative-frequency bits of some fields. Then

$$:ABC\cdots:\equiv\left(\underbrace{A'B'C'\cdots}_{\text{only } \mathbf{a}^\dagger\text{s}}\underbrace{\cdots}_{\text{only } \mathbf{a}\text{s}}\right)$$

Peskin writes $N(\cdots)\equiv:\cdots:$. Notice that $\langle 0| : \text{anything} : |0\rangle = 0$.

A comment about fermions. Later we will use *anticommuting* operators, which have

$$\{\mathbf{c}_k, \mathbf{c}_p^\dagger\} = \not{\delta}(k-p), \quad \{\mathbf{c}_k^\dagger, \mathbf{c}_p^\dagger\} = 0.$$

In particular, the equation $\left(\mathbf{c}_p^\dagger\right)^2 = 0$ is an algebraic realization of the Pauli principle. The cost is that even the ϕ^- bits generate signs when they move through each other. In that case, we define the normal ordered product as

$$: ABC \cdots :\equiv \left(\underbrace{A'B'C' \cdots}_{\text{only } \mathbf{a}^\dagger\text{s}} \underbrace{\cdots}_{\text{only } \mathbf{a}\text{s}} \right) (-1)^P$$

where P is the number of fermion interchanges required to get from $ABC\cdots$ to $A'B'C'\cdots$. Keeping track of these signs, and replacing commutators with anticommutators, everything below goes through for fermion fields.

Let's go back to (3.9). Because $[\phi^\pm, \phi^\pm] = 0$, the order in the last two terms doesn't matter. This can differ from the time-ordered product only in the first or second term. If $y^0 > x^0$, it differs by $[\phi^-(x), \phi^+(y)] = -\Delta^+(x-y)$, and if $x^0 > y^0$, it differs by $[\phi^-(y), \phi^+(x)] = +\Delta^+(x-y)$. Altogether:

$$: \phi(x)\phi(y) := \mathcal{T}\left(\phi(x)\phi(y)\right) - \Delta_F(x-y) \equiv \mathcal{T}\left(\phi(x)\phi(y)\right) - \overbracket{\phi(x)\phi}(y). \tag{3.10}$$

More generally, writing $\phi_a \equiv \phi_a(x_a)$, Wick's theorem says

$$: \phi_1 \cdots \phi_n := \mathcal{T}\left(\phi_1 \cdots \phi_n\right) + (\text{all contractions})$$

where a *contraction* is defined as the price for moving a pair of operators through each other to repair the time ordering, as in (3.10), and denoted by the symbol in (3.10).

For example, for four fields, the theorem says

$$\mathcal{T}(\phi_1 \cdots \phi_4) =: \left(\phi_1 \cdots \phi_4 + (\overbracket{\phi_1\phi_2}\phi_3\phi_4 + 5 \text{ more}) \right) : + (\overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4} + 2 \text{ more})$$

Notes: The fully-contracted bits are numbers, so they are outside the normal-ordering symbol. For a product of n fields, there are $\binom{n}{2} + \binom{n}{4}\binom{4}{2} + \binom{n}{6}\binom{6}{4}\binom{4}{2} \cdots + \binom{n}{\lfloor n/2 \rfloor}$ (= many) contractions. But if we take the vacuum expectation value (VEV) of the BHS, most terms go away.

Here's the idea of the proof (Peskin page 90), which is by induction on the number of fields m in the product. We showed $m = 2$ above. Assume WLOG that $x_1^0 \geq \cdots \geq x_m^0$, or else relabel so that this is the case. Wick for $\phi_2 \cdots \phi_m$ says

$$\mathcal{T}\left(\phi_1 \cdots \phi_m\right) = \underbrace{\phi_1}_{=\phi_1^+ + \phi_1^-} : \phi_2 \cdots \phi_m : + \left(\text{all contractions w/o } \phi_1 \right)$$

The ϕ_1^- term is already in the right place and can slip for free inside the normal-ordering sign. The ϕ_1^+ needs to move past all the uncontracted $\phi_{j\geq 2}^-$s; this process will add a term for every possible contraction involving ϕ_1. ■

3.4 Time-ordered correlation functions

Time-ordered correlation (or Green's) functions of local operators will be useful:

$$G^{(n)}(x_1 \cdots x_n) \equiv \langle \Omega | \mathcal{T}\left(\phi_1^H(x_1) \cdots \phi_n^H(x_n)\right) | \Omega \rangle .$$

Here, the operators are in Heisenberg picture for the full hamiltonian, and Ω is its actual lowest-energy eigenstate, $H |\Omega\rangle = E_0 |\Omega\rangle$. The fourier transform is also useful:

$$\tilde{G}^{(n)}(p_1 \cdots p_n) \equiv \int d^{d+1}x_1 \cdots \int d^{d+1}x_n \; e^{-\mathbf{i}\sum_i^n p_i x_i} G^{(n)}(x_1 \cdots x_n) .$$

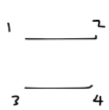
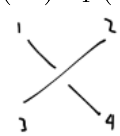
In the free theory of a real scalar, we know something about these:

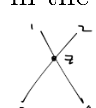
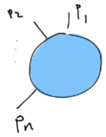
$$G^{(2)}_{\text{free}}(x_1, x_2) = \Delta_F(x_1 - x_2) =$$ 

$$\tilde{G}^{(2)}(p_1, p_2) = \not{\delta}^{d+1}(p_1 + p_2)\frac{\mathbf{i}}{p^2 - m^2 + \mathbf{i}\epsilon} = \not{\delta}^{d+1}(p_1 + p_2) \cdot$$  (3.11)

The higher correlations are Gaussian, in the sense that they are sums of products of the two point functions:

$$G^{(4)}_{\text{free}}(x_1 \cdots x_4) = \Delta_F(12)\Delta_F(34) + \Delta_F(13)\Delta_F(24) + \Delta_F(14)\Delta_F(23)$$

$=$  $+$  $+$  . (3.12)

Expectations. Our next goal is to construct a perturbative expansion in the case of $V = \int d^dz \frac{\lambda}{4!}\phi^4(z)$. We expect a correction of order λ of the form:  In momentum space, we have blobs (unspecified sums of diagrams) with external lines labelled by p_i^μ:  Notice that there is no need to restrict their values to the mass shell $p_i^2 = m^2$, that is, $\tilde{G}(p)$ is nonzero even when p_i are off-shell: these Green's functions contain "off-shell" information, more information than is available in just the scattering matrix. However, something special will happen when the external legs are on-shell. As you can see from the free two-point function, (3.11), they *blow up* on the mass-shell. The existence of a singularity of $\tilde{G}$ on the mass-shell is a general fact, and their residues give the S-matrix elements:

$$\tilde{G}(p_1 \cdots p_n) \stackrel{p_i^2 \to m_i^2}{\to} \prod_i \frac{\mathbf{i}}{p_i^2 - m_i^2 + \mathbf{i}\epsilon} S(p_1 \cdots p_n).$$

(This is the LSZ theorem, about which more later.)

Perturbative expansion of time-ordered correlators. We'll do this in three steps: (1) Relate $|\Omega\rangle$ to $|0\rangle$. (2) Relate ϕ_H to ϕ_I. (3) Wick expand and organize the diagrams.

Step (1): [Peskin page 86-87] Some preparations:

- Fix the additive normalization of the hamiltonian by $H_0 |0\rangle = 0$.
- Label the spectrum of H by $|n\rangle$, so $\mathbb{1} = \sum_n |n\rangle \langle n|$. This is a very scary sum over the whole QFT Hilbert space, really an integral.
- Assume that $\langle \Omega | 0 \rangle \neq 0$. A necessary condition for this is that the actual Hamiltonian is in the same phase as the H_0. Also, let's keep the volume finite for awhile.

Now consider

$$\langle 0| e^{-\mathbf{i}HT} = \sum_n \langle 0|n\rangle \langle n| e^{-\mathbf{i}HT} = \sum_{n \neq \Omega} \langle 0|n\rangle \langle n| e^{-\mathbf{i}E_n T} + \langle 0|\Omega\rangle \langle \Omega| e^{-\mathbf{i}E_0 T}.$$

Since $E_0 < E_n$ for all other n, by given T a large negative imaginary part, $T \to \infty(1 - \mathbf{i}\epsilon)$ we can make the contribution of Ω arbitrarily larger than the others. Multiplying by $e^{\mathbf{i}E_0 T} / \langle 0|\Omega\rangle$ gives

$$\langle \Omega| = \lim_{T \to \infty(1-\mathbf{i}\epsilon)} \left(\frac{\langle 0| e^{-\mathbf{i}HT} e^{\mathbf{i}E_0 T}}{\langle 0|\Omega\rangle} \right) \stackrel{\langle 0|H_0 = 0}{=} \lim_{T \to \infty(1-\mathbf{i}\epsilon)} \left(\frac{\langle 0| e^{\mathbf{i}H_0 T} e^{-\mathbf{i}HT} e^{\mathbf{i}E_0 T}}{\langle 0|\Omega\rangle} \right)$$

Since T is infinite anyway, we can shift it to $T \to T - t_0$ without change:

$$\langle \Omega| = \lim_{T \to \infty(1-\mathbf{i}\epsilon)} \left(\frac{\langle 0| e^{\mathbf{i}H_0 (T - t_0)} e^{-\mathbf{i}H(T-t_0)} e^{\mathbf{i}E_0 (T-t_0)}}{\langle 0|\Omega\rangle} \right) = \lim_{T \to \infty(1-\mathbf{i}\epsilon)} \left(\frac{\langle 0| U_I(t, t_0) e^{\mathbf{i}E_0(T-t_0)}}{\langle 0|\Omega\rangle} \right). \tag{3.13}$$

Do the same for $|\Omega\rangle$.

Now step (2): Consider

$$\begin{aligned}
G^{(2)}(x,y) &= \langle\Omega|\,\mathcal{T}\,(\phi_H(x)\phi_H(y))\,|\Omega\rangle \\
&\overset{(3.13)}{=} \lim_{T\to\infty(1-\mathbf{i}\epsilon)} \Big[\left(e^{-\mathbf{i}E_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}\langle 0|\,U_I(T,t_0) && \langle\Omega| \\
&\cdot \quad U_I^\dagger(x^0,t_0)\phi_I(x)U_I(x^0,t_0)\cdot U_I^\dagger(y^0,t_0)\phi_I(y)U_I(y^0,t_0) && \phi_H\phi_H \\
&\cdot \quad U_I(t_0,-T)\,|0\rangle\left(e^{-\mathbf{i}E_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1}\Big] && |\Omega\rangle \\
&= \lim_{T\to\infty(1-\mathbf{i}\epsilon)}\left(e^{-\mathbf{i}2E_0T}|\langle 0|\Omega\rangle|^2\right)^{-1}\langle 0|\,\underbrace{U(T,x^0)\phi(x)U(x^0,y^0)\phi(y)U(y^0,-T)}_{=\mathcal{T}(\phi(x)\phi(y)U(T,-T))}\,|0\rangle
\end{aligned}$$

In the last expression I've gone back to implying the I subscripts on the interaction picture fields. In observing that the big underbraced product is time ordered we are appealing to the Dyson formula for the interaction-picture evolution operators, *e.g.* $U_I(t,t') = \mathcal{T}\left(e^{-\mathbf{i}\int_{t'}^{t}dt''V(t'')}\right)$ – so it *is* a sum of time-ordered products, evaluated between the times in the argument. Notice that I did something slippery in the first step by combining the factors into one big limit; this is OK if each factor converges separately.

What's the denominator? The norm of the vacuum is one, but we can assemble it from these ingredients (3.13):

$$1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-\mathbf{i}\epsilon)}\left(e^{-\mathbf{i}2E_0T}|\langle 0|\Omega\rangle|^2\right)^{-1}\langle 0|\,\underbrace{U(T,t_0)U(t_0,-T)}_{=U(T,-T)}\,|0\rangle$$

Therefore

$$G^{(2)}(x,y) = \lim_{T\to\infty(1-\mathbf{i}\epsilon)}\frac{\langle 0|\,\mathcal{T}\left(\phi(x)\phi(y)e^{-\mathbf{i}\int_{-T}^{T}dt'V(t')}\right)|0\rangle}{\langle 0|\,\mathcal{T}\left(e^{-\mathbf{i}\int_{-T}^{T}dt'V(t')}\right)|0\rangle}$$

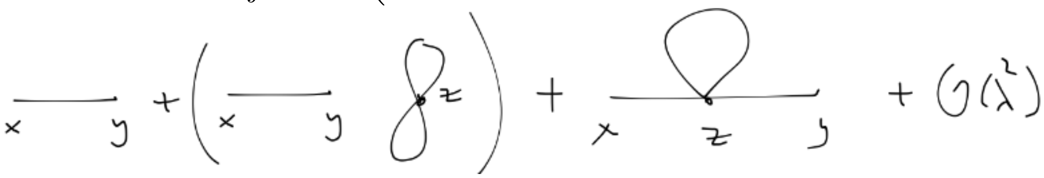
[End of Lecture 9]

The same methods give the analogous formula for $G^{(n)}(x_1\cdots x_n)$ for any number of any local operators. Now we can immediately perturbate to our hearts' content by expanding the exponentials. Let's do some examples, then I will comment on the familiarity of the prescription for T, and we will see that the denominator is our friend because it cancels annoying (disconnected) contributions in the numerator.

Examples. For $V = \frac{\lambda}{4!}\phi^4$, let's study the numerator of $G^{(2)}(x,y)$ in the first few orders of λ:

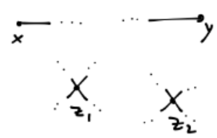
$$G^{(2)}_{\text{num}}(x,y) = \langle 0|\,\mathcal{T}\left(\phi(x)\phi(y)e^{-\mathbf{i}\int_{-T}^{T}d^{d+1}z\frac{\lambda}{4!}\phi^4(z)}\right)|0\rangle$$

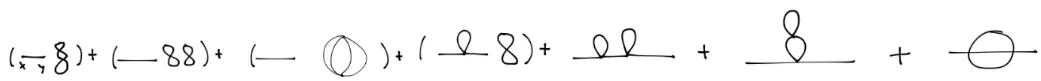
$$= \langle 0|\, \mathcal{T}\phi(x)\phi(y)\,|0\rangle + \frac{-\mathbf{i}\lambda}{4!}\int d^{d+1}z\, \langle 0|\, \mathcal{T}\left(\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\right)|0\rangle + \mathcal{O}(\lambda^2)$$
$$= \Delta_F(x-y) + \frac{-\mathbf{i}\lambda}{4!}\int d^{d+1}z \left(3\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z) + 4\cdot 3\phi(x)\phi(z)\phi(y)\phi(z)\phi(z)\phi(z)\right) +$$

$=$  $+\, \mathcal{O}(\lambda^2)$

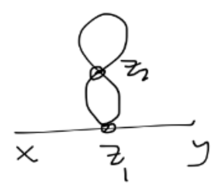
The $\mathcal{O}(\lambda^2)$ contribution is

$$\frac{1}{2!}(-\mathbf{i}\lambda)^2 \int d^{d+1}z_1 d^{d+1}z_2\, \langle 0|\, \mathcal{T}\left(\phi(x)\phi(y)\phi(z_1)^4\phi(z_2)^4\right)|0\rangle$$

With ten fields, there will be five propagators in each diagram. The ingredients which we must connect together are:  The answer is



For example,

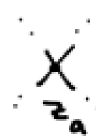
$$\phi(x)\phi(z_1)\phi(y)\phi(z_1)\left(\phi(z_1)\phi(z_2)\right)^2\phi(z_2)\phi(z_2) \propto$$ 

up to the symmetry factor.

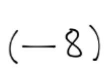
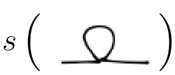
Feynman rules for ϕ^4 theory in position space. The set of diagrams is made by drawing external vertices for each x_i, and n internal vertices, and connecting them in all possible ways with propagators,

$$\{\text{diagrams}\} \equiv \{A\} = \{A_0\} \cup \{A_1\} \cup \cdots$$

where A_m gives contributions proportional to λ^m. Then the associated amplitude is $\mathcal{M}_A$ and the Green's function is $G^{(n)}(x_1 \cdots x_n) = \sum_A \mathcal{M}_A$. To get $\mathcal{M}_A$,

- Put a $-\mathbf{i}\lambda \int d^{d+1}z_a$ for each vertex  (notice no $\frac{1}{4!}$).

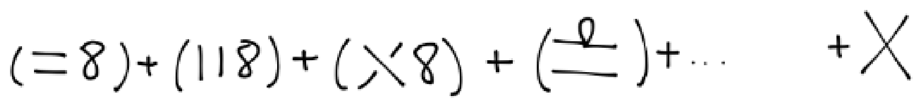
- Put a $\Delta_F(y_i - y_j)$ for each propagator  where y may be an internal point x_i or an internal point z_a.
- Multiply by the symmetry factor $s(A)$. The symmetry factor is defined to be $s(A) = |\mathrm{Aut}(A)|^{-1}$, the inverse of the order of the automorphism group of the diagram. Symmetries of the diagram mean that the sum over contractions fails to completely cancel Dyson's wretched $\frac{1}{n!}$. For example:

  s  $= \frac{1}{4!} \cdot 3 = \frac{1}{8}$, $\quad s$  $= \frac{1}{4!} \cdot 4 \cdot 3 = \frac{1}{2}$.

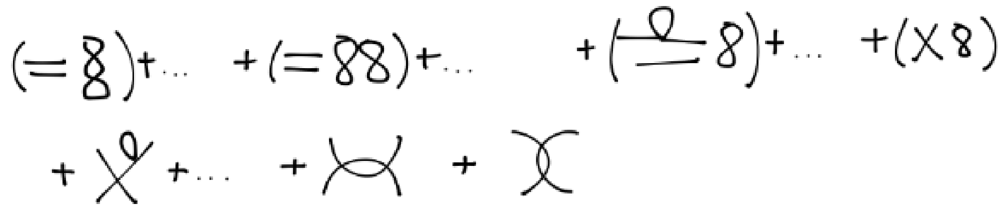
  Do not get hung up on this right now.

Let's do the numerator of $G^{(4)}$ through order λ^2:

$\mathcal{O}(\lambda^0)$: 

$\mathcal{O}(\lambda^1)$: 

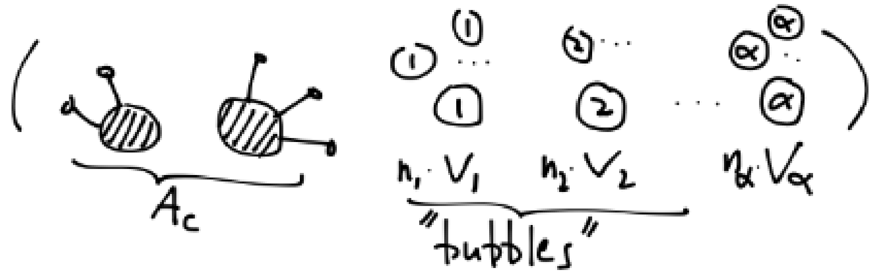
Notice that only the last term here is "fully connected" in the sense that you can't divide the diagram up into disjoint pieces without cutting propagators. The other diagrams follow a simple pattern: the first three are obtained from the $\mathcal{O}(\lambda^0)$ diagrams by multiplying by a figure-eight bubble. The second set is obtained by mutiplying $G_0^{(2)} \cdot G_1^{(2)}$, where $G_m^{(n)}$ denotes the order-λ^m fully-connected contribution to $G^{(n)}$.

$\mathcal{O}(\lambda^2)$: 

(3.14)

Here's an easy one: $G^{(n)} = 0$ when n is odd. Technically, we can see this from the fact that there is always a ϕ left over after all contractions, and $\langle 0|\phi|0\rangle = 0$. Slightly more deeply, this is because of the $\phi \to -\phi$ symmetry.

The exponentiation of the disconnected diagrams. [Peskin page 96] There are some patterns in these sums of diagrams to which it behooves us to attend. (The following discussion transcends the ϕ^4 example.) The general diagram has the form:



Only some of the components are attached to the external legs; for a given diagram A, call the factor associated with these components A_c (note that A_c need not be *fully* connected). The rest of the diagram is made of a pile of 'bubbles' of various types V_i (each one internally connected) and multiplicities n_i (*e.g.* V_1 could be a figure eight, and there could be $n_1 = 2$ of them, as in the second term indicated in (3.14)). These bubbles (or 'vacuum bubbles') would be there even if we didn't have any external lines, and they would have the same value; they are describing the fluctuations intrinsic to the vacuum. The amplitude associated with the general diagram is then

$$\mathcal{M}_A = \mathcal{M}_{A_c} \cdot \frac{V_1^{n_1}}{n_1!} \cdot \frac{V_2^{n_2}}{n_2!} \cdots \frac{V_\alpha^{n_\alpha}}{n_\alpha!}$$

where the $n_i!$ factors are the most important appearance of symmetry factors: they count the number of ways to permute the identical copies of V_i amongst themselves.

The numerator of $G^{(n)}$ is then

$$\begin{aligned} G^{(n)}_{\text{numerator}} = \langle 0| \mathcal{T}\left(\phi_1 \cdots \phi_n e^{-\mathbf{i}\int V}\right)|0\rangle = \sum_A \mathcal{M}_A &= \sum_{A_c} \mathcal{M}_{A_c} \sum_{\{n_i = 0\}} \frac{V_1^{n_1}}{n_1!} \cdot \frac{V_2^{n_2}}{n_2!} \cdots \frac{V_\alpha^{n_\alpha}}{n_\alpha!} \\ &= \sum_{A_c} \mathcal{M}_{A_c} \cdot e^{V_1} \cdot e^{V_2} \cdots e^{V_\alpha} \\ &= \sum_{A_c} \mathcal{M}_{A_c} e^{\sum_i V_i} \end{aligned} \tag{3.15}$$

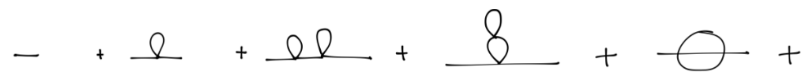
– the bubbles always exponentiate to give the same factor of $e^{\sum_i V_i}$, independant of the external data in G. In particular, consider the case of $n = 0$, where there are no external lines and hence no A_c:

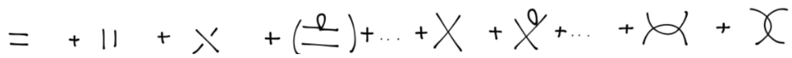
$$G^{(0)}_{\text{numerator}} = \langle 0| \mathcal{T} e^{-\mathbf{i}\int V}|0\rangle = 1 \cdot e^{\sum_i V_i}$$

But we care about this because it is the denominator of the actual Green's function:

$$G^{(n)} = \frac{\langle 0| \mathcal{T}\left(\phi_1 \cdots \phi_n e^{-\mathbf{i}\int V}\right)|0\rangle}{\langle 0| \mathcal{T} e^{-\mathbf{i}\int V}|0\rangle} = \frac{G^{(n)}_{\text{numerator}}}{G^{(0)}_{\text{numerator}}} = \sum_{A_c} \mathcal{M}_{A_c} . \tag{3.16}$$

And with that we can forget all about the bubbles. So for example,

$G^{(2)} =$ 

$G^{(4)} =$ 

Notice that in this manipulation (3.16) we are adding terms of many orders in perturbation theory in the coupling λ. If we want an answer to a fixed order in λ, we can regard anything of higher order as zero, so for example, it makes perfect sense to write

$$G^{(2)} = \frac{x_1 \text{———} x_2 \cdot (1+8+88+\cdots)}{(1+8+88+\cdots)} + \mathcal{O}(\lambda) = x_1 \text{———} x_2 \cdot \frac{e^V}{e^V} + \mathcal{O}(\lambda) = x_1 \text{———} x_2 + \mathcal{O}(\lambda).$$

(I only drew one kind of bubble in the previous expression since that one was easy to type.)

Momentum space Green's functions from Feynman diagrams. In translation-invariant problems, things are usually a little nicer in momentum space. Let's think about

$$\tilde{G}^{(n)}(p_1 \cdots p_n) \equiv \prod_{i=1}^{n} \int d^{d+1}x_i e^{-\mathbf{i}p_i x_i} G^{(n)}(x_1 \cdots x_n).$$

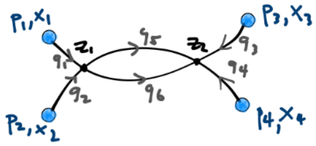
Again, this an off-shell Green's function, a function of general p, not necessarily $p^2 = m^2$. It will, however, vanish unless $\sum_i p_i^\mu = 0$ by translation invariance. Consider a fully-connected contribution to it, at order λ^N. (We'll get the others by multiplying these bits.) In ϕ^4 theory, we need to make a diagram by connecting n external position vertices x_i to N 4-valent vertices z_a using Feynman propagators $\Delta_F(y_A - y_B) = \int đ^{d+1}q_r e^{-\mathbf{i}(y_A - y_B)q_r} \frac{\mathbf{i}}{q_r^2 - m^2 + \mathbf{i}\epsilon}$. Since each propagator has two ends, the number of internal lines (by the fully-connected assumption) is

$$N_I = \frac{\#\text{ of ends of lines}}{2} = \frac{n + 4N}{2} = \frac{n}{2} + 2N.$$

The associated amplitude is then

$$A^N_{FC} = \int d^{d+1}x_1 \cdots d^{d+1}x_n e^{-\mathbf{i}\sum_i p_i x_i} (-\mathbf{i}\lambda)^N \cdot s(FC) \int d^{d+1}z_1 \cdots \int d^{d+1}z_N \prod_{r=1}^{N_I} \int đ^{d+1}q_r e^{-\mathbf{i}(y_A - y_B)q_r} \frac{\mathbf{i}}{q_r^2 - m^2}$$

For example, consider the particular contribution with $n = 4$ external legs and $N = 2$ interaction vertices:



has $\quad N_I = \dfrac{4 + 2 \cdot 4}{2} = 6$

Notice that we are doing a silly thing here of labelling the momenta of the external lines (the first n momenta $q_{i=1...n}$). Here's why it's silly: Look at the integral over x_1. Where is the dependence on x_1? There is the external factor of $e^{-\mathbf{i}p_1 x_1}$ that we put to do the Fourier transform, and there is the propagator taking x_1 to z_1, $\Delta_F(x_1 - z_1) = \int \bar{d}^{d+1} q_1 e^{-\mathbf{i}(x_1-z_1)q_1} \frac{\mathbf{i}}{q_1^2 - m^2 + \mathbf{i}\epsilon}$. So the integral over x_1

$$\int d^{d+1}x_1 e^{-\mathbf{i}x_1(p_1-q_1)} = \not{\delta}^{d+1}(p_1 - q_1)$$

just sets $p_1 = q_1$, and eats the $\int \bar{d}^{d+1} q_1$. The same thing happens for each external line, and reduces the number of momentum integrals to $N_I - n$.

Where is the dependence on z_2?

$$\int d^{d+1}z_2 \; e^{-\mathbf{i}z_2(q_3+q_4+q_5+q_6)} = \not{\delta}^{d+1}(q_3 + q_4 + q_5 + q_6)).$$

Similarly, the z_1 dependence is all in the exponentials:

$$\int d^{d+1}z_1 \; e^{-\mathbf{i}z_1(-q_3-q_4+q_1+q_2)} = \not{\delta}^{d+1}(q_3 + q_4 - q_1 - q_2)).$$

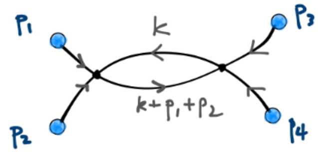
These two factors combine to set $q_1 + q_2 = q_3 + q_4 = -q_5 - q_6$: momentum is conserved at the vertices. Notice that in the example $q_5 - q_6$ is *not* determined.

Each internal vertex reduces the number of undetermined momenta by one, except one combination is already fixed by overall momentum conservation so we have left

$$N_I - n - (N - 1) = N - \frac{n}{2} + 1$$

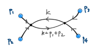
momentum integrals. This number is ≥ 0 for fully connected diagrams, and it is the number of *loops* in the diagram. (This counting is the same as in a Kirchoff's law resistor network problem.) In the example, $N_L = 2 - 2 + 1 = 1$ which agrees with one undetermined momentum integral.

[End of Lecture 10]

In practice now, we need not introduce all those extra qs. Label the external lines by $p_1 \cdots p_n$, and the loop momenta by $k_\alpha, \alpha = 1..N_L$. In the example, we might do it like this: 

for which the amplitude is

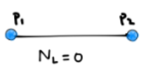
$$A_{FC}(p_1 \cdots p_n) = (-\mathbf{i}\lambda)^N \cdot s(FC) \not{\delta}^{(d+1)}\left(\sum p_i\right) \int \prod_{\text{loops},\alpha=1}^{N_L} đ^{d+1}k_\alpha \prod_{\text{lines},r} \frac{\mathbf{i}}{q_r^2 - m^2 + \mathbf{i}\epsilon}$$



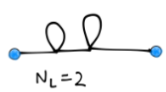
$$= \frac{(-\mathbf{i}\lambda)^2}{2!} \not{\delta}^{d+1}\left(\sum_{i=1}^{4} p_i\right) \prod_{i=1}^{n=4} \frac{\mathbf{i}}{p_i^2 - m^2 - \mathbf{i}\epsilon} \int đ^{d+1}k \frac{\mathbf{i}}{k^2 - m^2 + \mathbf{i}\epsilon} \frac{\mathbf{i}}{(p_1 + p_2 + k)^2 - m^2 + \mathbf{i}\epsilon}$$

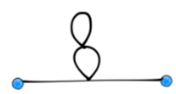
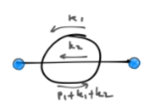
(You might notice that the integral over k is in fact formally infinite, since at large k it goes like $\int^\Lambda \frac{d^4k}{k^2} \sim \log(\Lambda)$. Try to postpone that worry.) For now, let's celebrate my successful prediction, for this particular graph, that there would be poles when the external particles are on-shell, $p_i^2 = m^2$. (It would be more correct to call it Lehmann, Symanzik and Zimmerman's successful prediction.)

The whole two point function in momentum space is then (through order λ^2) :

$\tilde{G}^{(2)} =$ 

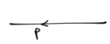
$+$ 

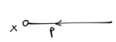
$+$ 

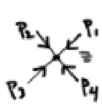
$+$  $+$ 

$+ \mathcal{O}(\lambda^3)$ (3.17)

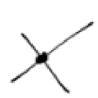
I draw the blue dots to emphasize the external propagators. So here are the momentum space Feynman rules for Green's function in ϕ^4 theory:

- An internal line is  $= \frac{\mathbf{i}}{p^2 - m^2 + \mathbf{i}\epsilon} = \tilde{\Delta}_F(p)$. Notice that since $\Delta_F(x - y) = \Delta_F(y - x)$, the choice of how we orient the lines is not so fateful.

- An external line at fixed position,  $= e^{-\mathbf{i}px}$ More generally, external vertices are associated with the wavefunctions of the states we are inserting; here they are plane waves.

- An internal vertex gives 

$\rightsquigarrow (-\mathbf{i}\lambda) \int d^{d+1}z e^{-\mathbf{i}\sum_i p_i z} = (-\mathbf{i}\lambda)\not{\delta}^{d+1}(\sum_i p_i)$, momentum conservation at each vertex. So, set $\sum_i p_i = 0$ at each vertex (I've assumed the arrows are all pointing toward the vertex). After imposing momentum conservation, the remaining consequence of the vertex is

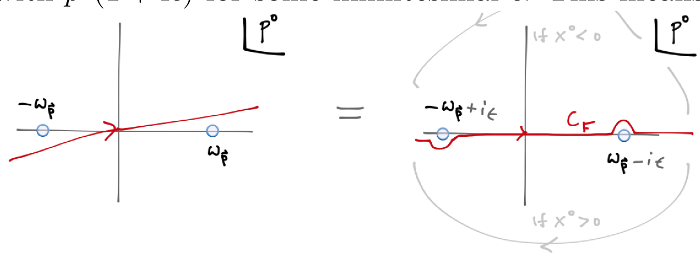
 $= -\mathbf{i}\lambda.$

- Integrate over the loop momenta $\prod_{\alpha=1}^{N_L} đ^{d+1} q_\alpha$ for each undetermined momentum variable. There is one for each loop in the diagram. You should think of these integrals as just like the Feynman path integral: if there is more than one way to get from here to there, we should sum over the amplitudes.
- Multiply by the wretched symmetry factor $s(A)$.
- For $\tilde{G}(p)$, multiply by an overall $\not{\delta}^{d+1}(\sum p)$ in each diagram.

Comment on $T \to \infty(1 - \mathbf{i}\epsilon)$. What happened to the limit on T? It's hidden in the integrals over the vertices:

$$\int d^{d+1}z\ e^{-\mathbf{i}z(\sum_i q_i)} \cdots = \lim_{T\to\infty(1-\mathbf{i}\epsilon)} \int dz^0 d^d z\ e^{-\mathbf{i}\left(z^0\left(\sum_i q_i^0 - \sum_i \vec{z}\cdot\vec{p}_i\right)\right)} \cdots$$

One end of the integral $z^0 = \pm\infty$ is going to be infinite unless $\sum_i p_i^0 z^0 \in \mathbf{i}\mathbb{R}$, in which case it just oscillates. This seems scary. We can make ourselves feel better about it if we just replace every p^0 with $p^0(1 + \mathbf{i}\epsilon)$ for some infinitesimal ϵ. This means that the integrals will look like:

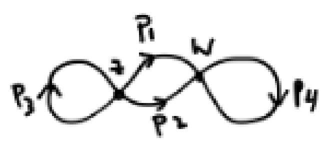


That is: if we use the Feynman contour for every propagator

$$\Delta_F(x) = \int_{C_F} đ^{d+1} p e^{-\mathbf{i}px} \frac{\mathbf{i}}{p^2 - m^2 + \mathbf{i}\epsilon_F}$$

with $\epsilon_F = \epsilon$ then this problem goes away.

The factors of T give another perspective on the exponentiation of the vacuum bubbles. Consider the diagram:



$$= (-\mathbf{i}\lambda)^2 \prod_{i=1}^{4} \int đ^{d+1} p_i \not{\delta}^{d+1}(p_1 + p_2) \not{\delta}^{d+1}(p_1 + p_2) \cdots$$

The two delta functions come from the integrals over $z_{1,2}$, and we can restore sense by remembering this:

$$\left(\not{\delta}^{d+1}(p_1 + p_2)\right)^2 = \not{\delta}^{d+1}(p_1 + p_2) \int d^{d+1} z_2 = \not{\delta}^{d+1}(p_1 + p_2) 2TV$$

where V is the volume of space. This factor arises because this process can happen anywhere, anytime. There is one such factor for each connected component of a collection of vacuum bubbles. But the free energy $\propto \log Z = \log G^{(0)}$ should be extensive, $\propto VT$. Therefore, the vacuum bubbles must exponentiate.

3.5 Interlude: old-fashioned perturbation theory

[Schwartz, chapter 4] I want to take a brief break from the inexorable building of theoretical machines to demonstrate some virtues of those machines. It will explain what I was really mumbling about when I said that the Feynman propagator involves antiparticles going backwards in time.

Consider a system which is a small perturbation of a solvable system $H = H_0 + V$. Suppose that the initial system H_0 has a continuous spectrum, so that there are eigenstates at every nearby energy. Then given an eigenstate of H_0, $H_0 |\varphi\rangle = E |\varphi\rangle$, we expect an eigenstate of H with the same energy $H |\psi\rangle = E |\psi\rangle$. Palpating the previous equation appropriately gives

$$|\psi\rangle = |\varphi\rangle + \underbrace{\frac{1}{E - H_0 + \mathbf{i}\epsilon}}_{\equiv \Pi} V |\psi\rangle$$

(the Lippmann-Schwinger equation). This represents the perturbed eigenstate as the free one plus a scattering term, in terms of the 'propagator' Π. The $\mathbf{i}\epsilon$ is a safety factor which helps us negotiate the fact that $E - H_0$ is not actually invertible. To write this entirely in terms of the known free state $|\varphi\rangle$, iterate. Let $V |\psi\rangle \equiv T |\varphi\rangle$ where T is the *transfer matrix*:

$$|\psi\rangle = |\varphi\rangle + \Pi T |\varphi\rangle .$$

Now act on both sides with V to get $V |\psi\rangle = T |\varphi\rangle = V |\varphi\rangle + V\Pi T |\varphi\rangle$, which is true for any φ so

$$T = V + V\Pi T = V + V\Pi(V + V\Pi T) = V + V\Pi V + V\Pi V + V\Pi V\Pi V + \cdots = \left(\frac{1}{1 - V\Pi}\right) V$$

Given a complete set of eigenstates of H_0, with $\sum_i |\varphi_i\rangle \langle\varphi_i| = \mathbb{1}$,

$$T_{fi} \equiv \langle\varphi_f| T |\varphi_i\rangle = V_{fi} + V_{fj}\Pi(j)V_{ji} + V_{fj}\Pi(j)V_{jk}\Pi(k)V_{ki} + \cdots$$

where $V_{fi} \equiv \langle\varphi_f| V |\varphi_i\rangle$ gives the first Born approximation, and $\Pi(j) \equiv \frac{1}{E - E_j}$, and $E = E_i = E_f$, energy is conserved.

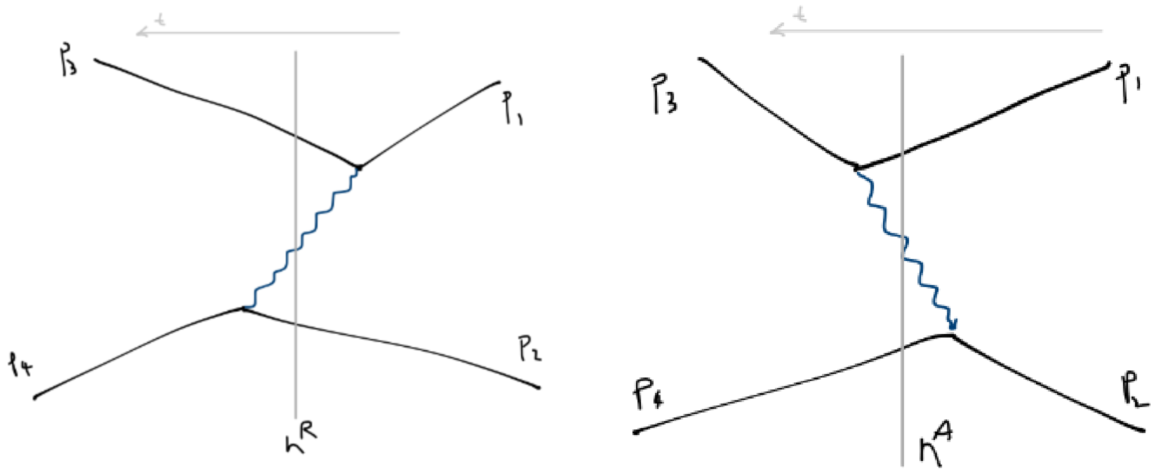
For a vivid example, consider the mediation of a force by a boson field. Let

$$V = \frac{1}{2} e \int d^d x \Psi_e(x)\phi(x)\Phi_e(x)$$

where 'ϕ' is for 'photon' and 'e' is for 'electron' but we've omitted spin and polarization information, and got the statistics of the electron wrong, for simplicity. Consider the free eigenstates $|i\rangle = |\vec{p}_1, \vec{p}_2\rangle$, $\langle f| = \langle\vec{p}_3, \vec{p}_4|$. Then

$$T_{fi} = \underbrace{V_{fi}}_{=0} + \sum_n V_{fn} \frac{1}{E_i - E_n} V_{ni} + \cdots .$$

What are the possible intermediate states $|n\rangle$? It has to be two e and one ϕ, as in the following visualization (*not* a Feynman diagram in the sense we've been discussing):



Time goes to the left, as always. You see that there are two classes of possibilities: $|n_R\rangle = |p_3, p_\gamma; p_2\rangle, |n_A\rangle = |p_1; p_\gamma, p_4\rangle$. Consider them from the point of view of particle 2. In the first (R) case, e_2 feels a photon emitted by particle 1, after the emission happens:

$$\begin{aligned}
V^R_{ni} &= \langle p^3, p_\gamma, p^2 | \, V \, |p_1, p_2\rangle = \langle p^3, p_\gamma | \, V \, |p_1\rangle \underbrace{\langle p_2 | p_2\rangle}_{=1} \\
&= \frac{e}{2}\int d^d x \, \langle p^3, p_\gamma | \, \Psi_e(x)\phi(x)\Psi_e(x) \, |p^1\rangle = \frac{e}{2}\int d^d x \, \underbrace{\langle p_\gamma | \, \phi(x) \, |0\rangle}_{=e^{-\mathbf{i}\vec{p}_\gamma\cdot\vec{x}}} \underbrace{\langle p^3 | \, \Psi(x)^2 \, |p^1\rangle}_{=2e^{-\mathbf{i}(\vec{p}_3-\vec{p}_1)\cdot x}} \\
&= e\not{\delta}^d(\vec{p}_1 - \vec{p}_3 - \vec{p}_\gamma)
\end{aligned} \tag{3.18}$$

– momentum is conserved. Note that energy is not, $E_i \neq E_m$ (or else the denominator is zero).

The other possibility is $|n_A\rangle = |p_1; p_\gamma, p_4\rangle$, which means e_2 feels the effects of a photon it emitted, which is *later* absorbed by e_1 (!!). [End of Lecture 11]

$$V^A_{ni} = \langle p^4 p_\gamma | \, V \, |p^2\rangle = e\not{\delta}^d(\vec{p}_2 - \vec{p}_4 - \vec{p}_\gamma).$$

Altogether, to leading nonzero order,

$$T_{fi} = \sum_n V_{fn}\frac{1}{E_i - E_n}V_{ni} = \sum_{n=R,A}\int d^d p_\gamma \not{\delta}^d(p_1 - p_3 - p_\gamma)\not{\delta}^d(p_2 - p_4 + p_\gamma)\frac{e^2}{E_i - E_n}.$$

A bit of kinematics: let the ϕ have mass m_γ, so for a given $\vec{p}_\gamma$, $E_\gamma = \sqrt{|\vec{p}_\gamma|^2 + m_\gamma^2}$. Notice that these are *real* particles, they satisfy the equations of motion. For the R case, the intermediate energy is

$$E^R_n = E_3 + E^R_\gamma + E_2 = E_3 + \sqrt{|\vec{p}_1 - \vec{p}_3|^2 + m_\gamma^2}$$

so

$$E_i - E_n^R = E_1 + E_2 - (E_3 + E_\gamma^R + E_2) = E_1 - E_3 - E_\gamma^R = -\Delta E - E_\gamma$$

where $\Delta E \equiv E_3 - E_1 = E_4 - E_2$ (by overall energy conservation). Momentum conservation means $p_2 - p_4 = p_1 - p_3$ so

$$E_\gamma^A = \sqrt{|\vec{p}_1 - \vec{p}_3|^2 + m_\gamma^2} = \sqrt{|\vec{p}_2 - \vec{p}_4|^2 + m_\gamma^2} = E_\gamma^R \equiv E_\gamma.$$

Therefore

$$E_i - E_n^A = \Delta E - E_\gamma.$$

The sum of these factors is

$$\sum_{n=R,A} \frac{e^2}{E_i - E_n} = \frac{e^2}{-\Delta E - E_\gamma} + \frac{e^2}{\Delta E - E_\gamma} = \frac{2E_\gamma e^2}{\Delta E^2 - E_\gamma^2} = 2E_\gamma \frac{e^2}{k^2 - m_\gamma^2}.$$

Here we defined $k^\mu = (\Delta E, \vec{p}_\gamma)^\mu$, and $k^2 = k_\mu k^\mu$ is a Lorentz-invariant inner product. Ignoring the normalization factor $2E_\gamma$, this is the Lorentz-invariant momentum-space propagator for the ϕ particle *with four-momentum* k^μ. Notice that there is no actual particle with that four-momentum! It is a superposition of a real particle going forward in time and its (also real) antiparticle going backward in time. If we followed the $\mathbf{i}\epsilon$ that would work out, too, to give the Feynman propagator.

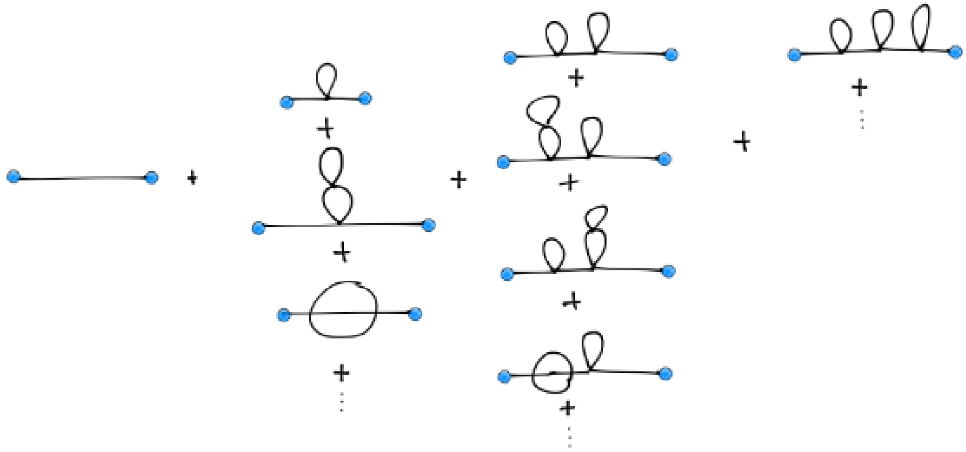
3.6 From correlation functions to the S matrix

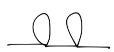
Now we resume our inexorable progress towards observable physics (such as cross sections and lifetimes). We would like to unpack the physics contained in the correlation functions which we've learned to compute in perturbation theory. The first interesting one is the two-point function.

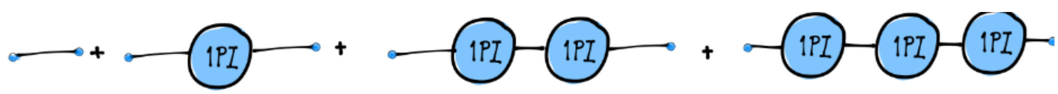
Recall our expression for the momentum-space two-point function (3.17) in terms of a sum of connected diagrams, ordered by the number of powers of λ. Let's factor out the overall delta function by writing:

$$\tilde{G}^{(2)}(p_1, p_2) \equiv \not{\delta}^{d+1}(p_1 + p_2)\tilde{G}^{(2)}(p_1).$$

It will be useful to re-organize this sum, in the following way:

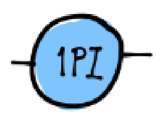


Here's the pattern: we define a diagram to be *one-particle irreducible* (1PI) if it cannot be disconnected by cutting through a single internal propagator. So for example,  is 1PI, but  is not; rather, the latter contributes to the bit with two 1PI insertions. Then

$\tilde{G}^{(2)}(p) =$ 

$+\cdots$

So that we may write equations without pictures, let

$-\mathbf{i}\Sigma(p) \equiv$ 

denote the 1PI two-point function. Σ being 1PI means that the external lines sticking out of it are 'nubbins,' placeholders where propagators may be attached. That's why there are no blue dots at the ends.

Now suppose we know Σ. It is known as the *self-energy*, for reasons we will see next. Then we can write

$$\begin{aligned}
\tilde{G}^{(2)}(p) &= \frac{\mathbf{i}}{p^2-m_0^2} + \frac{\mathbf{i}}{p^2-m_0^2}\left(-\mathbf{i}\Sigma(p)\right)\frac{\mathbf{i}}{p^2-m_0^2} + \frac{\mathbf{i}}{p^2-m_0^2}\left(-\mathbf{i}\Sigma(p)\right)\frac{\mathbf{i}}{p^2-m_0^2}\left(-\mathbf{i}\Sigma(p)\right)\frac{\mathbf{i}}{p^2-m_0^2} + \cdots \\
&= \frac{\mathbf{i}}{p^2-m_0^2}\left(1 + \frac{\Sigma}{p^2-m_0^2} + \left(\frac{\Sigma}{p^2-m_0^2}\right)^2 + \cdots\right) \\
&= \frac{\mathbf{i}}{p^2-m_0^2}\frac{1}{1-\frac{\Sigma}{p^2-m_0^2}} = \frac{\mathbf{i}}{p^2-m_0^2-\Sigma(p)}
\end{aligned} \tag{3.19}$$

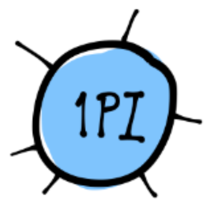
We see that the self-energy shifts the m^2 of the particle, it moves the location of the pole in the propagator. In the interacting theory, $m_0^2 + \Sigma(p)|_{\text{pole}}$ is the physical mass, while m_0 (what we've been calling m until just now) is deprecatingly called the 'bare mass'. For $p^2 \sim m^2$, we will write

$$\tilde{G}^{(2)}(p) \equiv \left(\frac{\mathbf{i}Z}{p^2 - m^2} + \text{regular bits} \right) \tag{3.20}$$

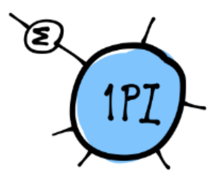
This equation defines the residue Z which is called the 'wavefunction renormalization factor'. It is 1 in the free theory, and represents the amplitude for the field to create a particle, and the other terms, which are not singular at $p^2 = m^2$, represent the amplitude for the field to do something else (such as create multiparticle states), and are absent in the free theory. Later (in 215C?) we will see that unitarity requires $Z < 1$. Notice that if we know Σ only to some order in perturbation theory, then (3.19) is still true, up to corrections at higher order.

The notion of 1PI extends to diagrams for $\tilde{G}^{(n>2)}(p_1 \cdots p_n)$. Let

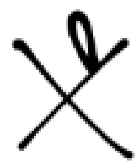
$$\tilde{G}^{(n)}_{1PI}(p_1 \cdots p_n) \equiv$$



where the blob indicates the sum over all 1PI diagrams with n external *nubbins* (notice that these do not have the blue circles that were present before). This means G_{1PI} does not include diagrams like:



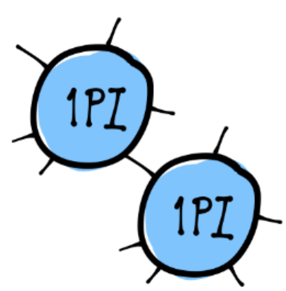
or



.

Notice that 1PI diagrams are *amputated* – their external limbs have been cut off.

This is almost what we need to make S-matrix elements. If we multiply the n-point function by $\prod_{i=1}^n \frac{p_i^2 - m^2}{\sqrt{Z}}$ we cancel out the propagators from the external legs. This object is naturally called the *amputated* n-point function. It differs from the 1PI n-point Green's function because of diagrams like this:



which is amputated but not 1PI. If we then take $p_i^2 \to m^2$, we keep only the part of $\tilde{G}$ which is singular on the mass-shell. And here's why we care about that:

Claim (the LSZ reduction formula):

$$S_{fi} \equiv \langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \cdots \vec{k}_m \rangle = \prod_{a=1}^{n+m} \left(\lim_{P_a^0 \to E_{\vec{P}_a}} \frac{P_a^2 - m^2}{\mathbf{i}\sqrt{Z}} \right) \tilde{G}^{(n+m)} \left(k_1 \cdots k_m, -p_1 \cdots - p_n \right) \tag{3.21}$$

where $P_a \in \{p_i, k_i\}$. In words: the S-matrix elements are obtained from Green's functions by amputating the external legs, and putting the momenta on-shell. Notice that choosing all the final momenta p_i different from all the initial momenta k_i goes a long way towards eliminating diagrams which are not fully connected.

This formula provides the bridge from time-ordered Green's functions (which we know how to compute in perturbation theory now) and the S-matrix, which collects probability amplitudes for things to happen to particles, in terms of which we may compute cross sections and lifetimes. Let us spend just another moment inspecting the construction of this fine conveyance.

Why is LSZ true? Here's the argument I've found which best combines concision and truthiness. [It is mainly from the nice book by Maggiore; I also like Schwartz' chapter 5; Peskin's argument is in section 4.6.] The argument has several steps. The field operators in this discussion are all in Heisenberg picture.

1. First, for a free field, the mode expansion implies that we can extract the ladder operators by:
$$\sqrt{2\omega_k} a_k = \mathbf{i} \int d^d x \; e^{\mathbf{i}kx} \left(-\mathbf{i}\omega_k + \partial_0 \right) \phi_{\text{free}}(x)$$
$$\sqrt{2\omega_k} a_k^\dagger = -\mathbf{i} \int d^d x \; e^{-\mathbf{i}kx} \left(+\mathbf{i}\omega_k + \partial_0 \right) \phi_{\text{free}}(x) \tag{3.22}$$
Notice that the LHS is independent of time, but the integrand of the RHS is not.

2. Now, recall our brontosaurus expedient (introduced previously after (3.7)): turn the interactions off at $t = \pm\infty$.[18] This allows us to write the field in terms of some pretend free fields
$$\phi(x) \begin{cases} \overset{t\to-\infty}{\rightsquigarrow} & Z^{\frac{1}{2}} \phi_{\text{in}}(x) \\ \overset{t\to+\infty}{\rightsquigarrow} & Z^{\frac{1}{2}} \phi_{\text{out}}(x) \end{cases}.$$

[18] Here's why this is really bad: everything we might scatter is a boundstate. For example: atoms, nuclei, nucleons etc... But if there are no interactions there are no boundstates.

The factors of $\sqrt{Z}$ are required to get the correct two point functions (3.20) near the mass shell. The mode operators for ϕ_{in} are called $a^{(\text{in})}$ etc. $\phi_{\text{in, out}}$ are free fields: their full hamiltonian is H_0. They are in Heisenberg picture, and the reference time for $\phi_{\text{in, out}}$ is $\pm\infty$ respectively. Since they are free fields, we can use (3.22) to write

$$\sqrt{2\omega_k}a^{(\text{in})\dagger} = -\mathbf{i}\int d^dx\, e^{-\mathbf{i}kx}\left(+\mathbf{i}\omega_k + \partial_0\right)\phi_{\text{in}}(x) = -\mathbf{i}Z^{-1/2}\int_{t\to-\infty} d^dx\, e^{-\mathbf{i}kx}\left(+\mathbf{i}\omega_k + \partial_0\right)\phi(x)$$

where in the second step we used the independence on time in (3.22), even though $\phi(x)$ is not a free field.

3. Now make this expression covariant using the fundamental theorem of calculus:

$$\begin{aligned}
\sqrt{2\omega_k}\left(a^{(\text{in})\dagger} - a^{(\text{out})\dagger}\right) &= \mathbf{i}Z^{-1/2}\int_{-\infty}^{\infty} dt\partial_t\left(\int d^dx\ e^{-\mathbf{i}kx}\left(\mathbf{i}\omega_k + \partial_0\right)\phi(x)\right) \\
&= \mathbf{i}Z^{-1/2}\int d^{d+1}x\left(e^{-\mathbf{i}kx}\partial_0^2\phi - \phi\cdot\underbrace{\partial_0^2\ e^{-\mathbf{i}k_\mu x^\mu}}_{(\vec\nabla^2-m^2)e^{-\mathbf{i}kx}}\right) \\
&\overset{\text{IBP}}{=} \mathbf{i}Z^{-1/2}\int d^{d+1}xe^{-\mathbf{i}kx}\left(\Box + m^2\right)\phi(x) \qquad (3.23)
\end{aligned}$$

In the last step we made a promise to only use wavepackets for external states, so that we can do IBP in space.

4. Now, here's where the S-matrix enters. Assume none of the incoming momenta k_i is the same as any outgoing momentum p_j.

$$\begin{aligned}
&\langle p_1\cdots p_n|\, S\, |k_1\cdots k_m\rangle \\
=\ & \prod_{p,k}\sqrt{2\omega}\,\langle 0|\prod a_p^{\text{out}} S\prod a_k^{\text{in}\dagger}|0\rangle \\
=\ & \prod_{p,k}\sqrt{2\omega}\,\langle 0|\mathcal{T}\left(\prod a_p^{\text{out}} S\prod a_k^{\text{in}\dagger}\right)|0\rangle \qquad a^{\text{out}}\text{ lives at } t=+\infty \\
=\ & \prod_{p,k}\sqrt{2\omega}\,\langle 0|\mathcal{T}\left(\prod a_p^{\text{out}} S\left(a_{k_1}^{\text{in}\dagger} - a_{k_1}^{\text{out}\dagger}\right)\prod_2^m a_k^{\text{in}\dagger}\right)|0\rangle \qquad \text{since } p_i\neq k_j\text{, use }\langle 0|\, a^{\text{out}\dagger} = 0 \\
\overset{(3.23)}{=}\ & \mathbf{i}Z^{-1/2}\int d^{d+1}xe^{-\mathbf{i}k_1x_1}\langle 0|\mathcal{T}\left(\prod a_p^{\text{out}} S\left(\Box + m^2\right)\phi(x_1)\prod_2^m a_k^{\text{in}\dagger}\right)|0\rangle \\
=\ & \mathbf{i}Z^{-1/2}\int d^{d+1}xe^{-\mathbf{i}k_1x_1}\left(\Box + m^2\right)\langle 0|\mathcal{T}\left(\prod a_p^{\text{out}} S\phi(x_1)\prod_2^m a_k^{\text{in}\dagger}\right)|0\rangle + \text{X}
\end{aligned}$$

In the last step, X comes from where the $\Box_{x_1}$ hits the time ordering symbol. This gives terms which will not matter when we take $k^2 \to m^2$, I promise.

5. Now do this for every particle to get

$$\langle p_1 \cdots p_n| S |k_1 \cdots k_m\rangle = \prod_{j=1}^m \int d^{d+1}y_j \; e^{+\mathbf{i}p_j y_j} Z^{-1/2} \left(\Box_j + m^2\right) \\ \prod_{i=1}^n \int d^{d+1}x_i \; e^{-\mathbf{i}k_i x_i} Z^{-1/2} \left(\Box_i + m^2\right) \; \langle 0| \mathcal{T} \phi(x_i) \cdots \phi(y_j) S |0\rangle + \mathrm{X}$$

The x and y integrals are just fourier transforms, and this says that near the mass shell,

$$\tilde{G}^{(n+m)}(k_1 \cdots k_m, -p_1 \cdots - p_n) = \prod_a^{n+m} \frac{\mathbf{i}\sqrt{Z}}{P_a^2 - m^2} \langle p_1 \cdots p_n| S |k_1 \cdots k_m\rangle + \text{regular}$$

(where $P_a \in \{p_j, k_i\}$) which is the same as (3.21).

Comment: In our discussion of QFT, a special role has been played by fields called ϕ. Suppose we have some other (say hermitian) local operator $\mathcal{O}$ such that

$$\langle p| \mathcal{O}(x) |\Omega\rangle = Z_{\mathcal{O}} e^{\mathbf{i}px}$$

where $\langle p|$ is a one-particle state made by our friend ϕ (we could put some labels, *e.g.* for spin or polarization or flavor, on both the operator and the state, but let's not). Such an $\mathcal{O}$ is called an 'interpolating field' or 'interpolating operator'. And suppose we have information about the correlation functions of $\mathcal{O}$:

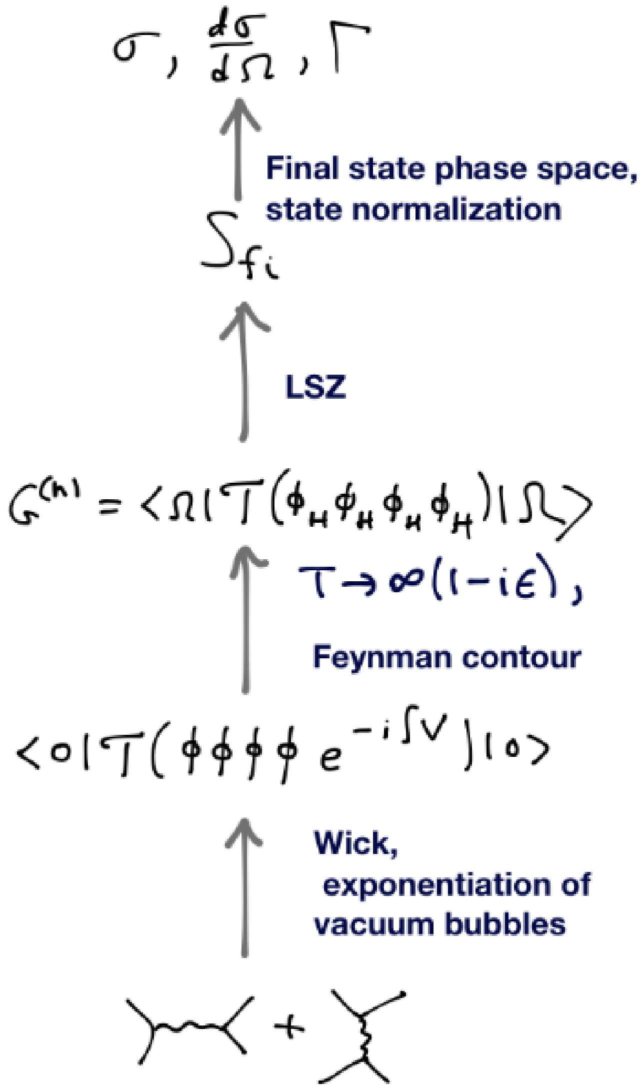
$$G_{\mathcal{O}}^{(n)}(1 \cdots n) \equiv \langle \Omega| \mathcal{T} \left(\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\right) |\Omega\rangle \,.$$

There is a more general statement of LSZ:

$$\prod_{a \in i} \left(Z_a^{-1/2} \mathbf{i} \int d^{d+1}x_a e^{-\mathbf{i}p_a x_a} \left(\Box_a + m_a^2\right)\right) \\ \prod_{b \in f} \left(Z_b^{-1/2} \mathbf{i} \int d^{d+1}x_b e^{+\mathbf{i}p_b x_b} \left(\Box_b + m_b^2\right)\right) \; G_{\mathcal{O}}^{(n)}(1 \cdots n) \\ = \langle \{p_f\}| S |\{p_a\}\rangle \tag{3.24}$$

This more general statement follows as above if we can write $\mathcal{O}_a \overset{t \to -\infty}{\rightsquigarrow} \sqrt{Z_a} \phi_{\text{in}}$.

Here is a summary of the long logical route connecting Feynman diagrams to measurable quantities:



One step is left.

[End of Lecture 12]

S-matrix from Feynman diagrams. The end result of the previous discussion is a prescription to compute S-matrix elements from Feynman diagrams. In a translation-invariant system, the S matrix always has a delta function outside of it. Also we are not so interested in the diagonal elements of the S matrix where nothing happens. So more useful than the S matrix itself are the scattering amplitudes $\mathcal{M}$ defined by

$$\langle f| (S - \mathbb{1}) |i\rangle \equiv (2\pi)^{d+1}\delta^{(d+1)}\left(\sum_f p_f - \sum_i p_i\right) \mathbf{i}\mathcal{M}_{fi} \ . \tag{3.25}$$

(The object $\mathbf{i}\mathcal{M}\not{\delta}^{d+1}(\sum p)$ is sometimes called the transfer matrix. The $\mathbf{i}$ is a convention.)

The rules for the Feynman diagram calculation of $\mathcal{M}$ (for ϕ^4 theory, as a representative example) are:

1. Draw all amputated diagrams with appropriate external nubbins for the initial and final states. For a diagram with N_L loops think of N_L letters that are like k or q or p to call the undetermined loop momenta.
2. For each vertex, impose momentum conservation and multiply by the coupling $(-\mathbf{i}\lambda)$.
3. For each *internal* line, put a propagator.
4. For each loop, integrate over the associated momentum $\int \text{đ}^{d+1}k$.

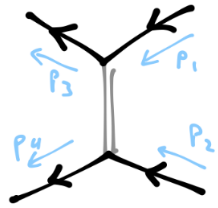
A comment about rule 1: For tree-level diagrams (diagrams with no loops), 'amputate' just means leave off the propagators for the external lines. More generally, it means leave off the resummed propagator (3.19). In particular, a diagram like  is already included by using the correct Z and the correct m.

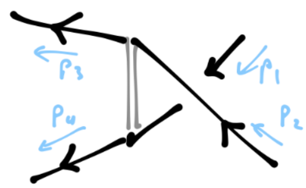
Example: snucleon scattering. [Here we follow Tong §3.5 very closely] Let's return to the example with a complex scalar field Φ and a real scalar field ϕ with Lagrangian (3.8). Relative to ϕ^4 theory, the differences are: we have two kinds of propagators, one of which is oriented, and instead of a 4-point vertex which costs $-\mathbf{i}\lambda$, we have a 3-point vertex for $\phi\Phi^\star\Phi$ which costs $-\mathbf{i}g$.

Let's consider $2 \to 2$ scattering of Φ particles [recall HW 5 or see Tong §3.3.3 for the artisanal version of this calculation], so

$$|i\rangle = |\vec{p}_1, \vec{p}_2\rangle \,, |f\rangle = |\vec{p}_3, \vec{p}_4\rangle \quad \text{with} \quad |\vec{p}_i, \vec{p}_j\rangle \equiv \sqrt{2E_{\vec{p}_i}}\sqrt{2E_{\vec{p}_j}}\mathbf{b}^\dagger_{\vec{p}_i}\mathbf{b}^\dagger_{\vec{p}_j}|0\rangle \, .$$

The Feynman rules above give, to leading nonzero order,

$$\mathbf{i}\mathcal{M} =$$ 

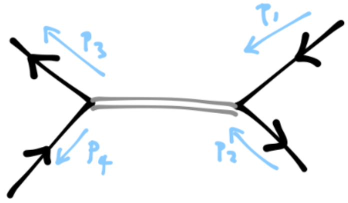
$+$ 

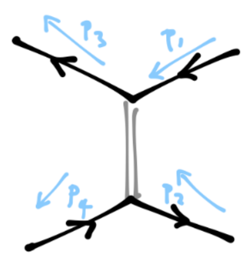
$$= (-\mathbf{i}g)^2 \left(\frac{\mathbf{i}}{(p_1 - p_3)^2 - M^2 + \mathbf{i}\epsilon} + \frac{\mathbf{i}}{(p_1 - p_4)^2 - M^2 + \mathbf{i}\epsilon} \right). \tag{3.26}$$

The diagrams depict two 'snucleons' Φ (solid lines with arrows indicating snucleons versus antisnucleons) exchanging a meson ϕ (double gray line, with no arrow) with momentum $k \equiv p_1 - p_3 = p_2 - p_4$ (first term) or $k \equiv p_1 - p_4 = p_2 - p_3$ (second term). Time goes to the left as always. Notice that here I am being careful about using arrows on the lines to indicate flow of particle number through the diagram, while the extra (light blue) arrows indicate momentum flow.

The meson in these diagrams is virtual, or off-shell, in the sense that it does not satisfy its equation of motion $k^2 \neq M^2$. As we saw in 3.5, each of these diagrams is actually the sum of retarded and advanced exchange of real on-shell particles. The two diagrams included in (3.26) make the amplitude symmetric under interchanging the two particles in the initial or final state, as it must be because they are indistinguishable bosons.

Two more examples with the same ingredients are useful for comparison. If we instead scatter a snucleon and an *anti-snucleon*, so $|i\rangle = \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}\mathbf{b}^\dagger_{\vec{p}_1}\mathbf{c}^\dagger_{\vec{p}_2}|0\rangle$, then the leading diagrams are

$$\mathbf{i}\mathcal{M} =$$ 

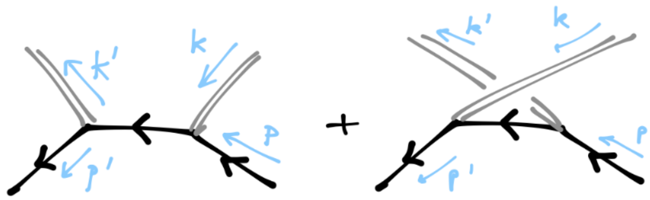
$+$ 

$$= (-\mathbf{i}g)^2 \left(\frac{\mathbf{i}}{(p_1 + p_2)^2 - M^2 + \mathbf{i}\epsilon} + \frac{\mathbf{i}}{(p_1 - p_3)^2 - M^2 + \mathbf{i}\epsilon} \right). \tag{3.27}$$

This one has a new ingredient: in the first diagram, the meson momentum is $k = p_1 + p_2$, which *can* be on-shell, and the $\mathbf{i}\epsilon$ matters. This will produce a big bump, a *resonance*, in the answer as a function of the incoming center-of-mass energy $\sqrt{s} \equiv \sqrt{(p_1 + p_2)^2}$.

Finally, we can scatter a meson and a snucleon:

$$\mathbf{i}\mathcal{M} =$$



$$= (-\mathbf{i}g)^2 \left(\frac{\mathbf{i}}{(p+k)^2 - m^2 + \mathbf{i}\epsilon} + \frac{\mathbf{i}}{(p-k')^2 - m^2 + \mathbf{i}\epsilon} \right). \tag{3.28}$$

Now the intermediate state is a snucleon.

There is a common notation for the Lorentz-invariant combinations of the momenta appearing in these various processes, called Mandelstam variables, of which s is one. A concise summary appears in §3.5.1 of Tong's notes.

3.7 From the S-matrix to observable physics

Now, finally, we extract some physics that can be measured from all the machinery we've built.

Mediation of forces. Consider the non-relativistic (NR) limit of the snucleon-snucleon scattering amplitude (3.26). In the center-of-mass frame $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$ and $\vec{p}' \equiv \vec{p}_3 = -\vec{p}_4$. In the NR limit, $|\vec{p}| \ll m$, and so $p_1^0 = m(1 + \frac{1}{2}\left(\frac{|\vec{p}|}{m}\right)^2 + \cdots)$. Energy-momentum conservation says $p_1 + p_2 = p_3 + p_4$, so $|\vec{p}'| = |\vec{p}| \ll m$ as well. In this limit, the meson propagator (in the first diagram) depends on $(p_1 - p_3)^2 = (p_1^0 - p_3^0)^2 - (\vec{p} - \vec{p}')^2 = -(\vec{p} - \vec{p}')^2$, so the amplitude reduces to

$$\mathbf{i}\mathcal{M} = +\mathbf{i}g^2 \left(\frac{1}{(\vec{p} - \vec{p}')^2 - M^2} + \frac{1}{(\vec{p} + \vec{p}')^2 - M^2} \right).$$

Now compare to NR QM. The scattering amplitude in the COM frame for two particles with relative position $\vec{r}$ and potential $U(\vec{r})$ is, in the first Born approximation, [19]

$$\mathbf{i}\mathcal{A}_{\text{Born}}(\vec{p} \to \vec{p}') = {}_{\text{NR}}\langle \vec{p}' | U(\mathbf{\vec{r}}) | \vec{p} \rangle_{\text{NR}} = -\int d^d r U(\vec{r}) e^{-\mathbf{i}(\vec{p} - \vec{p}') \cdot \vec{r}}$$

where the two-particle state with NR normalization is

$$|\vec{p}\rangle_{NR} = \frac{1}{\sqrt{2E_1}\sqrt{2E_2}} |p_1, p_2\rangle = \frac{1}{2m} |p_1, p_2\rangle.$$

[19] Please beware my signs here; thanks to Ken Intriligator for pointing out a problem here.

The two diagrams in the relativistic answer come from Bose statistics, which means we can't distinguish $\vec{p} \to \pm\vec{p}'$ from each other; to infer the potential we can just compare the first diagram, $(2m)^2 \mathbf{i}\mathcal{A}_{\text{Born}}(\vec{p} \to \vec{p}') = +\mathbf{i}g^2 \frac{1}{(\vec{p}-\vec{p}')^2 - M^2}$ to find:

$$\int d^d r \; U(\vec{r}) e^{-\mathbf{i}(\vec{p}-\vec{p}')\cdot\vec{r}} = -\int đ^d q \mathcal{M}(q) e^{\mathbf{i}\vec{q}\cdot\vec{r}} = -\frac{\left(\frac{g}{2m}\right)^2}{(\vec{p}-\vec{p}')^2 + M^2}$$

which means, in $d = 3$,[20]

$$U(\vec{r}) = -\frac{\left(\frac{g}{2m}\right)^2}{4\pi r} e^{-Mr}.$$

This is the Yukawa potential. (You encountered this potential on the homework, for the same reason, by a different approach.) It has a range, M^{-1}, determined by the mass of the exchanged particle. If we take $M \to 0$, it becomes the Coulomb potential. The sign means that it is attractive, even though this is the potential between particle and particle; this is a general consequence of scalar exchange. Notice that in $d = 3$, the Yukawa coupling between scalars has $1 = [\int d^4 x g \phi \Phi^2] = -4 + 3 + [g]$ so g/m is dimensionless.

A brief warning: while it is satisfying to make contact with something familiar here, the way we actually measure any such potential is by scattering the particles and measuring cross-sections.

[End of Lecture 13]

[20] For convenience, here's the integral:

$$\begin{aligned}
\int đ^3 k \frac{e^{\mathbf{i}\vec{k}\cdot\vec{x}}}{\vec{k}^2 + M^2} &\overset{y\equiv\cos\theta}{=} \frac{1}{(2\pi)^2}\int_0^\infty \frac{k^2 dk}{k^2+M^2} \underbrace{\int_{-1}^{1} dy e^{\mathbf{i}kyr}}_{=2\frac{\sin kr}{kr}} = \frac{1}{(2\pi)^2 r}\int_{-\infty}^{\infty} \frac{dk k \sin kr}{k^2+M^2} \\
&= \frac{1}{(2\pi)^2 r}\left(\frac{1}{2\mathbf{i}}\int_{-\infty}^{\infty} dk \frac{k e^{\mathbf{i}kr}}{k^2+M^2} + h.c.\right) \qquad \text{close contour in UHP for free} \\
&= \frac{1}{(2\pi)^2 r}\frac{1}{2\mathbf{i}} 2\pi\mathbf{i}\frac{\mathbf{i}M e^{\mathbf{i}(\mathbf{i}M)r}}{2\mathbf{i}M}\cdot 2 = \frac{e^{-Mr}}{4\pi r}.
\end{aligned}$$

Lifetimes. [Schwartz, chapter 4] How do we compute the lifetime of an unstable particle in QFT? Consider such a particle in its rest frame, $p^\mu = (M, \vec{0})^\mu$. Let dP be the probability that the particle decays (into some set of final states f) during a time T. The *decay rate* is then $d\Gamma \equiv \frac{1}{T}dP$, the probability per unit time. I put a $d\Gamma$ to indicate a *differential* decay rate into some particular set of final states. If we sum over all possible final states, we can make a practical, frequentist definition of the decay rate, with the idea that we have a big pile of particles and we just count how many go away in some time window:

$$\Gamma \equiv \frac{\#\text{ of decays per unit time}}{\#\text{ of particles}} \equiv \frac{1}{\tau} \tag{3.29}$$

where τ is the lifetime.

Fortunately for us, particles which are stable in the free theory can decay because of weak interactions; in such a case, we can relate dP to an S matrix element for a process which takes one particle to n particles, $S_{n\leftarrow 1}\left(\{p_j\}_{j=1}^n \leftarrow (M, \vec{0})\right)$. So:

$$d\Gamma \equiv \frac{1}{T}dP = \frac{1}{T}\frac{|\langle f| S |i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} d\Pi_f \tag{3.30}$$

Here are two new ingredients:

(1) $d\Pi_f$ is the volume of the region of final-state phase space, $d\Pi_f \propto \prod_{j=1}^n đ^d p_j$. We are allowing, as we must, for imperfect measurements. We will normalize the density of final states so that $\int d\Pi = 1$. Putting back the IR and UV walls of our padded room as in (1.3), we take the continuum limit ($N \to \infty$) of

$$x_i = \frac{i}{N}L, \quad p^i = \frac{2\pi}{L}\frac{i}{N}, \quad i = 1 \cdots N$$

which requires, for each spatial dimension,

$$\Delta x \sum_i = \frac{L}{N}\sum_i \overset{N\to\infty, L\text{fixed}}{\rightsquigarrow} \int dx = L \;\text{ and }\; \frac{1}{2\pi}\Delta p \sum_i = \frac{1}{2\pi}\frac{2\pi}{LN}\sum_i \overset{N\to\infty, L\text{fixed}}{\rightsquigarrow} \int đp = L^{-1}.$$

This gives

$$d\Pi = \prod_{j=1}^n V đ^d p_j,$$

a factor of the volume of space $V = L^d$ for each final-state particle.

(2) The normalization factors $\langle f|f\rangle \langle i|i\rangle$ are not so innocent as they look, because of our relativistic state normalization. Recall that $|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}}\mathbf{a}_{\vec{p}}^\dagger |0\rangle$ the price for

the relativistic invariance of which is

$$\left\langle \vec{k}|\vec{p}\right\rangle = \sqrt{2\omega_{\vec{p}}2\omega_{\vec{k}}} \quad \cdot \underbrace{\langle 0|\,\mathbf{a}_k\mathbf{a}_p^\dagger\,|0\rangle}_{=\langle 0|[\mathbf{a}_k,\mathbf{a}_p^\dagger]|0\rangle=\delta\!\!\!{}^-{}^d(\vec{k}-\vec{p})} \quad = 2\omega_{\vec{p}}\delta\!\!\!{}^-{}^d(\vec{p}-\vec{k})$$

Therefore,

$$\langle\vec{p}|\vec{p}\rangle = 2\omega_{\vec{p}}\delta\!\!\!{}^-{}^d(0) = 2\omega_{\vec{p}}\left(\int dx\ e^{i(p=0)x}\right)^d = 2\omega_{\vec{p}}V.$$

Therefore,

$$\begin{aligned} |i\rangle = \sqrt{2M}\mathbf{a}_{\vec{0}}^\dagger\,|0\rangle \qquad &\implies \qquad \langle i|i\rangle = 2MV \\ |f\rangle = |\{\vec{p}_j\}\rangle \qquad &\implies \qquad \langle f|f\rangle = \prod_j (2\omega_j V) \end{aligned} \tag{3.31}$$

where I've abbreviated $\omega_j \equiv \omega_{\vec{p}_j}$.

Now it is time to square the quantum amplitude

$$\langle f|S-\mathbb{1}|i\rangle = \mathbf{i}\delta\!\!\!{}^-{}^{d+1}(p_T)\,\langle f|\,\mathcal{M}\,|i\rangle$$

($p_T = \sum p_i - \sum p_f$ is the total momentum change) to get the probability (3.30). Again we encounter a δ^2, and again we use $(2\pi)^{d+1}\delta^{d+1}(0) = TV$, so as long as $f \neq i$, we have

$$|\,\langle f|\,(S-\mathbb{1})\,|i\rangle\,|^2 = \delta\!\!\!{}^-{}^{d+1}(0)\delta\!\!\!{}^-{}^{d+1}(p_T)|\,\langle f|\,\mathcal{M}\,|i\rangle\,|^2 = VT\delta\!\!\!{}^-{}^{d+1}(p_T)|\mathcal{M}|^2$$

so that

$$\begin{aligned} dP &= TV\delta\!\!\!{}^-{}^{d+1}(p_T)\frac{1}{2M\prod_j^n(2\omega_j V)}|\mathcal{M}|^2\prod_j^n V d\!\!\!{}^-{}^d p_j \\ &= \frac{T}{2M}|\mathcal{M}|^2 d\Pi_{LI} \end{aligned} \tag{3.32}$$

where all the factors of V went away (!), and

$$d\Pi_{LI} \equiv \prod_{\text{final state},j}\frac{d\!\!\!{}^-{}^d p_j}{2\omega_j}\delta\!\!\!{}^-{}^{d+1}(p_T)$$

is a Lorentz-invariant measure on the allowed final-state phase space. You can see that this is the case by the same calculation that led us to stick those $2\omega_j$s in the states. One more step to physics:

$$d\Gamma = \frac{1}{T}dP = \frac{1}{T}\frac{T}{2M}|\mathcal{M}|^2 d\Pi_{LI} = \underbrace{|\mathcal{M}|^2}_{\text{dynamics}}\underbrace{\frac{1}{2M}d\Pi_{LI}}_{\text{kinematics}}$$

$$\boxed{d\Gamma = |\mathcal{M}|^2 \frac{1}{2M} d\Pi_{LI}}\ .$$

On the RHS is all stuff we know how to calculate (recall the Feynman rules for $\mathcal{M}$ that we listed after (3.25)), and on the LHS is a particle decay rate.

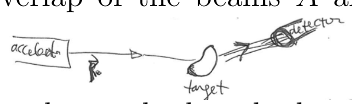
The boxed formula gives the decay rate in the rest from of the unstable particle. In other frames, the lifetime gets time-dilated. This must be true on general grounds of special relativity, but we can see this directly since in a general frame, the normalization of the initial state is not $\langle i|i\rangle_{\text{rest frame}} = \sqrt{2m}$ but $\langle i|i\rangle = \sqrt{2E}$. Therefore

$$\frac{\Gamma^{\text{rest frame}}}{\Gamma} = \frac{E}{m} = \gamma \leq 1$$

and $\tau = \tau^{\text{rest frame}}/\gamma \geq \tau^{\text{rest frame}}$.

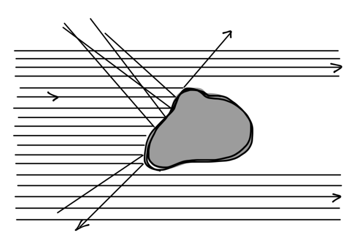
Cross sections. If we are not in the convenient situation of having in our hands a big pile of particles which are stable in the free theory and decay because of not-too-strong interactions, we need to be more proactive to get physics to come out: we have to smash the particles together. When doing this, we send *beams* of particles at each other and see what comes out. We will treat these beams as perfectly collimated momentum eigenstates; if something goes wrong, we'll make a more accurate representation and put them in better-localized wavepackets. A quantity which is good because it is intrinsic to the particles composing the beams is the *scattering cross section*, σ, defined by

$$\text{Number of events of interest} \equiv \frac{N_A N_B}{A}\sigma$$

where A is the common area of overlap of the beams A and B, and $N_{A,B}$ are the number of particles in each beam. 

(Peskin does a bit more worrying at this point, for example, about whether the beams have constant density of particles.) By 'events of interest' I mean for example those particles which end up going in a particular direction, for example in a solid angle $d\Omega(\theta,\varphi)$. Restricting to events of interest in particular direction gives the *differential cross section*, $\frac{d\sigma}{d\Omega}$. The notation is motivated by the idea that $\sigma = \int d\Omega \frac{d\sigma}{d\Omega}$.

The cross-section is the effective cross-sectional area of the beam taken out of the beam and put into the particular state of interest. Here is a picture (adapated from Schwartz' book) which I think makes vivid the idea behind the definition of a cross section:



Now we relate σ to the S-matrix. The scattering rate $dw_{fi} \equiv \frac{dP_{fi}}{T}$ is the scattering probability per unit time, for some fixed initial and final particle states. In a beam, this is related to the cross section by

$$dw = jd\sigma \tag{3.33}$$

where j is the particle current density, which for the case of scattering from an initial state with two particles $A + B \to ...$ is

$$j = \frac{\text{relative velocity of } A \text{ and } B}{\text{volume}} = \frac{v_{AB}}{V}.$$

The number of particles in each beam does not appear in (3.33) because the BHS is intensive. Putting together these statements, we can relate the cross section to the scattering probability:

$$d\sigma = \frac{1}{T}\frac{1}{j}\underbrace{\frac{dN}{N_{\text{incoming}}}}_{=dP_{fi}} = \frac{V}{T}\frac{1}{|\vec{v}_A - \vec{v}_B|}dP_{fi}. \tag{3.34}$$

The first equation in (3.34) is a practical frequentist origin of (3.33), analogous to (3.29) for decay rates. And just as in the discussion of lifetimes above,

$$dP = \frac{|\langle f|\, S\, |i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} d\Pi_f.$$

Everything is as before except for the different initial state:

$$|i\rangle = |\vec{p}_A, \vec{p}_B\rangle \qquad \Longrightarrow \qquad \langle i|i\rangle = (2\omega_A V)(2\omega_B V).$$

Squaring the amplitude gives

$$dP = \frac{T}{V}\frac{1}{2\omega_A 2\omega_B}|\mathcal{M}|^2 d\Pi_{LI}\ ;$$

the only difference is that we replace $\frac{1}{2M}$ with the factors for the 2-particle initial state. Finally, $d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_A - \vec{v}_B|}dP$ gives

$$\boxed{d\sigma = \frac{1}{2\omega_A 2\omega_B}\frac{1}{|\vec{v}_A - \vec{v}_B|}|\mathcal{M}|^2 d\Pi_{LI}}.$$

Again all the IR-divergent factors of V and T went away in the intrinsic physical quantity, as they must.

3.7.1 Two-body phase space

[Schwartz §5.1] To make the formulae of the previous section more concrete, let's simplify them for the case of $n = 2$: two particles in the final state, whose momenta we'll call p_1, p_2. Note that overall momentum conservation implies $p_1 + p_2 = p_{CM}$; we can use this to eliminate p_2. In that case

$$\begin{aligned}
d\Pi_{LI} &= \not{\delta}^{d+1}(p_T)\frac{\text{đ}^d p_1}{2E_1}\frac{\text{đ}^d p_2}{2E_2} && E_i \equiv \sqrt{|p_i|^2 + m_i^2} \\
&= \frac{1}{4(2\pi)^{2d-(d+1)}}\frac{1}{E_1E_2}\delta(E_1 + E_2 - E_{CM}) \quad \underbrace{d^d p_1}_{=d^{d-1}\Omega p_1^{d-1}dp_1} && p_1 \equiv |p_1| > 0 \\
&= \frac{1}{4(2\pi)^{d-1}}\frac{d^{d-1}\Omega p_1^{d-1}dp_1}{E_1E_2}\theta(p_1)\delta(x) && x(p_1) \equiv E_1(p_1) + E_2(p_2 = p_{CM} - p_1) - E_{CM} \\
&= \frac{1}{4(2\pi)^{d-1}}\frac{d^{d-1}\Omega p_1^{d-2}}{E_1E_2}\frac{E_1E_2}{E_{CM}}\underbrace{dx\delta(x)}_{=1}\theta(E_{CM} - m_1 - m_2) && dp_1 = \frac{dp_1}{dx}dx = \frac{E_1E_2}{E_1 + E_2}\frac{dx}{p_1}
\end{aligned}$$

In the last step, we used the fact that $p_1 \geq 0$ means $E_1(p_1) \geq m_1, E_2(p_2 = p_{CM} - p_1) \geq m_2$.

$2 \to 2$ **scattering in** $d = 3$**.** In the special case where the initial state also consists of two particles, we can also simplify the formula for the cross section. Let the initial momenta be k_A, k_B. In particular, the relative velocity factor is

$$|v_A - v_B| \overset{\text{COM}\vec{k}_A = -\vec{k}_B}{=} \left|\frac{|k_A|}{E_{k_A}} + \frac{|k_B|}{E_{k_B}}\right| = |k_A|\frac{E_{CM}}{E_{k_A}E_{k_B}}$$

Therefore

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{COM}} = \frac{1}{64\pi^2 E_{CM}^2}\frac{|\vec{p}_1|}{|\vec{k}_A|}|\mathcal{M}|^2\theta(E_{CM} - m_1 - m_2). \tag{3.35}$$

Warning: for identical particles in the final state, one must be careful about overcounting in the integral over angles, since a rotation by π exchanges them. In this case $\sigma = \frac{1}{2}\int_{4\pi} d\Omega \frac{d\sigma}{d\Omega}$.

[End of Lecture 14]

4 Spinor fields and fermions

[Peskin chapter 3] Now we need to confront the possibility of fields which transform in more interesting ways under Lorentz transformations. To do so let's back up a bit.

4.1 More on symmetries in QFT

Lightning summary of group theory. A *group* $G = \{g_i\}$ is a set of abstract elements,

1. two of which can be multiplied to give a third $g_1 g_2 \in G$.
2. The product is associative $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
3. and has an identity element $g_0 g_i = g_i$ for all g_i
4. and every element has an inverse, $\forall i, \exists g_i^{-1}$ such that $g_i g_i^{-1} = g_0$. (we will need to worry about distinct left and right inverses).

The *order* of G, denoted $|G|$, is the number of elements of the group. G is *abelian* if the product is commutative.

A *Lie group* is a group whose elements depend smoothly on continuous parameters, $g(\theta)$. These then provide local coordinates on the group manifold. The dimension of a Lie group is the number of coordinates (to be distinguished from $|G|$, which is continuously infinite for a Lie group).

A (linear) *representation* R of a group assigns to each abstract element g of the group a linear operator $\hat{D}_R(g)$ on some vector space $\mathcal{H}$, $R : g \mapsto \hat{D}_R(g)$ in a way which respects the group law (it is a group homomorphism): meaning that $\hat{D}_R(g_0) = \mathbb{1}$ and $\hat{D}_R(g_1)\hat{D}_R(g_2) = \hat{D}_R(g_1 g_2)$.[21] If we choose a basis of the vector space, then $\hat{D}_R(g)$ is a matrix. Two representations R and R' are regarded as the same $R \simeq R'$ if they are related by a change of basis on $\mathcal{H}$, $D_R(g) = S^{-1} D_{R'}(g) S$ (with S independent of g!). A *reducible representation* is one for which the matrices can be made block diagonal by a basis change. A reducible representation is equivalent to $R_1 \oplus R_2 \oplus ...$ a direct sum of irreducible representations, $D_R \simeq \begin{pmatrix} D_{R_1} & 0 & \\ 0 & D_{R_2} & \\ & & \ddots \end{pmatrix}$.

[21] Since the overall phase of a vector in $\mathcal{H}$ is unphysical, quantum mechanics allows for *projective representations* where the group law is only satisfied up to phases. We'll see an example below.

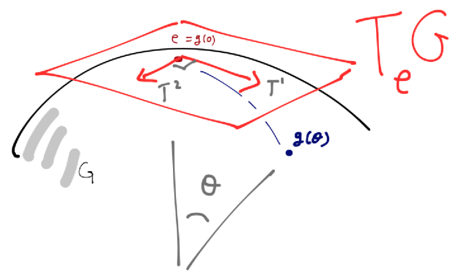
The dimension of R is the dimension of $\mathcal{H}$ as a vector space. Notice that different representations of the same group G can have different dimensions!

What properties of G are inherent in all of its representations? For the case of Lie groups, one answer is the *Lie algebra* relations. Consider a (say n-dimensional) representation of a group element near the identity (which let's label the identity element $g_0 \equiv e \equiv g(\theta = 0)$ by the coordinate value $\theta = 0$):

$$D_R(g(\theta \sim 0)) = \mathbb{1} + \mathbf{i}\theta_a T^a_R + \mathcal{O}(\theta^2), \quad i.e. \;\; T^a_R \equiv -\mathbf{i}\partial_{\theta_a} D_R(g(\theta))|_{\theta=0}$$

where T^a_R are the *generators* of G in the representation R. In a basis for the vector space, they are $n \times n$ matrices.

The generators T^a determine a basis of the tangent space of G at the identity, T_eG (or equivalently, by the group action, at any other point). A finite transformation (in the component of the Lie group which is continuously connected to the identity element) can be written as



$$D_R(g(\theta)) = e^{\mathbf{i}\theta_a T^a_R}$$

which is unitary if $T = T^\dagger$.

Given two such elements $D_R(g(\theta_1)) = e^{\mathbf{i}\theta^1_a T^a_R}$ and $D_R(g(\theta_2)) = e^{\mathbf{i}\theta^2_a T^a_R}$, their product must give a third:

$$D_R(g_1)D_R(g_2) = D_R(g_1 g_2) = e^{\mathbf{i}\theta^3_a T^a_R} \tag{4.1}$$

for some θ^3. Expanding the log of the BHS of (4.1) to second order in the θs (see Maggiore chapter 2.1 for more detail), we learn that we must have

$$\theta^3_a = \theta^1_c + \theta^2_c - \frac{1}{2}\theta^1_b \theta^2_c f^{bc}{}_a + \mathcal{O}(\theta^3)$$

which implies that

$$[T^a, T^b] = \mathbf{i} f^{ab}_c T^c$$

which relation is called the Lie algebra g of G, and the fs are called *structure constants* of g or G. f does not depend on the representation. For those of you comfortable with differential geometry, an easy way to see this is that the commutator is the Lie bracket between two tangent vectors (which gives another tangent vector). Note that the normalization of the T^a is ambiguous, and rescaling T rescales f. A common convention is to choose an orthonormal basis

$$\mathrm{tr} T^a T^b = \frac{1}{2}\delta^{ab}. \tag{4.2}$$

Notice that we often use lowercase letters to denote the algebra and uppercase letters to denote the group, which is natural since the algebra generates small group transformations. The Lie algebra is defined in the neighborhood of the identity element, but by conjugating by finite transformations, the tangent space to any point on the group has the same structure, so it determines the local structure. It doesn't know about global, discrete issues, like disconnected components, so different groups can have the same Lie algebra.

A *casimir* of the algebra is an operator made from the generators which commutes with all of them. Acting on an irreducible representation ($\equiv$ one which is not reducible $\equiv$ irrep), where all the states can be made from each other by the action of products of generators, it is proportional to the identity.

Example: representations of the rotation group. This will be a fancy packaging of familiar stuff which will make the step to Lorentz transformations painless (I hope). Recall from QM that generators of rotations about the axes $x, y, z = 1, 2, 3$, $\mathbf{J}^{i=1,2,3}$, satisfy the algebra $\mathsf{so}(3) = \mathsf{su}(3)$:

$$[\mathbf{J}^i, \mathbf{J}^j] = \mathbf{i}\epsilon^{ijk}\mathbf{J}^k. \tag{4.3}$$

So the structure constants are $f^{ij}_k = \epsilon^{ijl}\delta_{lk}$. A Casimir of this algebra is $\mathbf{J}^2 = \sum_i \left(\mathbf{J}^i\right)^2$, which acts of $j(j+1)$ on the spin-j representation, whose dimension is $2j+1$, any non-negative integer. A finite rotation on $\mathcal{H}$ is

$$D(\hat{n}, \theta) = e^{\mathbf{i}\theta\hat{n}\cdot\vec{\mathbf{J}}}$$

where $\hat{n}$ is a unit vector and θ is an angle, so three real parameters. Familiar matrix solutions of (4.3) are its action on vectors, where the generators are 3×3 matrices:

$$\left(J^i_{(j=1)}\right)_{jk} = \mathbf{i}\epsilon^{ijk}$$

and its $2d$ representation on the Hilbert space of a spin-$\frac{1}{2}$ object:

$$J^i_{\left(j=\frac{1}{2}\right)} = \frac{1}{2}\sigma^i \ .$$

Also, its one-dimensional representation, on a scalar, has $J^i_{(j=0)} = 0$, so $e^{\mathbf{i}\theta J_{(j=0)}} = \mathbb{1}$. More generally, the $2j+1$ dimensional representation is $D_{(j)}(\theta) = e^{-\mathbf{i}\theta^a J^a}$ with

$$\left(J^3\right)_{mm'} = \delta_{mm'}m, \quad \left(J^{\pm}\right)_{mm'} \equiv \left(J^1 \pm \mathbf{i}J^2\right)_{mm'} = \delta_{m',m\mp 1}\sqrt{(j \mp m)(j \pm m + 1)},$$

with the basis labels taking the $2j+1$ values $m, m' \in \{-j, -j+1 \cdots j-1, j\}$.

Notice that the rotation algebra (4.3) is the statement that $\mathbf{J}^i$ itself transforms as a vector $(j = 1)$ under infinitesimal rotations. What I mean by this is: the action of G

on $\mathcal{H}$ by $|\psi\rangle \to D_R|\psi\rangle$ implies an action on linear operators on $\mathcal{H}$ by $\mathcal{O} \mapsto D_R \mathcal{O} D_R^\dagger$. Relabelling the reference axes x, y, z that we used to label $\mathbf{J}^i$ by a rotation g produces a rotation by the same angle in the 3d representation. The reference axes themselves transform in the spin-1 representation:

$$D_{(j=1)}(g)^k_j \mathbf{J}^j = D_R(g)\mathbf{J}^k D_R(g)^\dagger,$$

the infinitesimal version of which is (4.3) (or maybe its complex conjugate). More generally, the equation

$$[J^i, K^j] = \mathbf{i}\epsilon^{ijk} K^k$$

is the statement that K transforms as a vector.

General d. Some of what I have said so far about rotations is special to rotations in $d = 3$. In particular, the notion of "axis of rotation" is ($d = 3$)-centric. More generally, a rotation is specified by a (2d) *plane* of rotation; in $d = 3$ we can specify a plane by its normal direction, the one that's left out, $J^i \equiv \epsilon^{ijk} J^{jk}$, in terms of which the $\mathsf{so}(3)$ lie algebra is (using ϵ identities)

$$[J^{ij}, J^{kl}] = \mathbf{i}\left(\delta^{jk}J^{il} + \delta^{il}J^{jk} - \delta^{ik}J^{jl} - \delta^{jl}J^{ik}\right). \tag{4.4}$$

The vector representation is

$$\left(J^{ij}_{(1)}\right)^k{}_l = \mathbf{i}\left(\delta^{ik}\delta^j_l - \delta^{jk}\delta^i_l\right) \tag{4.5}$$

(that is, there is a $-\mathbf{i}$ in the ij entry and an $\mathbf{i}$ in the ji entry and zeros everywhere else). In $d = 3$, the spinor representation is

$$J^{ij}_{\left(\frac{1}{2}\right)} = \epsilon^{ijk}\frac{1}{2}\sigma^k = \frac{\mathbf{i}}{4}[\sigma^i, \sigma^j]. \tag{4.6}$$

For general d, we can make a spinor representation of dimension k if we find d $k \times k$ matrices γ^i which satisfy the *Clifford algebra* $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ (as the Paulis do). More on this soon.

Define the group $\mathsf{O}(d)$ by its action on the d-dimensional vector representation: $d \times d$ real matrices $\mathcal{O}$ preserving lengths of vectors under $n_i \mapsto \mathcal{O}_i{}^j n_j$: $|\mathcal{O}n|^2 \stackrel{!}{=} |n|^2 = n^i n^j \delta_{ij}, \forall n^i$:

$$\mathcal{O}^t\mathcal{O} = 1\!\!1 \quad \text{or, in a basis,} \quad \left(\mathcal{O}^t\right)_i{}^j \delta_{jk}\mathcal{O}^k{}_l = \delta_{il}. \tag{4.7}$$

In words: $\mathsf{O}(d)$ transformations preserve the bilinear form δ_{ij}. Looking in the connected component with the identity[22], $\mathcal{O} = e^{-\mathbf{i}\theta^{ij}J^{ij}}$, (4.7) implies that the generators J^{ij},

[22] For real matrices $\mathcal{O}^t\mathcal{O} = 1\!\!1$ says $1 = \det \mathcal{O}^t\mathcal{O} = (\det \mathcal{O})^2$, so $\det \mathcal{O} = \pm 1$ gives two disconnected components. The component with $\det \mathcal{O} = 1$ (containing the identity) is called $\mathsf{SO}(d)$. The other component is obtained by multiplying by a diagonal matrix with an odd number of minus signs on the diagonal.

$i, j = 1..d$ satisfy are antisymmetric and pure imaginary. There are $\frac{d(d-1)}{2}$ of them, and a good basis is given by (4.5). This agrees with the $d = 3$ case above where there are $\frac{3\cdot 2}{2} = 3$ such generators. These satisfy (4.4).

A special case is $\mathsf{SO}(2)$ where the one generator is $T = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \sigma^2$, and the finite transformation is

$$e^{\mathbf{i}\beta T} = 1\!\!1 \cos\beta + \mathbf{i}\sigma^2 \sin\beta = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}.$$

$\mathsf{U}(N)$. Another important example is the Lie group $\mathsf{U}(N)$ defined by its N-dimensional representation as $N \times N$ complex unitary matrices $1\!\!1 = M^\dagger M = M M^\dagger$. This one doesn't arise as a spacetime symmetry, but is crucial in the study of gauge theory, and already arose as an example of a global symmetry on the homework. We can generate these $M = e^{-\mathbf{i}\beta^a T^a}$ by any *hermitian* $N \times N$ matrices T^a. A basis is given by the following set of generators satisfying (4.2):

$$T^1 = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \quad T^2 = \frac{1}{3}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}, \quad T^3 = \frac{1}{6}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -3 & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix}, \cdots$$

$$\text{for } i \neq j: \quad \left(T_x^{ij}\right)^k{}_l = \frac{1}{2}\left(\delta^{ik}\delta^j_l + \delta^{jk}\delta^i_l\right), \quad \left(T_y^{ij}\right)^k{}_l = \frac{\mathbf{i}}{2}\left(\delta^{ik}\delta^j_l - \delta^{jk}\delta^i_l\right), \quad T^{N^2} = \frac{1}{\sqrt{2N}} 1\!\!1_{N\times N}$$

Altogether there are $\frac{N(N-1)}{2} \cdot 2 + N = N^2$ of these. Only the last one has a nonzero trace. The ones called T_x^{ij} and T_y^{ij} only have nonzero entries in the ij and ji place, and are like σ^x and σ^y respectively. A finite transformation is

$$e^{-\mathbf{i}\beta^a T^a} = \underbrace{e^{-\mathbf{i}\sum_{a=1}^{N^2-1}\beta^a T^a}}_{=M} \underbrace{e^{-\mathbf{i}\beta^{N^2} T^{N^2}}}_{=e^{-\mathbf{i}\beta^{N^2}/\sqrt{2N}}}$$

where the first factor has $\det M = 1$ (since $\log\det M = \operatorname{tr}\log M = -\mathbf{i}\sum_{a=1}^{N^2-1}\beta^a \operatorname{tr}(T^a) = 0$) and the second is just a phase. The subgroup with $\det M = 1$ is called $\mathsf{SU}(N)$. This shows that $\mathsf{U}(N) = \mathsf{SU}(N) \times \mathsf{U}(1)$.

A special case is $\mathsf{SU}(2)$ which has $N^2 - 1 = 2^2 - 1 = 3$ generators, which are 2×2 and are $T^a = \frac{1}{2}\sigma^a$ Pauli matrices. So $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ have the same lie algebra. They are not the same group, though, since $\mathsf{SU}(2)$ is twice as big: a 2π rotation is not the identity (but squares to it). Half-integer spin representations of $\mathsf{SU}(2)$ are *projective* representations of $\mathsf{SO}(3)$ – the $\mathsf{SO}(3)$ group law is only satisfied up to a phase (in fact, a sign).

Lorentz group. The Lorentz group can be defined, like $\mathsf{O}(d)$ above, as the linear transformations preserving (proper) lengths of vectors. This implies

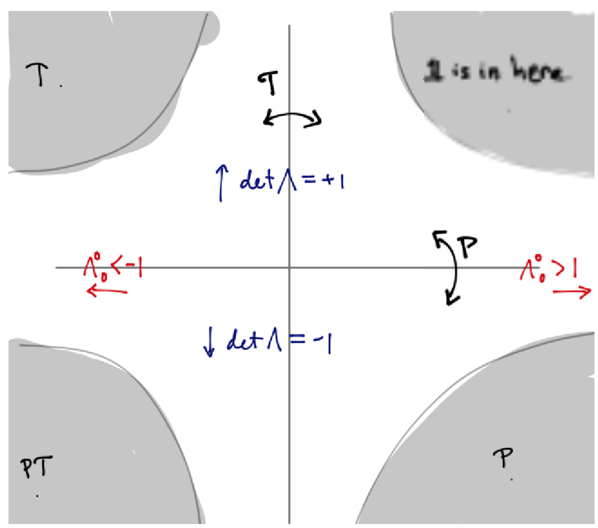
$$\eta = \Lambda^t \eta \Lambda \qquad i.e. \;\; \eta_{\mu\nu} = \left(\Lambda^t\right)_\mu{}^\rho \eta_{\rho\sigma} \Lambda^\sigma{}_\nu . \tag{4.8}$$

There are four disconnected components of solutions to this condition. As for $\mathsf{O}(d)$, taking the det of both sides of (4.8) implies that such a matrix has $\det \Lambda = \pm 1$; the two components are called proper and improper Lorentz transformations, respectively. The $\mu\nu = 00$ component of (4.8) says

$$1 = \left(\Lambda^0_0\right)^2 - \sum_i \left(\Lambda^i_0\right)^2 \qquad \Longrightarrow \qquad \left(\Lambda^0_0\right)^2 \geq 1$$

which has two components of solutions, $\Lambda^0_0 \geq 1$ (orthochronous) and $\Lambda^0_0 \leq -1$ (not orthochronous).

Below we will focus on the proper, orthochronous component. The other three components are obtained by multiplying one of these by one or both of the following extra discrete symmetries (whose action on (real) vectors is $P = \begin{pmatrix} 1 & \\ & -\mathbb{1}_{3\times3} \end{pmatrix}$ and $T = \begin{pmatrix} -1 & \\ & \mathbb{1}_{3\times3} \end{pmatrix}$. (A warning about time reversal: In order to preserve the time evolution operator $e^{-\mathbf{i}Ht}$ while reversing t and preserving the Hamiltonian, the time reversal transformation T must also be accompanied by complex conjugation $K : \mathbf{i} \to -\mathbf{i}$, and the combined operation $\mathcal{T} = T \otimes K$ is therefore *antilinear*.)



The identity component is called $\mathsf{SO}(1, d)$. More generally $\mathsf{O}(m, n)$ is the group of linear operations preserving the matrix with m -1s and n $+1$s on the diagonal, which I will also call $\eta_{\mu\nu}$. All the steps leading to the associated algebra (4.4) (and generators in the $m + n$ dimensional representation (4.5)) are the same as for $\mathsf{SO}(d)$ with the replacement $\delta_{ij} \mapsto \eta_{\mu\nu}$. We will nevertheless resort to some special features of the case $d = 3$ to build representations.

[End of Lecture 15]

4.2 Representations of the Lorentz group on fields

Consider a Lorentz-invariant field theory of a collection of fields $\phi_r = (\phi_1 ... \phi_n, \psi_\alpha, A_\mu \cdots)_r$. Together they form a (in general reducible) representation of the Lorentz group

$$\phi_r(x) \mapsto D_{rs}(\Lambda)\phi_s(\Lambda x)$$

where $D_{rs}(\Lambda)$ is some matrix representation. So far we know two possibilities: the scalar (one-dimensional) representation, where $D(\Lambda) = 1$, and the vector ($d+1$-dimensional) representation, where $D(\Lambda) = \Lambda$ is a $(d+1)\times(d+1)$ matrix. (We can also take direct sums of these to make reducible representations.)

But there are other irreps. To find more, let's think about the algebra in more detail by extracting it from the representation on 4-vectors

$$V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu, \quad \Lambda(\theta^a, \beta^a) = \exp\left(-\mathbf{i}\theta^a \underbrace{T^a_{\text{rot}}}_{\equiv J^a} - \mathbf{i}\beta^a \underbrace{T^a_{\text{boost}}}_{\equiv K^a}\right).$$

Let's find the f^{ab}_c by building the Js and Ks: with the time component first, the matrix representations are

$$J^i = \begin{pmatrix} 0 & \\ & \mathbf{J}^i \end{pmatrix} \tag{4.9}$$

where the 3×3 matrix is $\left(\mathbf{J}^i\right)_{jk} = \mathbf{i}\epsilon^{ijk}$ and

$$\left(K^i\right)_{j0} = -\mathbf{i}\delta^i_j \tag{4.10}$$

and other components zero. To check this, consider a boost in the x direction:

$$e^{-\mathbf{i}\beta K^1} = \mathbb{1} - \mathbf{i}\beta K^1 + \mathcal{O}(\beta)^2 = \begin{pmatrix} 1 & -\beta & \\ -\beta & 1 & \\ & & \mathbb{1}_{2\times 2} \end{pmatrix} + \mathcal{O}(\beta^2) = \begin{pmatrix} \gamma & -\beta\gamma & \\ -\beta\gamma & \gamma & \\ & & \mathbb{1}_{2\times 2} \end{pmatrix} + \mathcal{O}(\beta^2). \tag{4.11}$$

That is, $\delta V^0 = \beta V^1, \delta V^1 = \beta V^0, \delta V^{2,3} = 0$. The others are related to this one by a rotation. In (4.11), we only checked the infinitesimal transformation; but this is enough, by the uniqueness of solutions of linear first-order differential equations: $\partial_\beta \Lambda(\beta) = -\mathbf{i}K\Lambda$ with initial condition $\Lambda(\beta = 0) = \mathbb{1}$ has a unique solution, and so our solution must be the correct one. We'll use this strategy several times below. [23]

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[23] In case you are wondering, the finite transformation is $e^{-\mathbf{i}\beta K^1} = \begin{pmatrix} \cosh\beta & -\sinh\beta & \\ -\sinh\beta & \cosh\beta & \\ & & \mathbb{1}_{2\times 2}. \end{pmatrix}$ Note that the parameter β here is the *rapidity*; it is additive under successive finite boosts, unlike the velocity (though they agree when infinitesimal $\sinh\beta = \frac{v}{c} + \mathcal{O}(v/c)^2$.)

Notice with slight horror that the boost generators are *not* hermitian, and hence the finite boost operator is *not* unitary. This is a symptom of the fact that the Lorentz group is *non-compact* (in the sense that its group manifold is not compact: think of the orbits of rotations on a 4-vector (a sphere, compact), and the orbits of a boost on a 4-vector (a hyperbola, non-compact)). For (faithful) representations of non-compact groups, 'unitary' and 'finite-dimensional' are mutually exclusive.

The commutators of these objects are[24]

$$[J^i, J^j] = \mathbf{i}\epsilon^{ijk}J^k, \quad [J^i, K^j] = \mathbf{i}\epsilon^{ijk}K^k \tag{4.12}$$

(which are respectively the statements that the rotation and boost generators each form a vector) and

$$[K^i, K^j] = -\mathbf{i}\epsilon^{ijk}J^k \tag{4.13}$$

which says two boosts commute to a rotation. Notice that these equations are not changed by the (parity-like) operation $K \to -K$.

Now consider $\vec{J}^{\pm} \equiv \frac{1}{2}\left(\vec{J} \pm \mathbf{i}\vec{K}\right)$. The observation that $K \to -K$ changes nothing implies that they satisfy

$$[J^i_+, J^j_-] = 0, \quad [J^i_\pm J^j_\pm] = \mathbf{i}\epsilon^{ijk}J^k_\pm,$$

two independent $\mathsf{su}(2)$ algebras, which will be called left and right. Formally, we've shown that as algebras over the complex numbers, $\mathsf{so}(1,3) \simeq \mathsf{su}(2)_L \times \mathsf{su}(2)_R$. But we know what the representations of $\mathsf{su}(2)_L \times \mathsf{su}(2)_R$ are! We just have to specify a representation of each. So we can label states in an irrep by (j_+, m_+, j_-, m_-) with $m_\pm \in \{-j_\pm \cdots + j_\pm\}$; this has dimension $(2j_+ + 1)(2j_- + 1)$.

Let me emphasize here that we are identifying the possible ways that the Lorentz group can act on *fields*, not on the particle excitations of such fields. The resulting unitaries on the Fock space will come later.

Weyl spinors. Let's focus on the first nontrivial entry in the table. This is a

[24] Like we did for $\mathsf{O}(d)$, we can slick this up, and generalize to other $\mathsf{SO}(1,d)$ by collecting the generators into an antisymmetric matrix $J^{\mu\nu}$ with components $J^{ij} = \epsilon^{ijk}J^k, J^{0i} = K^i = -J^{i0}$ (exactly as $\vec{E}, \vec{B}$ are collected into $F^{\mu\nu}$). This object satisfies the direct analog of (4.4)

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathbf{i}\left(\eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} - (\mu \leftrightarrow \nu)\right)$$

and the fundamental ($d+1$-dimensional vector) representation matrices solving this equation are

$$(J^{\mu\nu})^\rho{}_\sigma = \mathbf{i}\left(\eta^{\nu\rho}\delta^\mu_\sigma - (\mu \leftrightarrow \nu)\right).$$

(j_+, j_-)	dim	Preview of physics
$(0,0)$	1	scalar
$(\frac{1}{2},0)$	2	left-handed Weyl spinor
$(0,\frac{1}{2})$	2	right-handed Weyl spinor
$(\frac{1}{2},0)\otimes(0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$	$2\times 2=4$	4-vector
$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$2+2=4$	Dirac spinor (reducible)
$(1,0)\oplus(0,1)$	$3+3=6$	$V^{\mu\nu}=\pm\epsilon^{\mu\nu}{}_{\rho\sigma}V^{\rho\sigma}, V^{\mu\nu}=-V^{\nu\mu}$, antisymmetric tensor

Table 1: Lorentz representations on fields.

2-component field $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ on which 2×2 Lorentz matrices act as

$$D_{(\frac{1}{2},0)}(\theta,\beta) = e^{-\mathbf{i}\left(\theta^i J^i + \beta^i K^i\right)}.$$

But the fact that it's a singlet of $\mathsf{SU}(2)_R$ means that

$$0 = J_-^i \psi = \frac{1}{2}(J-\mathbf{i}K)^i\psi$$

that is $J=\mathbf{i}K$ when acting on ψ, which says that the nontrivial generators act as

$$J_+^i\psi = \frac{1}{2}(J+\mathbf{i}K)^i\psi = \frac{1}{2}(J+J)^i\psi = J^i\psi.$$

But we know a 2×2 representation of this object: $\vec{J}_{(\frac{1}{2})} = \frac{1}{2}\vec{\sigma}$ and hence $\vec{K} = -\mathbf{i}\frac{1}{2}\vec{\sigma}$. You can check that these satisfy the three relations (4.12), (4.13). Therefore

$$\psi_\alpha \mapsto \left(e^{-\mathbf{i}\frac{1}{2}\theta\cdot\sigma - \frac{1}{2}\beta\cdot\sigma}\right)_\alpha{}^\beta \psi_\beta = \left(e^{-\frac{1}{2}\sigma\cdot(\beta+\mathbf{i}\theta)}\right)_\alpha{}^\beta\psi_\beta \equiv M_\alpha{}^\beta\psi_\beta.$$

Notice that this matrix M is an ordinary rotation with a complexified angle; it is actually an $\mathsf{SL}(2,\mathbb{C})$ matrix, a general 2×2 complex matrix, with unit determinant. It is common to call the $(\frac{1}{2},0)$ representation a *left-handed* (L) Weyl spinor.

For the $(0,\frac{1}{2})$ or right-handed representation, χ, the same story obtains but now $\left(\vec{J}_+\right)_{\dot\alpha}{}^{\dot\beta}\chi_{\dot\beta} = 0$ and hence $J=-\mathbf{i}K$. Note the dotted indices to distinguish reps of the two $\mathsf{SU}(2)$s. Therefore

$$\chi_{\dot\alpha} \mapsto \left(e^{-\mathbf{i}\frac{1}{2}\theta\cdot\sigma + \frac{1}{2}\beta\cdot\sigma}\right)_{\dot\alpha}{}^{\dot\beta}\psi_{\dot\beta} = \left(e^{\frac{1}{2}\sigma\cdot(\beta-\mathbf{i}\theta)}\right)_{\dot\alpha}{}^{\dot\beta}.$$

Please don't get too hung up on dotting the indices, since as we'll see, there are ways to turn an L spinor into an R spinor. For example, the parity operation $K\to -K, J\to J$ interchanges the two.

Invariants. In order to write Lorentz-invariant local lagrangians, we need to know how to make Lorentz-invariant quantities out of products of fields and their derivatives. For example, given Lorentz vectors V^μ, U^μ, the object $V^\mu U_\mu = V^\mu U^\nu \eta_{\mu\nu}$ is a Lorentz scalar (by the defining property of the Lorentz matrices). Can we make a singlet from two Weyl spinors, $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)$? Yes: we know (*e.g.* from basic QM) that $\mathsf{SU}(2)$ representations combine as $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ where the triplet (spin one) part is symmetric and the singlet (spin 0) is the antisymmetric combination, $\uparrow\downarrow - \downarrow\uparrow$. More explicitly, $\psi_\alpha \xi_\beta \epsilon^{\alpha\beta} \equiv \psi_\alpha \xi^\alpha$ is a singlet. To see this explicitly:

$$\begin{aligned}\left(\mathbf{i}\sigma^2\psi\right) &\mapsto \mathbf{i}\sigma^2 e^{-\frac{1}{2}(\beta+\mathbf{i}\theta)\cdot\sigma}\psi \qquad \text{Insert } \mathbb{1} = \sigma^2\sigma^2 \text{ before } \psi \\ &= \exp\left(-\frac{1}{2}\left(\vec{\beta} + \mathbf{i}\vec{\theta}\right) \cdot \underbrace{\left(\sigma^2\vec{\sigma}\sigma^2\right)}_{=-\vec{\sigma}^t}\right)\left(\mathbf{i}\sigma^2\psi\right)\end{aligned}$$

which means that if $\psi_\alpha \mapsto M_\alpha{}^\beta \psi_\beta$, then

$$\psi^\alpha \equiv \left(\mathbf{i}\sigma^2\psi\right)^t{}_\alpha \mapsto \psi^\beta \left(e^{+\frac{1}{2}(\beta+\mathbf{i}\theta)\cdot\vec{\sigma}}\right)_\beta{}^\alpha \equiv \psi^\beta \left(M^{-1}\right)_\beta{}^\alpha$$

so $\psi^\alpha\psi_\alpha$ is invariant. Notice that on Weyl spinors, we raise and lower indices with $\epsilon^{\alpha\beta} \equiv \left(\mathbf{i}\sigma^2\right)^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\alpha\beta}$ which is a good idea because it is an invariant tensor, as we just showed.

The same story holds for the $(0, \frac{1}{2})$ "right-handed" Weyl representation, that is we can make a singlet using an epsilon tensor. It will be useful to write out the matrices:

$$\chi^{\dot{\alpha}} \equiv \epsilon^{\dot{\alpha}\dot{\beta}}\chi_{\dot{\beta}} \mapsto \left(e^{+\frac{1}{2}(\beta-\mathbf{i}\theta)\cdot\sigma^t}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\chi^{\dot{\beta}} = \chi^{\dot{\beta}}\left(e^{+\frac{1}{2}(\beta-\mathbf{i}\theta)\cdot\sigma}\right)_{\dot{\beta}}{}^{\dot{\alpha}} = \chi^{\dot{\beta}}\left(\left(M^\star\right)^{-1}\right)_{\dot{\beta}}{}^{\dot{\alpha}}.$$

Next, we will show that the $(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ representation is indeed a 4-vector. To see why this might be, notice that we just showed that if a left-handed Weyl spinor transforms as $\psi \to M\psi$ then $\psi^\star \to M^\star\psi^\star$ transforms like a right-handed Weyl spinor.

Introduce the following 'intertwiners':

$$\sigma^\mu_{\alpha\dot{\alpha}} \equiv \left(\mathbb{1}_{\alpha\dot{\alpha}}, \vec{\sigma}_{\alpha\dot{\alpha}}\right)^\mu, \quad \bar{\sigma}^\mu_{\dot{\alpha}\alpha} \equiv \left(\mathbb{1}_{\dot{\alpha}\alpha}, -\vec{\sigma}_{\dot{\alpha}\alpha}\right)^\mu.$$

Our next job is to show that these objects eat a L and an R Weyl spinor and spits out a vector, or vice versa. So for example, I claim that if V_μ is a vector then $V_\mu\sigma^\mu_{\alpha\dot{\alpha}}$ transforms as $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$. And given any two L and R Weyl spinors ψ, χ, $\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\chi^{\dot{\alpha}}V_\mu$ is a singlet.

To summarize:

$$(\frac{1}{2},0) \ni \psi_L \mapsto M_L\psi_L = e^{\frac{1}{2}(-\mathbf{i}\theta-\beta)\cdot\sigma}\psi_L$$
$$(0,\frac{1}{2}) \ni \psi_R \mapsto M_R\psi_L = e^{\frac{1}{2}(-\mathbf{i}\theta+\beta)\cdot\sigma}\psi_R$$

You can see from this expression and $\sigma^2\sigma^i\sigma^2 = -(\sigma^i)^\star$ that $\sigma^2 M_L^\star \sigma^2 = M_R$ and therefore $\sigma^2\psi_L^\star \in (0,\frac{1}{2})$.

Claim: for any ξ_R, ψ_R $\xi_R^\dagger \sigma^\mu \psi_R$ is a (complex) 4-vector. To see this, first notice (using $\sigma^\dagger = \sigma$) that $\xi_R^\dagger \mapsto \xi_R^\dagger e^{\frac{1}{2}(+\mathbf{i}\theta+\beta)\cdot\sigma}$, so

$$\xi_R^\dagger\sigma^\mu\psi_R \mapsto \xi_R^\dagger \underbrace{e^{\frac{1}{2}(+\mathbf{i}\theta+\beta)\cdot\sigma}\sigma^\mu e^{\frac{1}{2}(-\mathbf{i}\theta+\beta)\cdot\sigma}}_{\stackrel{?}{=}\Lambda(\theta,\beta)^\mu{}_\nu\sigma^\nu}\psi_R$$

where

$$\Lambda(\theta,\beta)^\mu{}_\nu = \left(e^{\mathbf{i}(\theta\cdot J+\beta\cdot K)}\right)^\mu{}_\nu$$

is the vector representation of the Lorentz transformation with rotation $\vec\theta$ and boost $\vec\beta$. To check this, it suffices to check the infinitesimal version (by the uniqueness of solutions to linear first-order ODEs):

$$\begin{aligned}
\delta\left(\xi_R^\dagger\sigma^\mu\psi_R\right) &= \delta\xi_R^\dagger\sigma^\mu\psi_R + \xi_R^\dagger\sigma^\mu\delta\psi_R \\
&= \xi_R^\dagger\left(\frac{1}{2}(\mathbf{i}\theta+\beta)^j\sigma^j\sigma^\mu + \sigma^\mu\frac{1}{2}(-\mathbf{i}\theta+\beta)^j\sigma^j\right)\psi_R \\
&= \begin{cases} \xi_R^\dagger\frac{1}{2}2\beta_j\sigma^j\psi_R & \text{...if } \mu=0 \\ \xi_R^\dagger\frac{1}{2}\left(\beta_j\underbrace{\left(\sigma^j\sigma^i+\sigma^i\sigma^j\right)}_{=2\delta^{ij}} + \mathbf{i}\theta_j\underbrace{\left(\sigma^j\sigma^i-\sigma^i\sigma^j\right)}_{=-2\mathbf{i}\epsilon^{ijk}\sigma^k}\right)\psi_R & \text{...if } \mu=i \end{cases}
\end{aligned}$$

On the other hand, using the form of the vector Lorentz generators (4.9), (4.10), the transformation of a vector is

$$\delta V^\mu = \left(\mathbf{i}\beta_j\left(K^j\right)^\mu{}_\nu + \mathbf{i}\theta_j\left(J^j\right)^\mu{}_\nu\right)V^\nu = \begin{cases}\beta_j V^j & \text{...if } \mu=0 \\ \beta_i V^0 - \theta_j\epsilon_{jim}V^m & \text{...if } \mu=i\end{cases}$$

which is just the form we've found.

[End of Lecture 16]

A few more claims and consequences:

- Since the vector representation matrices K, J in (4.9), (4.10) are pure imaginary, the matrices are *real* for any θ, β and we can impose the reality condition $V^\mu = (V^\mu)^\star$ consistent with Lorentz orbits. This is not true of the Weyl spinors by themselves.

- Similarly, for any ξ_L, ψ_L $\xi_L^\dagger \bar\sigma^\mu \psi_L$ is a (complex) 4-vector. Notice that we need to use the $\bar\sigma^\mu$, so that

$$\delta\left(\xi_L^\dagger \bar\sigma^\mu \psi_L\right) = \left(-\beta_i \xi_L^\dagger \sigma_i \psi_L, \beta_i \xi_L^\dagger \psi_L + \epsilon_{ijk}\theta_j \xi_L^\dagger \sigma_k \psi_L\right)^\mu$$

- Our explicit calculation was about $\xi_R^\dagger \sigma^\mu \psi_R$. But we showed that $\xi_R^\dagger$ transforms like a χ_L. So $\chi_L^\alpha \sigma^\mu_{\alpha\dot\alpha}\psi_R^{\dot\alpha}$ is a vector, too. And $\chi_L^\alpha \sigma^\mu_{\alpha\dot\alpha}\psi_R^{\dot\alpha} V_\mu$ is Lorentz invariant.

- Notice that $\xi_R^\dagger \psi_R$ is *not* Lorentz invariant (even if $\xi = \psi$), it is rather the 0 component of a four-vector. The dotted index can be a bit helpful in reminding us that $\psi_L^\dagger \psi_L$ is not a singlet, since if the index on $(\psi_L)_\alpha$ is undotted then $\left(\psi_L^\dagger\right)^{\dot\alpha}$ has a dotted index.

  On the other hand, given an R and an L spinor,

$$\delta\xi_L^\dagger = \xi_L^\dagger\left(+\mathbf{i}\theta - \beta\right)\cdot\sigma/2$$

  the combination $\xi_L^\dagger \psi_R$ *is* Lorentz invariant, since

$$\delta\left(\xi_L^\dagger \psi_R\right) = \xi_L^\dagger\left(\frac{1}{2}\left(\mathbf{i}\theta - \beta\right)\cdot\sigma + \frac{1}{2}\left(-\mathbf{i}\theta + \beta\right)\cdot\sigma\right)\psi_R = 0. \tag{4.14}$$

- Two more occasionally-useful facts:

$$V^\mu \sigma_\mu = \begin{pmatrix} V^0 + V^3 & V^1 - \mathbf{i}V^2 \\ V^1 + \mathbf{i}V^2 & V^0 - V^3 \end{pmatrix} \mapsto M V^\mu \sigma_\mu M^\dagger$$

  Also, $\det V^\mu \sigma_\mu = V^\mu V_\mu$ is Lorentz invariant.

4.3 Spinor lagrangians

Given a Weyl spinor field ψ_R, we'd like to make a local lorentz-invariant lagrangian of the form $\mathcal{L}\left(\psi_R, \psi_R^\dagger, \partial_\mu\psi_R, \partial_\mu\psi_R^\dagger\right)$. The sort of obvious generalization of the KG lagrangian i $\psi_R^\dagger\left(\Box + m^2\right)\psi_R$ which transforms like $\psi_R^\dagger\psi_R$, which is not boost invariant. On the other hand, the object $\psi_R^\dagger\sigma^\mu\psi_R$ is a vector, and we can find another index with which to contract by taking a derivative:

$$\mathcal{L}_{\text{Weyl}} \equiv \psi_R^\dagger \sigma^\mu \mathbf{i}\partial_\mu \psi_R = \psi_R^\dagger \mathbf{i}\partial_t \psi_R + \psi_R^\dagger \vec\sigma\cdot\left(\mathbf{i}\vec\nabla\right)\psi_R$$

is a nice Lorentz invariant kinetic term. The factor of $\mathbf{i}$ is to make $\mathbf{i}\partial_\mu$ hermitian, so

$$\mathcal{L}_{\text{Weyl}}^\dagger = -\mathbf{i}\left(\partial_\mu\psi_R^\dagger\right)\left(\sigma^\mu\right)^\dagger\psi_R \overset{\text{IBP}}{=} \psi_R^\dagger\sigma^\mu\mathbf{i}\partial_\mu\psi_R = \mathcal{L}_{\text{Weyl}}.$$

Notice that we neglected the total derivative $\partial_\mu \left(\psi_R^\dagger \sigma^\mu \psi_R\right)$ which does not change the equations of motion.

For a left-handed field, the Lorentz-invariant Weyl lagrangian involves $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})^\mu$:

$$\mathcal{L}_{\text{Weyl}}(\psi_L) = \psi_L^\dagger \bar{\sigma}^\mu \mathbf{i} \partial_\mu \psi_L.$$

What about mass terms? For a single R Weyl field, we could use the ϵ tensor:

$$\psi_R \mathbf{i} \sigma^2 \psi_R + h.c.$$

is Lorentz invariant. It is not invariant under $\psi_R \to e^{\mathbf{i}\theta}\psi_R$, it violates the particle number. Neutrinos may have such a term, but electrons don't.

Dirac spinors. To make a particle-number-conserving Lorentz-invariant mass term, we need one of each L and R, and the *Dirac mass* pairs them up via the invariant $\psi_L^\dagger \psi_R + h.c.$. We can slick this up, by combining the two 2-component spinors into one 4-component spinor Ψ:

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) \ni \Psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

Now let

$$\bar{\Psi} \equiv \left(\psi_R^\dagger, \psi_L^\dagger\right) = \psi^\dagger \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} \equiv \Psi^\dagger \gamma^0.$$

Then we can package the whole thing beautifully as

$$\mathcal{L}_{\text{Dirac}} = \psi_R^\dagger \mathbf{i} \sigma^\mu \partial_\mu \psi_R + \psi_L^\dagger \mathbf{i} \bar{\sigma}^\mu \partial_\mu \psi_L - m \left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\right) = \bar{\Psi}\left(\mathbf{i}\gamma^\mu \partial_\mu - m\right)\Psi$$

with

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{4.15}$$

The equations of motion are

$$0 = \frac{\partial S}{\partial \bar{\Psi}} = \left(\mathbf{i}\gamma^\mu \partial_\mu - m\right)\Psi \equiv \left(\mathbf{i}\partial\!\!\!/ - m\right)\Psi.$$

Being explicit about indices, the Dirac equation is $0 = \left(\mathbf{i}\gamma^\mu_{ab}\partial_\mu - m\delta_{ab}\right)\Psi_b$ with $a, b = 1..4$.

Notice that by dimensional analysis of the kinetic terms, $[\Psi] = 3/2$, so $[m] = 1$, so m is indeed a mass. Its sign has not been fixed (and I will probably mix up m and $-m$ at various points).

- The Dirac (or gamma) matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

Any set of four matrices satisfying this equation can be combined to form a 4d representation of $\mathsf{so}(1, d)$ in the form

$$J^{\mu\nu}_{\text{Dirac}} \equiv \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu].$$

If you want to see the algebra involved in this statement, see David Tong's notes.

- The particular basis of gamma matrices we've chosen (4.15) is called the Weyl basis. It makes the reducibility of the Dirac rep manifest, since the resulting $J^{\mu\nu}$ are block diagonal:

$$\begin{aligned}
J^{\mu\nu}_{\text{Dirac}} &\overset{\text{Weyl basis}}{=} \frac{\mathbf{i}}{4}\left[\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix}\right] \\
&= \frac{\mathbf{i}}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix} \\
&= \begin{cases} \frac{\mathbf{i}}{4}\begin{pmatrix} -2\sigma^i & 0 \\ 0 & 2\sigma^i \end{pmatrix} = -\frac{\mathbf{i}}{2}\sigma^i \otimes \sigma^3 & \text{if } \mu = 0, \nu = i \\ \frac{\mathbf{i}}{4}\begin{pmatrix} -[\sigma^i, \sigma^j] & 0 \\ 0 & -[\sigma^i, \sigma^j] \end{pmatrix} = +\frac{1}{2}\epsilon^{ijk}\sigma^k \otimes 1\!\!1_{2\times 2} & \text{if } \mu = i, \nu = j \end{cases}
\end{aligned} \tag{4.16}$$

So you see that in the Weyl basis, we already know that these satisfy the $\mathsf{so}(1, \mathsf{d})$ algebra since it is just the Lorentz generators for the Weyl_L and Weyl_R representations in blocks.

- This 4-dimensional Dirac representation is *not* the 4d vector representation. We can see this in several ways: It is complex (the generators are not pure imaginary), though more on this below. It is reducible (we built it by adding together two irreps!). And it is definitely different because, using (4.16), we have, *e.g.* $J^{12} = \frac{1}{2}\sigma^3 \otimes 1\!\!1$, so

$$\Lambda_{\text{Dirac}}(\theta = 2\pi\hat z) = e^{-\mathbf{i}2\pi J^{12}} = e^{\mathbf{i}\pi\sigma^3\otimes 1\!\!1} = \cos\pi 1\!\!1 + \sin\pi\sigma^3 \otimes 1\!\!1 = -1\!\!1.$$

This is just as in the non-relativistic case.

- The Weyl spinors ψ_L, ψ_R are irreps. What's the big deal about the Dirac rep? Only that the electron is a Dirac spinor (and some other folks are too). Before we learned about neutrino masses, they could have been Weyl spinors. Now we have two possibilities: either there is a secret (heavy, non-interactive) partner with whom the neutrinos pair up by a Dirac mass, and/or lepton number is violated by a Majorana mass term (see the homework).

- Other bases of the gamma matrices are possible and sometimes useful. If we replace γ^μ with
$$\gamma^\mu \mapsto \tilde{\gamma}^\mu = U\gamma^\mu U^\dagger, \quad \Psi \mapsto \tilde{\Psi} = U\Psi$$
for some 4×4 unitary U then this gives an equivalent representation, since $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}$ still and hence $\tilde{S}^{\mu\nu} = US^{\mu\nu}U^\dagger$ will still solve $\mathsf{so}(1, d)$.

  A particular useful other basis is the *Majorana basis*
$$\gamma^0_m = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^1_m = \begin{pmatrix} \mathbf{i}\sigma^1 & 0 \\ 0 & \mathbf{i}\sigma^1 \end{pmatrix}, \gamma^2_m = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3_m = \begin{pmatrix} \mathbf{i}\sigma^3 & 0 \\ 0 & -\mathbf{i}\sigma^3 \end{pmatrix}.$$
They have the property that they are all imaginary, which means so is the resulting Lorentz generators $J^{\mu\nu}_m$, which means that all the matrix elements of $e^{\mathbf{i}\theta_{\mu\nu}J^{\mu\nu}_m}$ are real. So it is consistent to impose a reality condition on the spinors in this basis. $\eta_m = \eta^\star_m$. (The reality condition can be imposed in any basis, but in another $\gamma^\mu_m = U\tilde{\gamma}^\mu U^\dagger$, the condition looks like $(U^\star)^{-1} U\psi = \psi^\star$.) This 4d real representation is still different from the vector; a proof is that a 2π rotation is still -1. A good analogy is (real scalar):(complex scalar)::(majorana spinor):(Dirac spinor). For example, a Majorana spinor particle will be its own antiparticle, just like for a real scalar.

- The Dirac equation $(\mathbf{i}\not{\partial} - m)\,\Psi = 0$ implies the wave equation. Act on the BHS by
$$\begin{aligned}
(\mathbf{i}\not{\partial} + m)\,(BHS) \quad \implies 0 &= (\mathbf{i}\not{\partial} + m)\,(\mathbf{i}\not{\partial} - m)\,\Psi \\
&= \left(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2\right)\Psi \\
&= \left(-\frac{1}{2}\left(\underbrace{[\gamma^\mu, \gamma^\nu]}_{\text{antisymmetric}} + \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{=2\eta^{\mu\nu}}\right)\underbrace{\partial_\mu\partial_\nu}_{\text{symmetric}} - m^2\right)\Psi \\
&= -\left(\partial^2 + m^2\right)\Psi.
\end{aligned}$$

- The equation of motion for $\bar{\Psi}$ can be obtained by taking the dagger[25], or by IBP in $\mathcal{L}_{\text{Dirac}}$:
$$\mathcal{L}_{\text{Dirac}} \overset{\text{IBP}}{=} \bar{\Psi}\left(-\mathbf{i}\overleftarrow{\partial}_\mu\gamma^\mu - m\right)\Psi + \text{total deriv}$$
so $0 = \frac{\partial S_{\text{Dirac}}}{\partial\Psi} = \bar{\Psi}\left(-\mathbf{i}\overleftarrow{\partial}_\mu\gamma^\mu - m\right)$.

- The Dirac lagrangian is real if $m = m^\star$, since we already checked the kinetic terms.

[25] In doing so, note that $\gamma^0 = \left(\gamma^0\right)^\dagger$, but the spatial ones $(\vec{\gamma})^\dagger = -\vec{\gamma}$ are anti-hermitian. This compensates the fact that that the $\vec{\gamma}$s acquire a minus sign in moving through the γ^0 in $\bar{\Psi}$. x

- **Lorentz transformations of Dirac spinors.** We have $\Psi \mapsto e^{-\mathbf{i}\theta_{\mu\nu}J^{\mu\nu}_{\text{Dirac}}}\Psi \equiv \Lambda_{\frac{1}{2}}\Psi$ with $\Lambda_{\frac{1}{2}} = \begin{pmatrix} M & 0 \\ 0 & \sigma^2 M^\star \sigma^2 \end{pmatrix}$. The Dirac conjugate spinor transforms as

  $$\bar{\Psi} \mapsto \Psi^\dagger e^{+\mathbf{i}\theta_{\mu\nu}\left(J^{\mu\nu}_{\text{Dirac}}\right)^\dagger}\gamma^0 = \Psi^\dagger\gamma^0\Lambda^{-1}_{\frac{1}{2}} = \bar{\Psi}\Lambda^{-1}_{\frac{1}{2}}.$$

  Here we used that the ij components of J_{Dirac} are hermitian and commute with γ^0 (same for L, R), while the $0i$ components are antihermitean and anticommute with γ^0 (opposite sign for L, R). This makes it clear that the mass term $\bar{\Psi}\Psi$ is Lorentz invariant.

  The gamma matrices also provide nice packaging of the relation we showed above between vectors and bispinors[26]:

  $$\Lambda^{-1}_{\frac{1}{2}}(\theta)\gamma^\mu\Lambda_{\frac{1}{2}}(\theta) = \Lambda^\mu{}_\nu(\theta)\gamma^\nu. \tag{4.17}$$

  This means that any product of gamma matrices between two spinors $V^{\mu_1\cdots\mu_n} \equiv \bar{\Psi}\gamma^{\mu_1}\cdots\gamma^{\mu_n}\Psi$ is a tensor, in the sense that $V^{\mu_1\cdots\mu_n} \mapsto \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n}V^{\nu_1\cdots\nu_n}$. To see this, just use $V^{\mu_1\cdots\mu_n} \mapsto \bar{\Psi}\Lambda^{-1}_{\frac{1}{2}}\gamma^{\mu_1}\cdots\gamma^{\mu_n}\Lambda_{\frac{1}{2}}\Psi$, insert $\mathbb{1} = \Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}$ in between each pair of gammas, and use (4.17).

  Notice that any combination of $A_{\mu_1\cdots\mu_n}\bar{\Psi}\gamma^{\mu_1}\cdots\gamma^{\mu_n}\Psi$ which is symmetric under interchange of indices can be written using the Clifford algebra to a tensor with fewer indices. Let $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ be just the antisymmetric bit, and similarly for more indices. In fact, any bispinor $\Gamma_{ab}\bar{\Psi}_a\Psi_b$ can be decomposed as a sum of these tensors: $\sum_n A_{\mu_1\cdots\mu_n}\bar{\Psi}\gamma^{\mu_1\cdots\mu_n}\Psi$. This follows from counting: $4\times4 = 1+4+6+4+1$.

- Consider the object $\gamma^5 \equiv \mathbf{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{\mathbf{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$. The factor of $\mathbf{i}$ is chosen so that $(\gamma^5)^\dagger = \gamma^5$. Notice that $(\gamma^5)^2 = 1$ so its eigenvalues are ±1. Since it contains one of each of the other four gamma matrices, it anticommutes with each of them: $\{\gamma^5, \gamma^\mu\} = 0, \forall\mu$. Since the Lorentz generators $J^{\mu\nu}_{\text{Dirac}}$ are quadratic in γs, this implies $[\gamma^5, J^{\mu\nu}_{\text{Dirac}}] = 0$, *i.e.* γ^5 is a Casimir, proportional to the identity on irreps, indeed ±1. By direct calculation, in the Weyl basis,

  $$\gamma^5 \overset{\text{Weyl basis}}{=} \begin{pmatrix} -\mathbb{1} & \\ & \mathbb{1} \end{pmatrix}$$

[26] This is really the same equation as we showed above. In Dirac notation its infinitesimal version

$$(1 + \mathbf{i}\theta\cdot J_{\text{Dirac}})\,\gamma^\mu\,(1 - \mathbf{i}\theta\cdot J_{\text{Dirac}}) = (1 - \mathbf{i}\theta\cdot J_{\text{vector}})^\mu{}_\nu\gamma^\nu$$

follows from

$$[\gamma^\mu, J^{\mu\nu}_{\text{Dirac}}] = (J^{\rho\sigma}_{\text{vector}})^\mu{}_\nu\gamma^\nu.$$

is ± 1 on right-handed and left-handed spinors, respectively. This means that the *chirality projectors* $P_{R/L} \equiv \frac{1\pm\gamma^5}{2}$ project onto R/L spinors, respectively, $P_{R/L}^2 = P_{R/L}$. Notice that $P_L\gamma^\mu = \gamma^\mu P_R$.

- Our basis of bispinors can be rewritten using γ^5. Using $\gamma^{\mu\nu\rho\sigma} = -\mathbf{i}\epsilon^{\mu\nu\rho\sigma}\gamma^5$ and $\gamma^{\mu\nu\rho} = +\mathbf{i}\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$, we can make a basis of (hermitean) bispinors as in the table below. The modifier 'pseudo' here refers to the properties under parity; for

Bispinor	multiplicity	representation
$\bar{\Psi}\mathbb{1}\Psi$	1	scalar
$\bar{\Psi}\gamma^\mu\Psi$	4	vector
$\mathbf{i}\bar{\Psi}\gamma^{\mu\nu}\Psi$	6	antisymmetric tensor
$\mathbf{i}\bar{\Psi}\gamma^\mu\gamma^5\Psi$	4	pseudovector
$\mathbf{i}\bar{\Psi}\gamma^5\Psi$	1	pseudovector

example, in terms of $\Psi = \begin{pmatrix}\psi_L & \psi_R\end{pmatrix}$, consider $\mathbf{i}\bar{\Psi}\gamma^5\Psi = \mathbf{i}\left(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L\right)$. It is Lorentz invariant (since it is of the form (4.14)), but under parity $P : \psi_L \leftrightarrow \psi_R$ (that is, parity acts by γ^0 on Dirac spinors) it goes to minus itself.

Why care about these bispinors? One reason is that we can make 4-fermion interactions out of them. For example, $\bar{\Psi}\gamma^{\mu\nu}\Psi \cdot \bar{\Psi}\gamma_{\mu\nu}\Psi$ is a Lorentz-invariant local interaction term which we might add to our Lagrangian.

Another reason is that the vector combinations play an important role:

$$j^\mu \equiv \bar{\Psi}\gamma^\mu\Psi = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L$$

is the Noether current associated with the symmetry $\Psi \to e^{\mathbf{i}\alpha}\Psi$ of the Dirac Lagrangian. You can directly check that $\partial_\mu j^\mu = 0$ using the Dirac equation. Similarly, the *axial* current

$$j_5^\mu \equiv \bar{\Psi}\gamma^\mu\gamma^5\Psi = \psi_R^\dagger\sigma^\mu\psi_R - \psi_L^\dagger\bar{\sigma}^\mu\psi_L$$

would be the Noether current associated with the transformation $\Psi \to e^{\mathbf{i}\alpha\gamma^5}\Psi$. This a symmetry rotates the L and R bits oppositely, and is only a symmetry if $m = 0$. Indeed, the Dirac equation implies that $\partial_\mu j_5^\mu = 2\mathbf{i}m\bar{\Psi}\Psi$. For $E \gg m$, the breaking of this symmetry by m can be ignored and it is still useful. The combinations

$$j_{R/L}^\mu \equiv \bar{\Psi}\gamma^\mu\left(\frac{1\pm\gamma^5}{2}\right)\Psi$$

involve only the Weyl components and are separately conserved if both j^μ and j_5^μ are conserved.

Coupling to the electromagnetic field. Here's another purpose for the current. Suppose our spinor field is propagating in a background electromagnetic field with vector potential A_μ. The whole thing should be Lorentz invariant, so we should be able to couple them via a Lorentz-invariant Lagrangian. How can we resist adding $\mathcal{L}_{EM} = -ej^\mu A_\mu$ for some constant e (blame Ben Franklin for the sign). The full Lagrangian is then

$$\bar{\Psi}[\mathbf{i}\left(\partial_\mu + \mathbf{i}eA_\mu\right)\gamma^\mu - m]\Psi$$

and the Dirac equation is modified to

$$0 = \left(\mathbf{i}\gamma^\mu D_\mu - m\right)\Psi, \quad \text{where } D_\mu\Psi \equiv \left(\partial_\mu + \mathbf{i}eA_\mu\right)\Psi$$

is the gauge covariant derivative in the following sense: $D_\mu\Psi \mapsto e^{-\mathbf{i}\alpha(x)}D_\mu\Psi$ under $\Psi \to e^{-\mathbf{i}\alpha(x)}\Psi(x), A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha$. We could have used the demand that the action respects this transformation to determine the coupling to A_μ to replace $\partial_\mu \to D_\mu$.

The solutions of the Dirac equation in a background EM field are no longer solutions of the KG equation:

$$0 = \left(\mathbf{i}\not{D} + m\right)\left(\mathbf{i}\not{D} - m\right)\Psi = \left(\mathbf{i}D_\mu\mathbf{i}D_\nu\gamma^\mu\gamma^\nu - m^2\right)\Psi \ .$$

Whereas mixed partials commute, $[\partial_\mu, \partial_\nu] = 0$, the covariant derivatives need not:

$$[D_\mu, D_\nu] = e\mathbf{i}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = e\mathbf{i}F_{\mu\nu}$$

so the antisymmetric term matters:

$$\begin{aligned} 0 &= \left(\left(\partial_\mu + \mathbf{i}eA_\mu\right)^2 + e\frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} + m^2\right)\Psi \\ &\overset{\text{Weyl basis}}{=} \left(\left(\partial_\mu + \mathbf{i}eA_\mu\right)^2 - e\begin{pmatrix} \left(\vec{B} + \mathbf{i}\vec{E}\right)\cdot\vec{\sigma} & \\ & \left(\vec{B} - \mathbf{i}\vec{E}\right)\cdot\vec{\sigma} \end{pmatrix} + m^2\right)\Psi \ . \end{aligned} \tag{4.18}$$

In the last step we used the form of the Lorentz generators $J^{\mu\nu}_{\text{Dirac}} = \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu]$ in the Weyl basis. This extra term (relative to the gauge covariant scalar wave equation) is an intrinsic magnetic dipole moment of a Dirac particle. This is a consequence of the Dirac equation with implications for the non-relativistic limit.

[End of Lecture 17]

Notice that we could add extra terms coupling the spin to the EM field strength

$$\mathcal{L}_{\text{DM}} = F_{\mu\nu}\left(g_m\bar{\Psi}\mathbf{i}\gamma^{\mu\nu}\Psi + g_e\bar{\Psi}\mathbf{i}\gamma^{\mu\nu}\gamma^5\Psi\right)$$

but now $[g_{e,m}] = -1$ so these coefficients (which would change the magnetic and electric dipole moments respectively) are suppressed by the inverse of some new mass scale (a priori independent of m), which is presumably large or we would have noticed it, and hence we will ignore such terms for a while.

4.4 Free particle solutions of spinor wave equations

[Peskin §3.3] To understand a quantum scalar field, we had to know that the solutions of the KG equation were plane waves $e^{-\mathbf{i}px}$ with $p^2 = m^2$ (then we associated a mode operator $\mathbf{a}_{\vec{p}}$ with each solution and added them up). To do the analog for spinors, we'll need to know the free particle solutions.

Let's focus on the Dirac equation. This implies the wave equation, so solutions can be made from superpositions of plane waves with $p^2 = m^2$

$$\Psi(x) = e^{-\mathbf{i}px}u(p)$$

but the Dirac equation places a further restriction on the constant spinor $u(p)$:

$$0 = (\gamma_\mu p^\mu - m)\, u(p).$$

Let's assume $m \neq 0$ and solve this in the rest frame, $p_0 = (m, \vec{0})$. Then we can find the answer for general p^μ (with $p^0 > 0$) by a boost: $u(p) = \Lambda_{\frac{1}{2}} u(p_0)$.

$$0 = (m\gamma^0 - m)u(p_0) = m \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p_0)$$

which is solved by $u(p_0) \propto \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ for any 2-component spinor ξ. The fact that there are two solutions for each p is the "intrinsic two-valuedness" associated with spin $\frac{1}{2}$. It will be convenient to normalize the solutions by

$$u(p_0) = \sqrt{2m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \quad \xi^\dagger \xi = 1.$$

We can choose a basis for such ξs which diagonalize σ^3, *e.g.* $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the standard basis for the Paulis. ξ is an ordinary non-relativistic spinor.

Now, under a boost in the z direction (suppressing the x, y components which remain zero),

$$p_0 \mapsto \begin{pmatrix} E \\ p^3 \end{pmatrix} = \exp\left(\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix},$$

and the positive-energy solution of the Dirac equation becomes

$$u(p_0) \mapsto \Lambda_{\frac{1}{2}}(\eta)u(p_0) = \underbrace{\exp\left(\frac{1}{2}\eta \begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}\right)}_{\cosh(\eta/2)\mathbb{1} - \sinh(\eta/2)\begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{E+p^3}P_+ + \sqrt{E-p^3}P_-\right)\xi \\ \left(\sqrt{E-p^3}P_+ + \sqrt{E+p^3}P_-\right)\xi \end{pmatrix} \qquad P_\pm \equiv \frac{1}{2}\left(1 \pm \sigma^3\right)$$

$$= \begin{pmatrix} \left(\sqrt{E+p^3\sigma^3}P_+ + \sqrt{E-p^3\sigma^3}P_-\right)\xi \\ \left(\sqrt{E-p^3\sigma^3}P_+ + \sqrt{E+p^3\sigma^3}P_-\right)\xi \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar\sigma}\xi \end{pmatrix} .$$

In the final expression, we define the square root of a matrix by its action on eigenstates. The last expression also works for any boost direction since it's rotation invariant. Using the identity

$$(p\cdot\sigma)\,(p\cdot\bar\sigma) = p^2 \tag{4.19}$$

(check it on the homework) we can check directly that this expression actually solves the Dirac equation for general p:

$$(p_\mu\gamma^\mu - m)\begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar\sigma}\xi \end{pmatrix} = \begin{pmatrix} -m & \sigma\cdot p \\ \bar\sigma\cdot p & -m \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar\sigma}\xi \end{pmatrix} = \begin{pmatrix} -m\sqrt{p\cdot\sigma} + \sqrt{p^2}\sqrt{p\cdot\sigma} \\ \sqrt{\bar\sigma\cdot p}\sqrt{p^2} - \sqrt{\bar\sigma\cdot p}m \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \overset{p^2=m^2}{=} 0.$$

Negative-energy solutions. Just as for the KG equation, there are also negative-energy solutions with the same $\vec{p}$ (which are not related to the previous by any orthochronous Lorentz transformation):

$$\Psi(x) = v(p)e^{+\mathbf{i}p\cdot x}, \quad p^2 = m^2, p^0 > 0$$

where the Dirac equation further imposes

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s \end{pmatrix}, \quad s = 1, 2.$$

Normalization. A Lorentz-invariant normalization condition is

$$\bar u^s u^r = 2m\xi_s^\dagger\xi_r = 2m\delta_{sr} \qquad \bar v^r v^s = -2m\delta^{rs}.$$

This is equivalent to the following statements about the Lorentz-variant quantities:

$$u_r^\dagger(p)u_s(p) = 2E_p\xi_r^\dagger\xi_s = 2E_p\delta_{rs} \qquad v_r^\dagger v_s = +2E_p\eta_r^\dagger\eta_s = 2E_p\delta^{rs}.$$

Notice that for each p, $0 = \bar u^r(p)v^s(p) = \bar v^r(p)u^s(p)$ (but $(u^r)^\dagger(p)v^s(p) \neq 0$).

Completeness relations. Suppose we choose a basis

$$\mathbb{1}_{2\times 2} = \sum_{s=1,2}\xi^s\left(\xi^s\right)^\dagger . \tag{4.20}$$

Then

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \left((\xi^s)^\dagger \sqrt{p\cdot\bar{\sigma}}, (\xi^s)^\dagger \sqrt{p\cdot\sigma} \right) \stackrel{(4.20)}{=} \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \left(\sqrt{p\cdot\sigma}\right)^2 \\ \left(\sqrt{p\cdot\bar{\sigma}}\right)^2 & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$\stackrel{(4.19)}{=} \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = \gamma\cdot p + m \equiv \not{p} + m. \tag{4.21}$$

Similarly, $\sum_s v\bar{v} = \not{p} - m$. Note that these completeness relations for spinor polarizations are analogous to the relation I quoted for photon polarizations (1.19).

Helicity. Define the *helicity* operator, acting on one-particle states

$$\hat{h} \equiv \hat{p}\cdot\vec{S} \equiv \hat{p}^i \cdot J^i_{\text{Dirac}} = \hat{p}^i \frac{1}{2} \begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}.$$

$\hat{p} = \frac{\vec{p}}{|\vec{p}|}$ is a unit vector, so $\hat{h}^2 = 1/4$ and the eigenvalues are $\pm 1/2$, which are called right-handed and left-handed. (Sometimes people normalize the helicity so that $h = \pm 1$.) Naturally, positive-energy Weyl R/L spinors are helicity eigenstates, with $h = \pm 1/2$ respectively, since the R/L Weyl equation is $0 = p_0 \mathbb{1} \mp p^i\sigma^i$, and $p_0 = |\vec{p}|$. More generally, consider the ultra-relativistic limit of a Dirac spinor, where $E = \sqrt{\vec{p}^2 + m^2} \to |\vec{p}|$, with (WLOG) $\vec{p} = \hat{z}p^3$,

$$u(E_p, p^3) = \begin{cases} \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} \stackrel{E_p\to|\vec{p}|}{\to} \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, & \text{if } \sigma^3 = +1 \\ \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} \stackrel{E_p\to|\vec{p}|}{\to} \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, & \text{if } \sigma^3 = -1 \end{cases}$$

These are helicity eigenstates. More generally, any ultrarelativistic spinor wavefunction can be decomposed into a linear combination of such helicity eigenstates. For massive particles (away from this limit), you can switch the helicity by outrunning the particle; it is not Lorentz invariant.

4.5 Quantum spinor fields

What did we need to build the Fock space of a relativistic scalar field? A quick recapitulation: For each mode, we had ladder operators, $\mathbf{a} \neq \mathbf{a}^\dagger$, and a number operator $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a}$ which was hermitian and hence observable. The ladder operators are so called because

$$[\mathbf{N}, \mathbf{a}] = -\mathbf{a}, \quad [\mathbf{N}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger \ . \tag{4.22}$$

This says that given a number eigenstate $\mathbf{N}\left|n\right\rangle = n\left|n\right\rangle$, we can make others by $\mathbf{N}\left(\mathbf{a}\left|n\right\rangle\right) \overset{(4.22)}{=} (n-1)\left(\mathbf{a}\left|n\right\rangle\right)$ and $\mathbf{N}\left(\mathbf{a}^\dagger\left|n\right\rangle\right) \overset{(4.22)}{=} (n+1)\left(\mathbf{a}^\dagger\left|n\right\rangle\right)$. And we know $n \geq 0$ since $0 \leq \|\,\mathbf{a}\left|n\right\rangle\|^2 = \left\langle n\right|\mathbf{N}\left|n\right\rangle = n\left\langle n|n\right\rangle$, so there must exist a lowest n_0 that we can't lower any further, $\mathbf{a}\left|n_0\right\rangle = 0$, but then $n_0\left|n_0\right\rangle = \mathbf{N}\left|n_0\right\rangle = \mathbf{a}^\dagger\underbrace{\mathbf{a}\left|n_0\right\rangle} = 0$, so $n_0 = 0$. Hence, a ladder of eigenstates of $\mathbf{N}$ with eigenvalues $0, 1, 2, 3, \ldots$.

OK, but here's why I just did that: the necessary equation (4.22) did *not* require that $[\mathbf{a}, \mathbf{a}^\dagger] = 1$. In fact, (4.22) would also follow from the *anticommutation* relation

$$\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a} \equiv \{\mathbf{a}, \mathbf{a}^\dagger\} = 1, \quad \{\mathbf{a}, \mathbf{a}\} = 0 = \{\mathbf{a}^\dagger, \mathbf{a}^\dagger\}.$$

To see this, note the identity $[AB, C] = A\{B, C\} - \{A, C\}B$, so

$$[\mathbf{a}^\dagger\mathbf{a}, \mathbf{a}] = \mathbf{a}^\dagger\{\mathbf{a}, \mathbf{a}\} - \{\mathbf{a}^\dagger, \mathbf{a}\}\mathbf{a} = -\mathbf{a}.$$

But now $0 = \{\mathbf{a}, \mathbf{a}\} = 2\mathbf{a}^2$ means $\mathbf{a}^2 = 0$ and $\left(\mathbf{a}^\dagger\right)^2 = 0$: the ladder only has only one rung. $\left|0\right\rangle$ with $\mathbf{a}\left|0\right\rangle = 0$ and $\left|1\right\rangle = \mathbf{a}^\dagger\left|0\right\rangle$, and there is no $\left|2\right\rangle \propto \left(\mathbf{a}^\dagger\right)^2\left|0\right\rangle = 0$. This is Pauli exclusion.

For multiple modes, we can take

$$\{\mathbf{a}_i, \mathbf{a}_j^\dagger\} = \delta_{ij}, \quad \{\mathbf{a}_i, \mathbf{a}_j\} = 0 = \{\mathbf{a}_i^\dagger, \mathbf{a}_j^\dagger\}.$$

And states look like

$$\mathbf{a}_i^\dagger\mathbf{a}_j^\dagger\left|0\right\rangle = \pm\left|00001_i00...001_j0000\right\rangle = -\mathbf{a}_j^\dagger\mathbf{a}_i^\dagger\left|0\right\rangle$$

we have to remember the order in which the quanta are created to get the sign right (the overall choice is a convention, but relative signs are physics). This is Fermi statistics.

Given a field, how do we know whether to use commutators or anticommutators? A practical answer: try both and one of them will be *bad* somehow. The general answer, with Lorentz invariance, is the *spin-statistics* theorem: fields with half-integer spin are fermionic, and those with integer spin are bosonic. Schwartz, §12.4 has an illuminating argument for the connection between spin and statistics from the Lorentz-invariance of the S-matrix.

Anticommuting scalar fields? Consider a real scalar $\phi(x) = \int \frac{đ^d p}{\sqrt{2\omega_p}} \mathbf{a}_p e^{-\mathbf{i}px} +$ $h.c.$, with $\pi = \dot{\phi}$ and

$$H = \frac{1}{2} \int \left(\pi^2 + \left(\vec{\nabla}\phi \right)^2 + m^2 \phi^2 \right) = \int đ^d p \frac{\omega_p}{2} \left(\mathbf{a}_p^\dagger \mathbf{a}_p + \mathbf{a}_p \mathbf{a}_p^\dagger \right).$$

You see that if $\{\mathbf{a}_p, \mathbf{a}_{p'}^\dagger\} = \delta\!\!\!{}^{-d}(p - p')$ then we get a hamiltonian which is just an infinite constant, independent of the state. Not such a useful energy functional. We also get $\{\phi(x), \pi(y)\} = 0$.

The situation is worse for a complex scalar, with $\Phi(x) = \int \frac{đ^d p}{\sqrt{2\omega_p}} \left(\mathbf{a}_p e^{-\mathbf{i}px} + \mathbf{b}_p^\dagger e^{\mathbf{i}px} \right)$. Then with anticommutators we get

$$H = \int đ^d p \frac{1}{2} \omega_p \left(\mathbf{a}_p^\dagger \mathbf{a}_p - \mathbf{b}_p^\dagger \mathbf{b}_p \right)$$

and the energy is unbounded below. We could alternatively allow negative norm states or non-local anticommutators. Is that any better?

[End of Lecture 18]

Dirac Hamiltonian. From the Dirac lagrangian, we have the canonical momentum density $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = \mathbf{i}\bar{\Psi}\gamma^0 = \mathbf{i}\Psi^\dagger$ and the hamiltonian density

$$\mathfrak{h} = \Pi \dot{\Psi} - \mathcal{L} = \underbrace{\mathbf{i}\Psi^\dagger \partial_t \Psi}_{= \mathbf{i}\bar{\Psi}\gamma^0 \partial_t \Psi} - \mathcal{L} = \bar{\Psi} \left(\mathbf{i}\vec{\gamma} \cdot \vec{\nabla} + m \right) \Psi \overset{\text{eom}}{=} \Psi^\dagger \mathbf{i} \partial_t \Psi. \tag{4.23}$$

Following our nose and writing the operator-valued field as a sum over all solutions of the eom weighted by ladder operators, we have

$$\begin{aligned} \Psi(x) &= \int \frac{đ^3 p}{\sqrt{2\omega_p}} \sum_{s=1,2} \left(u^s(p) e^{-\mathbf{i}px} \mathbf{a}_p^s + v^s(p) e^{\mathbf{i}px} \mathbf{b}_p^{s\dagger} \right) \\ \bar{\Psi}(x) &= \int \frac{đ^3 p}{\sqrt{2\omega_p}} \sum_{s=1,2} \left(\bar{u}^s(p) e^{\mathbf{i}px} \mathbf{a}_p^{s\dagger} + \bar{v}^s(p) e^{-\mathbf{i}px} \mathbf{b}_p^s \right) \end{aligned} \tag{4.24}$$

where, as for the scalar, we implicitly set $p^0 = \omega_{\vec{p}}$.

The hamiltonian is then (using the last expression in (4.23))

$$\begin{aligned} H &= \int d^3\, \mathfrak{h} \\ &= \int d^3 x \int \frac{đ^3 p}{\sqrt{2\omega_p}} \int \frac{đ^3 q}{\sqrt{2\omega_q}} \sum_{ss'} \left(u^{s\dagger}(p) e^{\mathbf{i}px} \mathbf{a}_p^{s\dagger} + v^{s\dagger}(p) e^{-\mathbf{i}px} \mathbf{b}_p^s \right) \left(\omega_q u^s(q) e^{-\mathbf{i}qx} \mathbf{a}_q^s - \omega_q v^s(q) e^{\mathbf{i}qx} \mathbf{b}_q^{s\dagger} \right) \end{aligned}$$

This is of the form $u^\dagger u + v^\dagger v + u^\dagger v + v^\dagger u$. In the first two terms, the x integral is $\int d^3 x e^{\mathbf{i}(p-q)x} = \delta^{(3)}(p-q)$ and in the mixed uv terms, we have $\vec{q} = -\vec{p}$. We use the spinor identities

$$u_s^\dagger(p) u_{s'}(p) = v_s^\dagger(p) v_{s'}(p) = +2\omega_p \delta_{ss'}, \quad u_s^\dagger(p) v_{s'}(-p) = v_s^\dagger(p) u_{s'}(-p) = 0$$

and get

$$H = \int đ^3 p \; \omega_p \sum_s \left(\mathbf{a}_p^{s\dagger} \mathbf{a}_p^s - \mathbf{b}_p^s \mathbf{b}_p^{s\dagger} \right).$$

Now, if $[b, b^\dagger] = 1$, this is $\sum_p \omega_p (N_p^a - N_p^b)$ + constant and the world explodes in a spontaneous shower of antiparticles lowering the energy by coming from nowhere. If instead we have anticommutation relations,

$$\{b_s(p), b_{s'}(q)^\dagger\} = \delta\!\!\!{}^{-}{}^d(p-q)$$

then this is

$$H = \int đ^3 p \; \omega_p \sum_s (N_s^a(p) + N_s^b(p)) + \text{const}$$

and all is well, $H - E_0 \geq 0$. (Here we only used $\{b, b^\dagger\} = \delta$ You could ask why we can't have $[\mathbf{a}, \mathbf{a}^\dagger] = 1$ still. That would just be gross.)

This gives canonical equal time local anticommutators

$$\{\Psi(\vec{x})_a, \Pi(\vec{y})_b\}_{ET} = \mathbf{i}\delta^d(\vec{x} - \vec{y})\delta_{ab}$$

or

$$\{\Psi(\vec{x})_a, \bar{\Psi}(\vec{y})_b\}_{ET} = \mathbf{i}\gamma^0_{ab}\delta^d(\vec{x} - \vec{y}).$$

Dirac propagator. [Peskin §4.7] Now we will efficiently redo the story of interaction picture perturbation theory and Wick's theorem for Dirac spinor fields. The story differs by just a few very important signs.

Time ordering for fermions is defined with an extra minus sign:

$$\mathcal{T}(A_1(x_1) \cdots A_n(x_n)) \equiv (-1)^P A_{1'}(x_{1'}) \cdots A_{n'}(x_{n'}), \quad x^0_{1'} > x^0_{2'} > \cdots > x^0_{n'}$$

where $P \equiv$ the number of fermion interchanges required to get from the ordering $1...n$ to the ordering $1'...n'$ (mod two). Similarly, normal ordering is

$$: ABC \cdots : \equiv (-1)^P A'B'C' \cdots ,$$

where on the RHS all annihilation operators are to the right of all creation operators, and P is defined the same way. [27] Wick's theorem is then still

$$\mathcal{T}(ABC\cdots) =: ABC\cdots : + \text{all contractions.}$$

The possible contractions of Dirac fields are:

$$\overbracket{\Psi(x)\Psi}(y) = 0, \quad \overbracket{\bar{\Psi}(x)\bar{\Psi}}(y) = 0, \quad \overbracket{\Psi(x)\bar{\Psi}}(y) = S_F(x-y),$$

where S_F is the Feynman propagator for the Dirac field, if you like we could define it by this equation. It is

$$\begin{aligned}
S_F^{ab}(x-y) &= \langle 0|\, \mathcal{T}\left(\Psi^a(x)\bar{\Psi}^b(y)\right)|0\rangle \\
&= \theta(x^0-y^0)\langle 0|\,\Psi^a(x)\bar{\Psi}^b(y)\,|0\rangle - \theta(y^0-x^0)\langle 0|\,\bar{\Psi}^b(y)\Psi^a(x)\,|0\rangle \\
&= \theta(x^0-y^0)\underbrace{\langle 0|\,\Psi_a^{(+)}(x)\bar{\Psi}_b^{(-)}(y)\,|0\rangle}_{=\langle 0|\{\Psi_a^{(+)}(x),\bar{\Psi}_b^{(-)}(y)\}|0\rangle \equiv S^+} - \theta(y^0-x^0)\underbrace{\langle 0|\,\bar{\Psi}_b^{(+)}(y)\Psi_a^{(-)}(x)\,|0\rangle}_{\langle 0|\{\bar{\Psi}_b^{(+)}(y),\Psi_a^{(-)}(x)\}|0\rangle \equiv S^-}
\end{aligned}$$

$$\Psi \equiv \underbrace{\Psi^{(+)}}_{\mathbf{a}} + \underbrace{\Psi^{(-)}}_{\mathbf{b}^\dagger}$$

The S^+ bit, made only from $\mathbf{a}$s, is a c-number

$$\begin{aligned}
S_{ab}^+(x-y) &= \{\Psi_a^{(+)}(x), \bar{\Psi}_b^{(-)}(y)\} = \int \frac{đ^3p}{\sqrt{2\omega_{\vec{p}}}} e^{-\mathbf{i}px} \sum_{s=1,2} \int \frac{đ^3q}{\sqrt{2\omega_{\vec{q}}}} e^{+\mathbf{i}qy} \sum_{s'=1,2} u_a^s(p)\bar{u}_b^{s'}(q) \underbrace{\{\mathbf{a}_p^s, \mathbf{a}_q^{s'\dagger}\}}_{=\eth^d(p-q)\delta^{ss'}} \\
&= \int \frac{đ^3p}{2\omega_{\vec{p}}} e^{-\mathbf{i}p(x-y)} \underbrace{\sum_s u_a^s(p)\bar{u}_b^s(p)}_{\overset{(4.21)}{=}\left(\not{p}+m\right)_{ab}} \\
&= \int \frac{đ^3p}{2\omega_{\vec{p}}} \left(\mathbf{i}\not{\partial}_x + m\right)_{ab} e^{-\mathbf{i}p(x-y)} \\
&= \left(\mathbf{i}\not{\partial}_x + m\right)_{ab} \underbrace{\int \frac{đ^3p}{2\omega_{\vec{p}}} e^{-\mathbf{i}p(x-y)}}_{=\Delta^+(x-y)} \qquad \Delta^+(x-y) \overset{(2.3)}{=} \int_{C^+} đ^4p e^{-\mathbf{i}p(x-y)} \frac{\mathbf{i}}{p^2-m^2} \\
&= \int_{C^+} đ^4p e^{-\mathbf{i}p(x-y)} \frac{\mathbf{i}\left(\not{p}+m\right)_{ab}}{p^2-m^2}
\end{aligned} \tag{4.25}$$

The same calculation for S^-, the bit involving $\{\mathbf{b},\mathbf{b}^\dagger\}$, gives the same integrand, the only difference, as for the complex KG field, is the contour $C^+ \to C^-$. Getting the

[27] Why the extra signs? One way to see that they must be there is if they weren't everything would be zero. With the sign, the following two choices for a normal ordered product are equivalent:

$$: a_p a_q a_r^\dagger := (-1)^2 a_r^\dagger a_p a_q = (-1)^3 a_r^\dagger a_q a_p,$$

but without it, we would conclude that the LHS would be zero.

same integrand with the same sign required the relative minus (the red one above) which for bosons came from the sign in the commutator $[\mathbf{b}, \mathbf{b}^\dagger] = \mathbf{b}\mathbf{b}^\dagger - \mathbf{b}^\dagger\mathbf{b}$. Adding the two terms together, we learn that the momentum space Dirac propagator is

$$\tilde{S}(p) = \frac{\mathbf{i}\left(\not{p} + m\right)}{p^2 - m^2} = \frac{\mathbf{i}\left(\not{p} + m\right)}{\left(\not{p} + m\right)\left(\not{p} - m\right)} = \frac{\mathbf{i}}{\not{p} - m}. \tag{4.26}$$

(These matrices all commute with each other, so my cavalier manipulation of them can be done in the eigenbasis of $\not{p}$ without trouble.) It is not a coincidence that the numerator of the propagator is the polarization sum: $\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m$

The position-space Feynman propagator comes from integrating (4.26) over the Feynman contour, as for scalars:

$$S_F(x-y) = \int_{C_F} đ^4p \frac{\mathbf{i}}{\not{p} - m} e^{-\mathbf{i}p(x-y)}.$$

Fermions and causality. Earlier I made a big deal that we need commutators to vanish outside the lightcone to prevent acausal communication. But the Dirac field $\Psi(x)$ *doesn't* commute with $\bar{\Psi}(y)$ for spacelike $x-y$ (rather, they anticommute). Why is this OK? What saves the day is the fact that we can't measure a single fermion operator. The operators we can measure (such as the number density of fermions $\Psi^\dagger\Psi$, or their momentum density $\Psi^\dagger\vec{\nabla}\Psi$) are all made of even powers of Ψ and $\bar{\Psi}$. And these do commute outside the lightcone.

A principle which would make this restriction on what we can measure precisely true and inevitable is if *fermion parity is gauged*. By 'fermion parity' I mean the transformation which takes $\Psi \to -\Psi$ for every fermionic operator in the world. By 'is gauged' I mean that this transformation should be regarded as an equivalence relation, rather than a transformation which relates distinct physical configurations. In that case, a local operator with an odd number of fermions would not be gauge invariant. [28]

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[28]This point of view, that locality should be built into the Hilbert space, and that therefore that fermion parity should be gauged, is advocated forcefully by Wen, *Quantum Field Theory of Many-Body Systems* (Oxford, 2004) in Chapter 10. It is not clear how to include gravitational degrees of freedom in this principle.

5 Quantum electrodynamics

We must fill a small hole in our discussion, to help our fermions interact. In §1.3, we figured out a bit of the quantum theory of the radiation field. A few things we did not do: study the propagator, figure out the data on external states, and the relation of between the masslessness of the photon and gauge invariance. After that we will couple electrons and photons and study leading-order (tree-level) processed in the resulting theory of quantum electrodynamics (QED).

5.1 Vector fields, quickly

[We'll follow Ken Intriligator's efficient strategy for this discussion.] Consider the following Lagrangian for a vector field A_μ (which I claim is the most general quadratic Poincaré-invariant Lagrangian with at most two derivatives):

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A^\nu \partial_\mu A^\nu + a\underbrace{\partial_\mu A^\mu \partial_\nu A^\nu}_{=(\partial A)^2} + bA_\mu A^\mu + c\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma\right).$$

The sign is chosen so that spatial derivatives are suppressed, and the normalization of the first term is fixed by rescaling A. The last term is a total derivative, $\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma \propto \partial_\mu\left(\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma\right)$, and will not affect the EOM or anything at all in perturbation theory; it is called a θ term. The EOM are

$$0 = \frac{\delta}{\delta A^\nu(x)}\int \mathcal{L} = -\partial^2 A_\nu - a\partial_\nu\left(\partial\cdot A\right) + bA_\nu$$

which like any translation invariant linear equation is solved by Fourier transforms $A_\mu(x) = \epsilon_\mu e^{-\mathbf{i}kx}$ if

$$k^2\epsilon_\mu + ak_\mu\left(k\cdot\epsilon\right) + b\epsilon_\mu = 0.$$

There are two kinds of solutions: longitudinal ones with $\epsilon_\mu \propto k_\mu$ (for which the dispersion relation is $k^2 = -\frac{b}{1+a}$), and transverse solutions $\epsilon\cdot k = 0$ with dispersion $k^2 = -b$. The longitudinal mode may be removed by taking $b \neq 0$ and $a \to -1$, which gives the Proca Lagrangian:

$$\mathcal{L}_{a=-1,b=-\mu^2} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu$$

Note that the EOM (Proca equation) $0 = \partial^\cdot F_{\cdot\nu} + \mu^2 A_\nu$ implies $0 = \partial^\nu A_\nu$ by $0 = \partial^\mu\partial^\nu F_{\mu\nu}$. So each component of A_μ satisfies the KG equation, $k^2 = \mu^2$. In the rest frame, we can choose a basis of plane wave solutions which are eigenstates of

$$J^z = \mathbf{i}\begin{pmatrix} & & & \\ & & +1 & \\ & -1 & & \\ & & & \end{pmatrix}, \text{ namely, } \epsilon^{(\pm)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \mp\mathbf{i} \\ 0 \end{pmatrix}, \epsilon^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

They are normalized so that $\epsilon^{(r)} \cdot \epsilon^{(s)} = +\delta^{rs}$ and $\sum_{r=\pm1,0} \epsilon_\mu^{(r)\star}\epsilon_\nu^{(r)} = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$ so that they project out $\epsilon \propto k$. Notice that if $\vec{p} \propto \hat{z}$ (for example in the masselss case with $p^\mu = (E, 0, 0, E)^\mu$) then these ϵ are also helicity eigenstates: $h = \vec{S} \cdot \hat{p} = J^z$.

Canonical stuff: The canonical momenta are $\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i$ (as for electrodynamics in §1.3) and $\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$. This last bit is a little awkward, but it just means we can solve the equations of motion for A_0 algebraically in terms of the other (real) dofs:

$$0 = \frac{\delta S}{\delta A_0} = \vec{\nabla}\cdot\vec{E} - \mu^2 A_0 = (-\nabla^2 + \mu^2)A_0 + \vec{\nabla}\cdot\dot{\vec{A}} \implies A_0(\vec{x}) = \int d^3y e^{-\mu|\vec{x}-\vec{y}|}\frac{\left(-\vec{\nabla}\cdot\dot{\vec{A}}\right)}{4\pi|\vec{x}-\vec{y}|}. \tag{5.1}$$

So at each moment A_0 is determined by A_i. (Notice that this is still true for $\mu \to 0$.) The hamiltonian density is

$$\mathfrak{h} = +\frac{1}{2}\left(F_{0i}^2 + \frac{1}{2}F_{ij}^2 + \mu^2 A_i^2 + \mu^2 A_0^2\right) = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2 + \mu^2\vec{A}^2 + A_0\left(\vec{\nabla}\cdot\vec{E} - \mu^2 A_0\right)\right) \geq 0,$$

where positivity follows from the fact that it is a sum of squares of real things.

The canonical equal time commutators are then

$$[A_i(t,\vec{x}), F^{j0}(t,\vec{y})] = \mathbf{i}\delta_i^j \delta^{(3)}(\vec{x}-\vec{y})$$

which if we add up the plane wave solutions as

$$A_\mu(x) = \sum_{r=1,2,3}\int \frac{đ^3k}{\sqrt{2\omega_k}}\left(e^{-\mathbf{i}kx}\mathbf{a}_k^r\epsilon_\mu^{(r)} + e^{+\mathbf{i}kx}\mathbf{a}_k^{r\dagger}\epsilon_\mu^{(r)\star}\right)$$

give the bosonic ladder algebra for each mode

$$[\mathbf{a}_k^r, \mathbf{a}_p^{s\dagger}] = ð^{(3)}(\vec{k}-\vec{p})\delta^{rs}.$$

The normal-ordered hamiltonian is

$$: H := \sum_r \int đ^3k\ \omega_k \mathbf{a}_k^{r\dagger}\mathbf{a}_k^r.$$

The propagator for the $A_\mu(x)$ field is

$$\langle \mathcal{T} A_\mu(x) A_\nu(y) \rangle = \int đ^4 k e^{-\mathbf{i}k(x-y)} \left[\frac{-\mathbf{i}(\eta_{\mu\nu} - k_\mu k_\nu / \mu^2)}{k^2 - \mu^2 + \mathbf{i}\epsilon} \right]. \tag{5.2}$$

Notice that like in the spinor case the polarization sum $\sum_r \epsilon_\mu^{r\star} \epsilon_\nu^r$ appears in the numerator of the propagator. The quantity in square brackets is then the momentum-space propagator. Since $\langle 0| A_\mu(x) |k, r\rangle = \epsilon_\mu^r(k) e^{-\mathbf{i}kx}$, a vector in the initial state produces a factor of $\epsilon_\mu^r(k)$, and in the final state gives $\epsilon^\star$.

Massless case. In the limit $\mu \to 0$ some weird stuff happens. If we couple A_μ to some object j^μ made of other matter, by adding $\Delta\mathcal{L} = j^\mu A_\mu$, then we learn that $\partial_\mu A^\mu = \mu^{-2} \partial_\mu j^\mu$. This means that in order to take $\mu \to 0$, it will be best if the current is conserved $\partial_\mu j^\mu$.

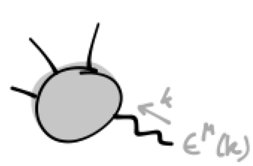
One example is the QED coupling, $j^\mu = \bar\Psi \gamma^\mu \Psi$. We saw that this coupling $A_\mu j^\mu$ arose from the 'minimal coupling' prescription of replacing $\partial_\mu \to D_\mu = \partial_\mu + \mathbf{i}eqA_\mu$ in the Dirac Lagrangian. In that case, the model had a local invariance under $A_\mu \to A_\mu + \partial_\mu \lambda(x)/e, \Psi(x) \to e^{\mathbf{i}q\lambda(x)}\Psi(x)$. For λ non-constant (and going to zero far away), this is a redundancy of our description rather than a symmetry (for example, they have the same configuration of $\vec{E}, \vec{B}, \oint A$). That is, configurations related by this gauge transformation should be regarded as equivalent.

Another example can be obtained by taking a complex scalar and doing the same replacement: $\mathcal{L} = D_\mu \Phi^\star D^\mu \Phi + ...$ Notice that in this case the vertex involves a derivative, so it comes with a factor of $-\mathbf{i}eq(p_\Phi + p_{\Phi^\star})^\mu$. Also, there is a $A_\mu A_\nu \Phi^\star \Phi$ coupling, which gives a vertex proportaional to $-\mathbf{i}e^2 q^2 \eta_{\mu\nu}$.

How do I know that configurations related by a gauge transformation should be regarded as equivalent? If not, the kinetic operator for the massless vector field $(\eta_{\mu\nu}(\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu) A^\nu = 0$ is not invertible, since it annihilates $A_\nu = \partial_\nu \lambda$.

[End of Lecture 19]

What's the propagator, then? One strategy is to simply ignore the gauge equivalence and use the same propagator (5.2) that we found in the massive case with $\mu \to 0$. Since the dynamics are gauge invariant, it will never make gauge-variant stuff, and the longitudinal bits $\propto k_\mu k_\nu$ in (5.2) (which depend on μ) will just drop out, and we can take $\mu \to 0$ in the denominator at the end. This actually works. The guarantee that it works is the QED Ward identity: any amplitude with an external vector $\epsilon(k)_\mu$ is of the form



$$= \mathbf{i}\mathcal{M} = \mathbf{i}\mathcal{M}^\mu(k)\epsilon_\mu(k)$$

and if all external fermion lines are on-shell then

$$\mathcal{M}^\mu(k)k_\mu = 0.$$

There is a complicated diagrammatic proof of this statement in Peskin; we will see some illustrations of it below (I also recommend Zee §II.7). It is basically a statement of current conservation: such an amplitude is made (by LSZ) from a correlation function involving an insertion of the electromagnetic current $j^\mu(k) = \int d^4x\, e^{-\mathbf{i}kx} j^\mu(x)$, in the form, $\mathcal{M}^\mu \sim ... \langle\Omega|....j^\mu(k)...|\Omega\rangle$, and $k_\mu j^\mu(k) = 0$ is current conservation. A systematic proof using this point of view is easy with the path integral (in 215C).

This property guarantees that we will not emit any longitudinal photons, since the amplitude to do so is the $\mu \to 0$ limit of

$$\mathcal{A}\begin{pmatrix}\text{emit } \epsilon^L_\lambda = \frac{1}{\mu}(k,0,0,-\omega)_\lambda \\ \text{with } k^\lambda = (\omega,0,0,k)^\lambda\end{pmatrix} \propto \epsilon^L_\mu \mathcal{M}^\mu = \frac{1}{\mu}\left(k\mathcal{M}^0 - \omega\mathcal{M}^3\right) = \frac{1}{\mu}\left(k\mathcal{M}^0 - \underbrace{\sqrt{k^2+\mu^2}}_{=k+\frac{\mu^2}{2k}+...}\mathcal{M}^3\right)$$

$$= \frac{1}{\mu}\underbrace{k_\mu\mathcal{M}^\mu}_{=0,\text{by Ward}} \underbrace{-\frac{\mu}{2k}\mathcal{M}^3 + \mathcal{O}(\mu^3)}_{\to 0 \text{ as } \mu\to 0} \overset{\mu\to 0}{\to} 0.$$

Gauge fixing. You might not be happy with the accounting procedure I've advocated above, where unphysical degrees of freedom are floating around in intermediate states and only drop out at the end by some formal trick. In that case, a whole zoo of formal tricks called *gauge fixing* has been prepared for you. Here's a brief summary to hold you over until 215B.

At the price of Lorentz invariance, we can make manifest the physical dofs, by choosing *Coulomb gauge*. That means we restrict $\partial_\mu A^\mu = 0$ (so far, so Lorentz invariant) *and also* $\vec{\nabla}\cdot\vec{A} = 0$. Looking at (5.1), we see that this kills off the bit of A_0 that depended on $\vec{A}$. We also lose the helicity-zero polarization $\vec{\nabla}\cdot\vec{A} \propto \epsilon^{(0)}$. But the Coulomb interaction is instantaneous action at a distance.

To keep Lorentz invariance, we can instead merely discourage configurations with $\partial\cdot A \neq 0$ by adding a term to the action

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial\cdot A\right)^2$$

for some arbitrary number ξ. Physics should not depend on ξ, and this is a check on calculations. The propagator is

$$\langle\mathcal{T}A_\mu(x)A_\nu(y)\rangle_\alpha = \int đ^4k\, e^{-\mathbf{i}k(x-y)}\left[\frac{-\mathbf{i}(\eta_{\mu\nu} - (1-\xi)k_\mu k_\nu/\mu^2)}{k^2-\mu^2+\mathbf{i}\epsilon}\right]$$

and again the bit with $k_\mu k_\nu$ must drop out. $\xi = 1$ is called Feynman gauge and makes this explicit. $\xi = 0$ is called Landau gauge and makes the propagator into a projector onto $k_\perp$.

It becomes much more important to be careful about this business in non-Abelian gauge theory.

[29]

5.2 Feynman rules for QED

First, Feynman rules for Dirac fermion fields, more generally[30]. As always in these notes, time goes to the left, so I draw the initial state on the right (like the ket) and the final state on the left (like the bra).

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[29] By the way, you might be bothered that we didn't go back to our table 1 of possible Lorentz representations on fields to think about spin one particles. Indeed, we could start with the $(1,0)\oplus(0,1)$ antisymmetric tensor $F_{\mu\nu}$ as our basic object. (See, for example, the book by Haag, page 47.) Indeed, in this way we could construct a theory of a free EM field. But don't we need a vector potential to couple to charged matter? The answer turns out to be 'no,' as explained by Mandelstam here. The price is that the charged fields depend on not just a point, but a choice of *path*; if we did introduce the vector potential, they would be related to our fields by

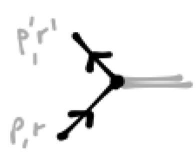
$$\Phi(x, P) = e^{\mathbf{i}\int_P A}\Phi(x),$$

where P is a path which ends at x and infinity. This *Wilson line* $e^{\mathbf{i}\int_P A}$ carries away the gauge transformation, so that $\Phi(x, P)$ is actually invariant under gauge transformations that fall off at infinity.

Thanks to Wei-Ting Kuo for asking about this.

[30] Another good example of a QFT with interacting fermions is the Yukawa theory theory of a Dirac fermion field plus a scalar ϕ and an interaction

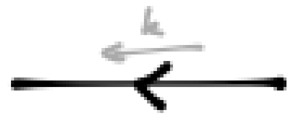
$$V = g\phi\bar{\Psi}\Psi \quad \Longrightarrow$$



$$= \quad -\mathbf{i}g\delta^{rr'}. \qquad (5.3)$$

Notice that in $3+1$ dimensions, $[g] = +4 - [\phi] - 2[\Psi] = 4 - 1 - 2\frac{3}{2} = 0$, the coupling is dimensionless. This describes more realistically the interactions between nucleons (which are fermions, as opposed to snucleons) and scalar pions, which hold together nuclei. It also is a crude sketch of the Higgs coupling to matter; notice that if ϕ is some nonzero constant $\langle\phi\rangle$, then there is a contribution to the mass of the fermions, $g\langle\phi\rangle$.

1. An internal fermion line gives

 $$= \frac{\mathbf{i}}{\not{k} - m_\Psi}$$

which is a matrix on the spinor indices.

There are four possibilities for an external fermion line of definite momentum.

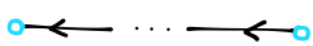
2.  $= \Psi|p, r\rangle = u^r(p)$

3.  $= \langle p, r|\bar{\Psi} = \bar{u}^r(p)$

4.  $= \bar{\Psi}|p, r\rangle = \bar{v}^r(p)$

5.  $= \langle p, r|\Psi = v^r(p)$

6. Some advice: When evaluating a Feynman diagram with spinor particles, always begin at the head of the particle-number arrows on the fermion lines, and keep going along the fermion line until you can't anymore. This will keep the spinor indices in the form of matrix multiplication. Why: every Lagrangian you'll ever encounter has fermion parity symmetry, under which every fermionic field gets a minus sign; this means fermion lines cannot end, except on external legs. The result is always of the form of a scalar function (not a matrix or a spinor) made by sandwiching gamma matrices between external spinors:

   $_{r'p'}$  $_{rp} \quad = \quad \sum_{a,b...=1..4} \bar{u}^{r'}(p')_a \, (\text{pile of gamma matrices})_{ab} \, u^r(p)_b$

   Furthermore, in S-matrix elements the external spinors $u(p), v(p)$ satisfy the equations of motion $(\not{p} - m)u(p) = 0$, a fact which can be used to our advantage to shrink the pile of gammas.

   There can also be fermion lines which form internal loops (though not at tree level, by definition). In this case, the spinor indices form a trace,

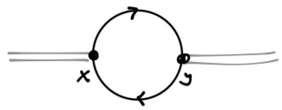
   $$\sum_a (\text{pile of gamma matrices})_{aa} \equiv \operatorname{tr}(\text{pile of gamma matrices}) .$$

   We'll learn to compute such traces below (around (5.5)); in fact, traces appear even in the case with external fermions if we do not measure the spins.

7. Diagrams related by exchanging external fermions have a relative minus sign.

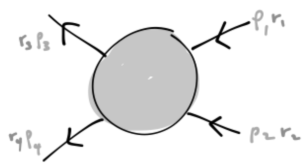
8. Diagrams with an odd number of fermion loops have an extra minus sign.

The last two rules are best understood by looking at an example in detail.

To understand rule 8 consider the following amplitude in the Yukawa theory with interaction (5.3): 

It is a contribution to the meson propagator. It is proportional to

$$\sum_{abcd} \bar{\Psi}_a(x)\Psi_c(x)\bar{\Psi}_c(y)\Psi_d(y) = (-1)\mathrm{tr}\Psi(x)\bar{\Psi}(y)\Psi(x)\bar{\Psi}(y) = (-1)\mathrm{tr}S_F(x-y)S_F(x-y)$$

[Peskin page 119] To understand rule 7 consider $\Psi\Psi \to \Psi\Psi$ (nucleon) scattering in the Yukawa theory:  The blob represents the matrix element

$$_0\langle p_3r_3; p_4r_4|\, \mathcal{T} e^{-\mathbf{i}\int V d^4z}\, |p_1r_1; p_2r_2\rangle_0$$

where the initial state is

$$|p_1r_1; p_2r_2\rangle_0 = \mathbf{a}^{r_1\dagger}_{p_1}\mathbf{a}^{r_2\dagger}_{p_2}|0\rangle$$

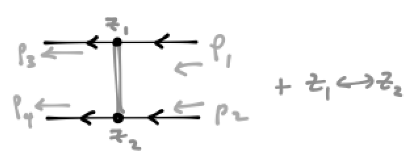
and the final state is

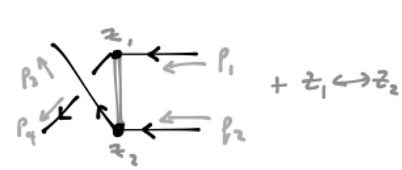
$$_0\langle p_3r_3; p_4r_4| = \left(|p_3r_3; p_4r_4\rangle_0\right)^\dagger = \langle 0|\, \mathbf{a}^{r_4}_{p_4}\mathbf{a}^{r_3}_{p_3} = -\langle 0|\, \mathbf{a}^{r_3}_{p_3}\mathbf{a}^{r_4}_{p_4}$$

where note that the dagger reverses the order.

The leading contribution comes at second order in V:

$$_0\langle p_3r_3; p_4r_4|\, \mathcal{T}\left(\frac{1}{2!}(\mathbf{i}g)^2\int d^4z_1\int d^4z_2\left(\bar{\Psi}\Psi\phi\right)_1\left(\bar{\Psi}\Psi\phi\right)_2\right)|p_1r_1; p_2r_2\rangle_0$$





To get something nonzero we must contract the ϕs with each other. The diagrams at right indicate best the possible ways to contract the fermions. Exchanging the roles of z_1 and z_2 interchanges two pairs of fermions so costs no signs and cancels the $\frac{1}{2!}$.

The overall sign is annoying but can be fixed by demanding that the diagonal bit of the S-matrix give

$$\langle p_3p_4|\,(\mathbb{1}+...)\,|p_1p_2\rangle = +\delta(p_1-p_3)\delta(p_2-p_4)+\cdots$$

The relative sign is what we're after, and it comes by comparing the locations of fermion operators in the contractions in the two diagrams at right. In terms of the contractions, these $t-$ and $u-$ channel diagrams are related by leaving the annihilation operators alone and switching the contractions between the creation operators and the final state. Denoting by $\mathbf{a}^\dagger_{1,2}$ the fermion creation operators coming from the vertex at $z_{1,2}$,

$$\begin{aligned} &\langle 0|\, \mathbf{a}_{p_4} \underbrace{\mathbf{a}_{p_3}\mathbf{a}^\dagger_1}\, \mathbf{a}^\dagger_2 ... + \langle 0|\, \mathbf{a}_{p_4} \underbrace{\mathbf{a}_{p_3}\mathbf{a}^\dagger_1 \mathbf{a}^\dagger_2} ... \\ = &\langle 0|\, \mathbf{a}_{p_4} \underbrace{\mathbf{a}_{p_3}\mathbf{a}^\dagger_1}\, \mathbf{a}^\dagger_2 ... - \langle 0|\, \underbrace{\mathbf{a}_{p_4}\mathbf{a}^\dagger_1}\, \underbrace{\mathbf{a}_{p_3}\mathbf{a}^\dagger_2} ... \end{aligned}$$

In the last expression the fermion operators to be contracted are all right next to each other and we see the relative minus sign.

While we're at it, let's evaluate this whole amplitude to check the Feynman rules I've claimed and get some physics out. It is

$$S_{fi} = -g^2 \int dz_1 dz_2 \int đ^4 q \frac{e^{-\mathbf{i}q(z_1-z_2)}\mathbf{i}}{q^2 - m^2 + \mathbf{i}\epsilon} \left(e^{-\mathbf{i}z_2(p_1-p_3)} \bar{u}^{r_3}(p_3) u^{r_1}(p_1) \cdot e^{-\mathbf{i}z_1(p_2-p_4)} \bar{u}^{r_4}(p_4) u^{r_2}(p_2) - (3 \leftrightarrow 4) \right).$$

In the first (t-channel) term, the integrals over $z_{1,2}$ gives $\not{\delta}(p_1 - p_3 - q)\not{\delta}(p_2 - p_4 - q)$, and the q integral then gives $\delta(p_1 + p_2 - p_3 - p_4)$, overall momentum conservation. In the second (u-channel) term, $q = p_1 - p_4 = p_3 - p_2$. Altogether,

$$S_{fi} = \mathbb{1} + \not{\delta}^4(p_T)\mathbf{i}\mathcal{M}$$

with

$$\mathbf{i}\mathcal{M} = -\mathbf{i}g^2 \left(\frac{1}{t - m^2} \left(\bar{u}_3 u_1\right)\left(\bar{u}_4 u_2\right) - \frac{1}{u - m^2} \left(\bar{u}_4 u_1\right)\left(\bar{u}_3 u_2\right) \right) \tag{5.4}$$

with $t \equiv (p_1 - p_3)^2, u \equiv (p_1 - p_4)^2$. This minus sign implements Fermi statistics.

Yukawa force revisited. In the non-relativistic limit, we can again relate this amplitude to the force between particles, this time with the actual spin and statistics of nucleons. In the COM frame, $p_1 = (m, \vec{p}), p_2 = (m, -\vec{p})$ and $p_3 = (m, \vec{p}'), p_4 = (m, -\vec{p}')$. The spinors become $u^r_p = \sqrt{m}\begin{pmatrix}\xi^r \\ \xi^r\end{pmatrix}$ so that $\bar{u}_3 u_1 \equiv \bar{u}(p_3)^{r_3} u(p_1)^{r_1} = 2m\xi^\dagger_{r_3}\xi_{r_1} = 2m\delta_{r_3 r_1}$. Let's simplify our lives and take two *distinguishable* fermions (poetically, they could be proton and neutron, but let's just add a label to our fermion fields; they could have different masses, for example, or different couplings to ϕ, call them g_1, g_2). Then we only get the t-channel diagram. The intermediate scalar momentum is $q = p_1 - p_3 = (0, \vec{p} - \vec{p}')$ so $t = (p_1 - p_3)^2 = -\vec{q}^2 = -\left(\vec{p} - \vec{p}'\right)^2$ and

$$\mathbf{i}\mathcal{M}_{NR,COM} = \mathbf{i}g_1 g_2 \frac{1}{\vec{q}^2 + m_\phi^2} 4m^2 \delta^{r_1 r_3}\delta^{r_2 r^4}.$$

Compare this to the NR Born approximation matrix element

$$
\begin{aligned}
2\pi\delta(E_p - E_{p'})\left(-\mathbf{i}\tilde{V}(\vec{q})\right) &= {}_{NR}\langle \vec{p}'|\, S\, |\vec{p}\rangle_{NR} \\
&= \sum_{r_4} \int \text{đ}^3 p' V \underbrace{\prod_{i=1}^{4} \frac{1}{\sqrt{2E_i}}}_{=\frac{1}{\sqrt{2m}^4}} S_{fi} \\
&= 2\pi\delta(E_p - E_{p'})\delta^{r_1 r_3} \frac{\mathbf{i} g_1 g_2}{\vec{q}^2 + m_\phi^2}
\end{aligned}
$$

where in the second line we summed over the spins of the second particle, and corrected the relativistic normalization, so that ${}_{NR}\langle \vec{p}'|\vec{p}\rangle_{NR} = \not{\delta}^3(p - p')$. This is completely independent of the properties of the second particle. We infer that the scalar mediates a force with potential $U(x) = -\frac{g_1 g_2 e^{-m_\phi r}}{4\pi r}$. It is attractive if $g_1 g_2 > 0$.

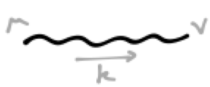
Back to QED. The new ingredients in QED are the propagating vectors, and the interaction hamiltonian $V = e\bar{\Psi}\gamma^\mu \Psi A_\mu$. The rest of the Feynman rules are

9. The interaction vertex gets a

 $= \quad -\mathbf{i}e\gamma^\mu$

10. An external photon in the initial state gets a $\epsilon^\mu(p)$, and in the final state gets a $\epsilon^{\mu\star}(p)$.

11. An internal photon line gets a

 $= \quad \frac{\mathbf{i}}{k^2 - m_\gamma^2}\left(-\eta^{\mu\nu} + (1-\xi)k^\mu k^\nu / k^2\right)$

where $m_\gamma = 0$ (it's sometimes useful to keep it in there for a while as an IR regulator) and the value of ξ is up to you (meaning that your answers for physical quantities should be independent of ξ).

Spinor trace ninjutsu.

$$\text{The trace is cyclic:} \qquad \text{tr} AB \cdots C = \text{tr} CAB. \tag{5.5}$$

$$\text{Our gamma matrices are } 4 \times 4 \text{, so} \qquad \text{tr}\mathbb{1} = 4.$$

$$\mathrm{tr}\gamma^\mu = \mathrm{tr}\left(\gamma^5\right)^2\gamma^\mu \overset{(5.5)}{=} \mathrm{tr}\gamma^5\gamma^\mu\gamma^5 \overset{\{\gamma^5,\gamma^\mu\}=0}{=} -\mathrm{tr}\gamma^\mu = 0. \tag{5.6}$$

The same works for any odd number of gammas.

$$\mathrm{tr}\gamma^\mu\gamma^\nu \overset{\text{clifford}}{=} -\mathrm{tr}\gamma^\nu\gamma^\mu + 2\eta^{\mu\nu}\mathrm{tr}\mathbb{1} \overset{(5.5)}{=} -\mathrm{tr}\gamma^\mu\gamma^\nu + 8\eta^{\mu\nu} \implies \mathrm{tr}\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}. \tag{5.7}$$

$$\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\sigma\mu}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma}\right). \tag{5.8}$$

Why is this? The completely antisymmetric bit vanishes because it is proportional to γ^5 which is traceless (by the same argument as (5.6)). If any pair of indices is the same then the other two must be too by (5.7). If adjacent pairs are the same they can just square to one and we get +1; if alternating pairs are the same (and different from each other) then we must move them through each other with the anticommutator. If they are all the same we get 4.

$$\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = -4\mathbf{i}\epsilon^{\mu\nu\rho\sigma}.$$

5.3 QED processes at leading order

Now we are ready to do *lots* of examples, nearly all of which (when pushed to the end) predict cross sections which are verified by experiments to about one part in 137.[31] Here $\frac{1}{137} \approx \alpha \equiv \frac{e^2}{4\pi}$ is the small number by which the next order corrections are suppressed. [32]

Did I mention that the antiparticle of the electron, predicted by the quantum Dirac theory (*i.e.* by Dirac), is the positron? It has the same mass as the electron and the opposite electromagnetic charge, since the charge density is the 0 component of the electromagnetic current, $j^\mu = \bar{\Psi}\gamma^\mu\Psi$, so the charge is

$$\int d^3x j^0(x) = \int \bar{\Psi}\gamma^0\Psi = \int \Psi^\dagger\Psi = \int đ^3p \sum_s \left(\mathbf{a}^\dagger_{p,s}\mathbf{a}_{ps} - \mathbf{b}^\dagger_{p,s}\mathbf{b}_{ps}\right).$$

So $\mathbf{b}^\dagger$ creates a positron.

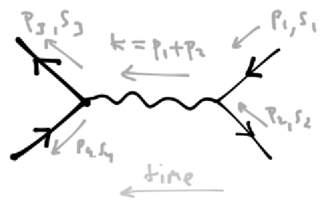
[Schwarz §13.3] Perhaps the simplest to start with is scattering of electrons and positrons. We can make things even simpler (one diagram instead of two) by including

[31] I guess it is this overabundance of scientific victory in this area that leads to the intrusion of so many names of physicists in the following discussion.

[32] This statement is true naively (in the sense that the next diagrams which are nonzero come with two more powers of e), and also true in fact, but in between naiveté and the truth is a long road of renormalization, which begins next quarter.

also the muon, which is a heavy version of the electron[33], and asking about the process $\mu^+\mu^- \leftarrow e^+e^-$. At leading order in e, this comes from

$$\mathbf{i}\mathcal{M}_{\mu^+\mu^-\leftarrow e^+e^-} =$$



$$= \left(-\mathbf{i}e\bar{u}^{s_3}(p_3)\gamma^\mu v^{s_4}(p_4)\right)_{\text{muons}} \frac{-\mathbf{i}\left(\eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2}\right)}{k^2} \left(-\mathbf{i}e\bar{v}^{s_2}(p_2)\gamma^\nu u^{s_1}(p_1)\right)_{\text{electrons}} \quad (5.9)$$

with $k \equiv p_1 + p_2 = p_3 + p_4$ by momentum conservation at each vertex. I've labelled the spinors according to the particle types, since they depend on the mass.

Ward identity in action. What about the $k_\mu k_\nu$ term in the photon propagator? The spinors satisfy their equations of motion, $\not{p}_1 u_1 = m_e u_1$ (where $u_1 \equiv u_{p_1}^{s_1}$ for short) and $\bar{v}_2 \not{p}_2 = -m_e \bar{v}_2$. The k_ν appears in

$$k_\nu \bar{v}_2 \gamma^\nu u_1 = \bar{v}_2 \left(\not{p}_1 + \not{p}_2\right) u_1 = \bar{v}_2 \not{p}_1 u_1 + \bar{v}_2 \not{p}_2 u_1 = (m - m)\bar{v}u = 0.$$

(The other factor is also zero, but one factor of zero is enough.) Therefore

$$\mathcal{M} = \frac{e^2}{s} \bar{u}_3 \gamma_\mu v_4 \cdot \bar{v}_2 \gamma^\mu u_1$$

where $s \equiv k^2 = (p_1 + p_2)^2 = E_{CoM}^2$ is the Mandelstam variable. And I am relying on you to remember which spinors refer to muons (3,4) and which to electrons (1,2).

Squaring the amplitude. We need to find $\mathcal{M}^\dagger$ (the dagger here really just means complex conjugate, but let's put dagger to remind ourselves to transpose and reverse the order of all the matrices). Recall the special role of γ^0 here:

$$\gamma_\mu^\dagger \gamma_0 = \gamma_0 \gamma_\mu, \quad \gamma_0^\dagger = \gamma_0.$$

This means that for any two Dirac spinors,

$$\left(\bar{\Psi}_1 \gamma^\mu \Psi_2\right)^\dagger = \bar{\Psi}_2 \gamma^\mu \Psi_1.$$

(This is the same manipulation that showed that the Dirac Lagrangian was hermitian.) So

$$\mathcal{M}^\dagger = \frac{e^2}{s} \left(\bar{v}_4 \gamma^\mu u_3\right) \left(\bar{u}_1 \gamma_\mu v_2\right).$$

[33]Who ordered *that*? (I. Rabi's reaction to learning about the muon.) I hope you don't find it too jarring that the number of 'elementary' particles in our discussion increased by three in the last two paragraphs. People used to get really disgruntled about this kind of thing. But here we have, at last, uncovered the true purpose of the muon, which is to halve the number of Feynman diagrams in this calculation (compare (5.14)).

and therefore

$$|\mathcal{M}_{\mu^+\mu^-\leftarrow e^+e^-}|^2 = \frac{e^4}{s^2}\underbrace{(\bar v_4\gamma^\mu u_3)(\bar u_3\gamma^\nu v_4)}_{\text{out}}\cdot\underbrace{(\bar u_1\gamma_\mu v_2)(\bar v_2\gamma_\nu u_1)}_{\text{in}}. \tag{5.10}$$

These objects in parentheses are just c-numbers, so we can move them around, no problem. I've grouped them into a bit depending only on the initial state (the electron stuff 1, 2) and a bit depending only on the final state (the muon stuff 3,4).

Average over initial, sum over final. In the amplitude above, we have fixed the spin states of all the particles. Only very sophisticated experiments are able to discern this information. So suppose we wish to predict the outcome of an experiment which does not measure the spins of the fermions involved. We must sum over the final-state spins using

$$\sum_{s_4} v_a^{s_4}(p_4)\bar v_b^{s_4}(p_4) = \left(\not p_4 - m_\mu\right)_{ab} = \sum_{s_4}\bar v_b^{s_4}(p_4)v_a^{s_4}(p_4)$$

(where I wrote the last expression to emphasize that these are just c-numbers) and

$$\sum_{s_3} u_a^{s_3}(p_3)\bar u_b^{s_3}(p_3) = \left(\not p_3 + m_\mu\right)_{ab}.$$

Looking at just the 'out' factor of $|\mathcal{M}|^2$ in (5.10), we see that putting these together produces a spinor trace, as promised:

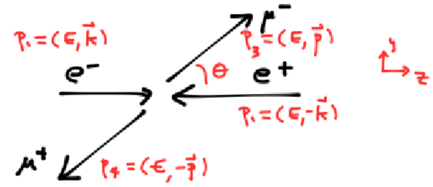
$$\begin{aligned}
&\sum_{s_3,s_4}\left(\bar u(p_3)_a^{s_3}\gamma^\mu_{ab}\underbrace{v(p_4)_b^{s_4})\left(\bar v(p_4)_c^{s_4}}_{(\not p_4-m_\mu)_{bc}}\gamma^\nu_{cd}u^{s_3}(p_3)_d\right)\\
&= \gamma^\mu_{ab}(\not p_4 - m_\mu)_{bc}\gamma^\nu_{cd}(\not p_3 + m_\mu)_{da}\\
&= \text{tr}\left(\gamma^\mu\left(\not p_4 - m_\mu\right)\gamma^\nu\left(\not p_3 + m_\mu\right)\right)\\
&= p_{4\rho}p_{3\sigma}\text{tr}\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma - m_\mu^2\text{tr}\gamma^\mu\gamma^\nu\\
&\overset{(5.7),(5.8)}{=} 4\left(p_4^\mu p_3^\nu + p_3^\nu p_4^\mu - \underbrace{p_3\cdot p_4}_{\equiv p_{34}}\eta^{\mu\nu} - m_\mu^2\right)
\end{aligned} \tag{5.11}$$

If also we don't know the initial (electron) spins, then the outcome of our experiment is the *average* over the initial spins, of which there are four possibilities. Therefore, the relevant probability for unpolarized scattering is

$$\begin{aligned}
\frac14\sum_{s_{1,2,3,4}}|\mathcal{M}|^2 &= \frac{e^4}{4s^2}\text{tr}\left(\gamma^\mu\left(\not p_4 - m_\mu\right)\gamma^\nu\left(\not p_3 + m_\mu\right)\right)\text{tr}\left(\gamma_\nu\left(\not p_2 - m_\mu\right)\gamma_\mu\left(\not p_1 + m_\mu\right)\right)\\
&\overset{(5.11)\text{ twice}}{=} \frac{4e^4}{s^2}\left(p_{13}p_{24} + p_{14}p_{23} + m_\mu^2 p_{12} + m_e^2 p_{34} + 2m_e^2 m_\mu^2\right)\\
&\overset{\text{algebra}}{=} \frac{2e^4}{s^2}\left(t^2 + u^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2\right)
\end{aligned} \tag{5.12}$$

where we used all the Mandelstam variables:

$$\begin{aligned}
s &\equiv (p_1+p_2)^2 = (p_3+p_4)^2 = \quad E_{CoM}^2 = 4E^2 \\
t &\equiv (p_1-p_3)^2 = (p_2-p_4)^2 = m_e^2 + m_\mu^2 - 2E^2 + 2\vec{k}\cdot\vec{p} \\
u &\equiv (p_1-p_4)^2 = (p_2-p_3)^2 = m_e^2 + m_\mu^2 - 2E^2 - 2\vec{k}\cdot\vec{p}
\end{aligned}$$



where the particular kinematic variables (in the rightmost equalities) are special to this problem, in the center of mass frame (CoM), and are defined in the figure at right. Really there are only two independent Lorentz-invariant kinematical variables, since $s+t+u=\sum_i m_i^2$.

Now we can use the formula (3.35) that we found for a differential cross section with a two-body final state, in the CoM frame:

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{CoM} &= \frac{1}{64\pi^2 E_{CoM}}\frac{|\vec{p}|}{|\vec{k}|}\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right) \\
&= \frac{\alpha^2}{16E^6}\frac{|\vec{p}|}{|\vec{k}|}\left(E^4 + |\vec{k}|^2|\vec{p}|^2\cos^2\theta + E^2(m_e^2+m_\mu^2)\right) \qquad (5.13)
\end{aligned}$$

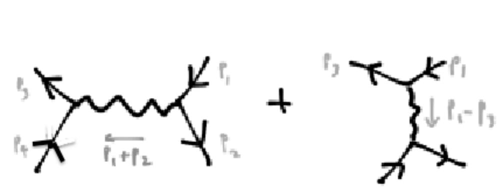
where $\alpha \equiv \frac{e^2}{4\pi}$ is the fine structure constant. This can be boiled a bit with kinematical relations $|\vec{k}| = \sqrt{E^2-m_e^2}, |\vec{p}| = \sqrt{E^2-m_\mu^2}$ to make manifest that it depends only on two independent kinematical variables, which we can take to be the CoM energy E and the scattering angle θ in $\vec{k}\cdot\vec{p} = |\vec{k}||\vec{p}|\cos\theta$ (best understood from the figure). It simplifies a bit if we take $E \gg m_e$, and more if we take $E \gg m_\mu \sim 200m_e$ to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\left(1+\cos^2\theta\right).$$

In fact, the two terms here come respectively from spins transverse to the scattering plane and in the scattering plane; see Schwartz §5.3 for an explanation.

There is a lot more to say about what happens when we scatter an electron and a positron! Another thing that can happen is that the final state could be an electron and positron again (Bhabha scattering[34]).

They are not necessarily the same e^- and e^+, though (except in the sense that they are all the same), because another way to get there at tree level is the second, t-channel, diagram, at right. The intermediate photon in that diagram has $k_t = (p_1-p_3)$, so that the denominator of the propagator is $t = k_t^2 = (p_1-p_3)^2$ instead of s.



[34]See figure 3 here. Now remember that a person doesn't have much control over their name. By the way, I totally believe the bit about *non-perturbative strings = lint*.

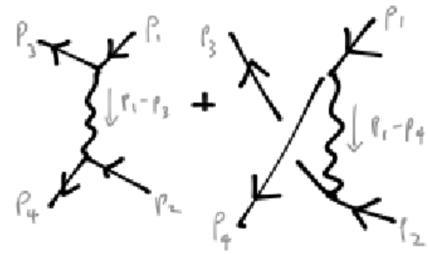
Squaring this amplitude gives

$$|\mathcal{M}_s + \mathcal{M}_t|^2 = |\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 + 2\mathrm{Re}(\mathcal{M}_s \mathcal{M}_t^\star), \tag{5.14}$$

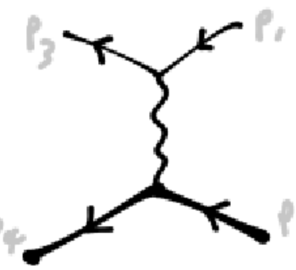
interference terms. Interference terms mean that you have to be careful about the overall sign or phase of the amplitudes. In this case, there is a relative minus sign, by exactly the calculation we did to get (5.4).

Above we've studied an *exclusive* cross-section in the sense that we fixed the final state to be exactly a muon and an antimuon. It has also been very valuable to think about *inclusive* cross-sections for e^+e^- scattering, because in this way you can make anything that the s-channel photon couples to, if you put enough energy into it.

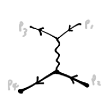
$e^-e^- \to e^-e^-$. What happens if instead we scatter two electrons (Möller scattering)? In that case, the leading order diagrams are the ones at right. Now the intermediate photons have $k_t = (p_1 - p_3)$ and $k_u = (p_1 - p_4)$ respectively, so that the denominator of the propagator is t and u in the two diagrams. The evaluation of these diagrams has a lot in common with the ones for $e^+e^- \to e^+e^-$, namely you just switch some of the legs between initial and final state.



The relation between such amplitudes is called *crossing symmetry*. Let's illustrate it instead for $e^-\mu^- \leftarrow e^-\mu^-$, where again there is only one diagram, related by crossing to (5.9). The diagram is the one at right. (The muon is the thicker fermion line.)



$$\mathbf{i}\mathcal{M} =$$



$$= (-\mathbf{i}e\bar{u}_3\gamma^\mu u_1))_{\text{electrons}} \frac{-\mathbf{i}\left(\eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2}\right)}{k^2} (-\mathbf{i}e\bar{u}_2\gamma^\nu u_4)_{\text{muons}} \tag{5.15}$$

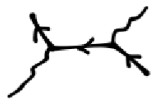
with $k \equiv p_1 - p_3 = p_2 - p_4$. It differs from (5.9) by replacing the relevant vs with us for the initial/final antiparticles that were moved into final/initial particles, and relabelling the momenta. After the spin sum,

$$\frac{1}{4}\sum_{s_{1,2,3,4}} |\mathcal{M}|^2 = \frac{e^4}{4t^2} \mathrm{tr}\left(\gamma^\mu \left(\not{p}_4 + m_\mu\right)\gamma^\nu \left(\not{p}_2 + m_\mu\right)\right) \mathrm{tr}\left(\gamma_\nu \left(\not{p}_3 + m_e\right)\gamma_\mu \left(\not{p}_1 + m_e\right)\right)$$

this amounts to the replacement $(p_1, p_2, p_3, p_4) \to (p_1, -p_3, p_4, -p_2)$; on the Mandelstam variables, this is just the permutation $(s,t,u) \to (t,u,s)$.

Electron-proton scattering. The answer is basically the same if we think of the heavy particle in (5.15) as a proton (we have to flip the sign of the charge but this gets squared away since there is no interference in this case). $ep \to ep$ is called Rutherford scattering, for good reason[35].

Electron-photon scattering. In the case of the process $e^-\gamma \leftarrow e^-\gamma$, [36] we meet a new ingredient, namely external photons:

$\mathbf{i}\mathcal{M} =$  $+$  $\equiv \mathbf{i}\mathcal{M}_s + \mathbf{i}\mathcal{M}_t$

$$= (-\mathbf{i}e)^2 \epsilon_1^\mu \epsilon_4^{\star\nu} \bar{u}_3 \left(\gamma_\nu \frac{\mathbf{i}\not{k}_s + m}{s - m^2} \gamma_\mu + \gamma_\mu \frac{\mathbf{i}\not{k}_t + m}{t - m^2} \gamma_\nu \right) u_2 \ . \tag{5.16}$$

The two amplitudes have a relative plus since we only mucked with the photon contractions, they just differ by how the gamma matrices are attached. If you don't believe me, draw the contractions on this:

$$\langle \gamma e | \, (\bar{\Psi} \not{A} \Psi)_1 (\bar{\Psi} \not{A} \Psi)_2 \, | \gamma e \rangle$$

(I'm not going to TeX it, thank you).

Now, if we don't measure the photon polarizations, we need

$$P = \frac{1}{4} \sum_{\text{polarizations, spins}} |\mathcal{M}|^2.$$

The key ingredient is the completeness relation

$$\sum_{i=1,2} \epsilon_\mu^{i\star}(k) \epsilon_\nu^j(k) = -\eta_{\mu\nu} + \text{something proportional to } k^\mu k^\nu.$$

We can do various incantations to find a definite coefficient, but *it will not matter* because of the Ward identity: anytime there is an external photon $\epsilon(k)_\mu$, the amplitude is $\mathcal{M} = \mathcal{M}_\mu \epsilon^\mu(k)$ and satisfies $k^\mu \mathcal{M}_\mu = 0$. Therefore, we can ignore the term about which I was vague and we have

$$\sum_{\text{polarizations}} |\mathcal{M}|^2 = \sum_i \epsilon_\mu^{i\star} \mathcal{M}^{\mu\star} \mathcal{M}^\nu \epsilon_\nu^i = -\eta_{\mu\nu} \mathcal{M}^{\mu\star} \mathcal{M}^\nu + (\text{terms with } \mathcal{M}_\mu k^\mu)$$

[35] If you don't know why, you should go read *Inward Bound*, by Abraham Pais, as soon as possible.

[36] which at high energy is called Compton scattering and at low energies is called Thomson scattering. Despite my previous curmudgeonly footnote chastising the innocent reader for a poor knowledge of the history of science, I do have a hard time remembering which name goes where. Moreover, as much as I revere the contributions of many of these folks, I find that using their names makes me think about the people instead of the physics. No one owns the physics! It's the same physics for lots of space aliens, too.

$$= -\mathcal{M}^\star_\mu \mathcal{M}^\mu.$$

Don't be scared of the minus sign, it's because of the mostly minus signature, and makes the thing positive. But notice the opportunity to get negative probabilities if the gauge bosons don't behave!

You get to boil the Compton scattering probability some more on the homework. Good luck!

To be continued ... here.