University of California at San Diego – Department of Physics – TA: Shauna Kravec

Quantum Mechanics C (Physics 212A) Fall 2015
Worksheet 6

Announcements

- The 212A web site is:

  http://physics.ucsd.edu/∼mcgreevy/f15/ .

  Please check it regularly! It contains relevant course information!

Problems

1. **Dilatations**

   Let's think about transformations of the form $x \to x' = e^{\lambda} x$ where λ is a real parameter.

   This is known as a *scale* transformation. Quantum mechanically we should be able to realize this transformation with a unitary operator: $U = e^{-\mathbf{i}\lambda\hat{D}}$ where $\hat{D}$ is hermitian.

   This is known as the *dilatation* generator.

   (a) Consider the action of $U|x\rangle$ to determine $[\hat{D}, \hat{x}]$ (Hint: Use the BCH expansion)

   (b) Use the Jacobi identity $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$ to derive $[\hat{D}, \hat{p}]$

   (c) Check that $\hat{D} = \frac{1}{2\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x})$ satisfies these relations [1]

   Now[2] consider a Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ for which $\hat{D}V = 2\mathbf{i}V$

   Such a potential would be *scale invariant*

   (d) Calculate $[\hat{D}, \hat{H}]$ explicitly

   (e) Show that $V(x) = \frac{1}{x^2}$ is an example of such a potential

   (f) Consider an energy eigenstate $|E\rangle$ and calculate $\langle E|[\hat{D}, \hat{H}]|E\rangle$ directly and with the result above. Do you see a problem?

2. **Bogliubov Transformation**

   In our last discussion we solved a new Hamiltonian by defining a set of transformed creation/annihilation operators which satisfy the same algebra $[A, A^{\dagger}] = \mathbb{1}$

   More generally consider $\hat{b} = \hat{a}\cosh\eta + \hat{a}^{\dagger}\sinh\eta$

[1] One might wish to add a term like $-t\hat{H}$ to the definition to make it a conserved charge.

[2] Set $\hbar = 1$.

(a) Show that $[\hat{b}, \hat{b}^\dagger] = \mathbb{1}$

(b) Show that $\hat{b} = U\hat{a}U^\dagger$ for $U = e^{\frac{\eta}{2}(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger)}$

(c) Show for fermionic operators $\hat{c}^2 = 0 = (\hat{c}^\dagger)^2$ and $\{\hat{c}, \hat{c}^\dagger\} = \mathbb{1}$ that $\hat{d} = \hat{c}\cos\theta + \hat{c}^\dagger \sin\theta$ is the analogous operator

Now consider the Hamiltonian

$$\hat{H} = \omega\hat{a}^\dagger\hat{a} + \frac{V}{2}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger) \tag{1}$$

(c) Diagonalize the Hamiltonian (1) using the $\hat{b}$ operators for suitably chosen η

(d) Show there is a limit on V for which this Hamiltonian makes physical sense