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Quantum Mechanics C (Physics 212A) Fall 2015
Worksheet 5

Announcements

- The 212A web site is:

  http://physics.ucsd.edu/∼mcgreevy/f15/ .

  Please check it regularly! It contains relevant course information!

Problems

1. **Give it a Kick**

   Consider the $D = 1$ simple harmonic oscillator in its groundstate. Suppose something kicks the system imparting an additional momentum p_0. What's the probability the system remains in the ground state?

   (a) What's the new Hamiltonian for the system? Express this in terms of the usual ladder operators $\hat{a}$ and $\hat{a}^\dagger$

   (b) Define a new operator $\hat{A} \equiv \hat{a} - \beta$ where $\beta \equiv \frac{1}{\mathbf{i}\omega}\frac{p_0}{m}\sqrt{\frac{m\omega}{2}}$.
   Show that the $\hat{A}$ are ladder operators: $[\hat{A}, \hat{A}^\dagger] = 1$

   (c) Rewrite the new Hamiltonian in terms of these operators, what do you find?

   (d) Relate the original groundstate $|0\rangle$ to the new groundstate $|\beta\rangle$

   (e) Using $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$ compute $P = |\langle 0|\beta\rangle|^2$
   Hint: Insert identity and use the relation above.

2. **Supersymmetry?**

   Consider a spin-$\frac{1}{2}$ particle on a line and in a magnetic field. The Hamiltonian is:

   $$\hat{H} = [\frac{1}{2}\hat{P}_x^2 + V(x)]\mathbb{1} + B(x)S^z \quad (1)$$

   Suppose your wrote your $V(x)$ and $B(x)$ as the following:

   $$V(x) = \frac{1}{2}(\partial_x W)^2 \qquad B(x) = \partial_x^2 W \qquad \hat{P}_x = -\mathbf{i}\partial_x \qquad S^z = \frac{1}{2}\boldsymbol{\sigma}^z$$

Where $W(x)$ is an arbitrary function known as the superpotential. Now define the following operators:

$$Q = (\hat{P}_x - \mathbf{i}\partial_x W)\boldsymbol{\sigma}^+ \qquad Q^\dagger = (\hat{P}_x + \mathbf{i}\partial_x W)\boldsymbol{\sigma}^-$$

Where recall $\boldsymbol{\sigma}^\pm = \frac{1}{2}(\boldsymbol{\sigma}^x \pm \mathbf{i}\boldsymbol{\sigma}^y)$ are raising and lowering operators. Q and $Q^\dagger$ are known as supercharges.

(a) Show that $Q^2 = 0 = (Q^\dagger)^2$

(b) Show that $\{Q, Q^\dagger\} = 2\hat{H}$ where recall $\{A, B\} = AB + BA$

(c) Show that $[Q, \hat{H}] = 0 = [Q^\dagger, \hat{H}]$

(d) We can also define $F = \boldsymbol{\sigma}^-\boldsymbol{\sigma}^+$ which is also a symmetry of (1). Show $[F, H] = 0$ What are $[F, Q]$ and $[F, Q^\dagger]$?

(e) Note that the operators $Q, Q^\dagger$ aren't Hermitian but we can define $Q_1 = \frac{1}{2}(Q+Q^\dagger)$ and $Q_2 = \frac{1}{2\mathbf{i}}(Q - Q^\dagger)$. What algebra do they satisfy?

Now what does supersymmetry do? Consider an eigenstate $|\Psi\rangle$ with energy E.

(f) Compute $\langle\Psi|\hat{H}|\Psi\rangle$. What does this tell us about the ground state of the system?

(g) Now also suppose that $W(x) \to \infty$ as $x \to \pm\infty$. Using the above constraint on the ground state construct the wavefunction $\Psi_0(x) = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix}$ in terms of $W(x)$.

Hint: It may be helpful to write:

$$Q = \begin{pmatrix} 0 & \hat{P}_x - \mathbf{i}W'(x) \\ 0 & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ \hat{P}_x + \mathbf{i}W'(x) & 0 \end{pmatrix}$$

3. **Fermions and Bosons**

You may already know that SUSY mixes bosons and fermions. How does that appear in this model? Let's think about the spin Hilbert space $\mathcal{H}_2 = \text{span}\{|0\rangle, |1\rangle\}$

I encourage you to think about these vectors as being labeled by occupation number: $|0\rangle$ has no fermion (it is bosonic) and $|1\rangle$ has a single fermion.[1]

The fermionic creation and annihilation operators are then simply: $\hat{\psi}^\dagger \equiv \boldsymbol{\sigma}^- \quad \hat{\psi} \equiv \boldsymbol{\sigma}^+$

All states in the Hilbert space can be written as: $|\Psi\rangle = f_0(x)|0\rangle + f_1(x)\hat{\psi}^\dagger|0\rangle$

(a) Convince yourself the above is true and that $F = \hat{\psi}^\dagger\hat{\psi}$ is a number operator

Now let's show something non-trivial. I claim that all the excited ($E \neq 0$) states are two fold degenerate into bosonic ($F = 0$) and fermionic ($F = 1$) pairs.

Let's do this explicitly:

[1] I'm being slick with notation as $|0\rangle, |1\rangle$ are the computer science way of denoting the eigenvectors of σ^z for a spin system.

(b) Define $|b\rangle$ to be a state of the form $\Psi(x) = \begin{pmatrix} \Psi_+(x) \\ 0 \end{pmatrix}$ and $|f'\rangle \equiv Q^\dagger |b\rangle$. Show that $|f'\rangle$ is fermionic and degenerate with $|b\rangle$

(c) Show also that the properly normalized states are $|f\rangle = \frac{1}{\sqrt{2E}}|f'\rangle$

4. **A Certain Magical Index**

This model gives another interesting example of an *index*

It's going to be useful to define the operator $(-1)^F$, the fermionic *parity*.

(a) Prove that $[(-1)^F, H] = 0 = \{(-1)^F, Q\}$

And finally the object known as the *Witten index*

$$\text{Tr}\left[(-1)^F e^{-\beta H}\right] \tag{2}$$

The object above is interesting as it only depends on the space of groundstates which is independent of β and is invariant[2] under deforming $W(x) \to \lambda W(x)$

(b) Show that (2) is equal to the number of bosonic groundstates minus the number of fermionic groundstates.

So for non-zero Witten index there must be some zero modes annihilated by the supercharges. This implies SUSY is not *spontaneously broken*.

5. **The Harmonic Oscillator Redux**

Consider $W(x) = \frac{\omega}{2}x^2$ in the above problem.

(a) Write the Hamiltonian. What are $V(x)$ and $B(x)$?

(b) What's the groundstate wavefunction? How does it depend on sign$[\omega]$? What's the spectrum of the Hamiltonian?

(c) Calculate the Witten index (2) as well as the partition function $Z(\beta) \equiv \text{Tr}\ e^{-\beta \hat{H}}$

There's a lot more to this story, including some very beautiful mathematics, but we'll have to pause here without the technology of the path integral.

[2] In this sense it is topological.