University of California at San Diego – Department of Physics – TA: Shauna Kravec

# Quantum Mechanics C (Physics 212A) Fall 2015
# Worksheet 4

## Announcements

- The 212A web site is:

  **http://physics.ucsd.edu/∼mcgreevy/f15/** .

  Please check it regularly! It contains relevant course information!

## Problems

1. **Talking 'bout My Generation** Let's think about translations.
   (a) Define the operator $T(x)$ by the following:
   $$T(x)|y\rangle = |y+x\rangle \tag{1}$$
   Show that $T(x)$ is unitary by considering the action of $T^{-1}(x) = T(-x)$
   (b) Given the above we can express 1 by $T(x) = e^{-iKx}$ where $K$ is hermitian.
   Let $|k\rangle$ be an eigenbasis of $K$; $K|k\rangle = k|k\rangle$. What is the action of $T(x)|k\rangle$?
   Denote $\langle y|k\rangle = \phi_k(y)$. Note that by unitarity:
   $$\langle y|T(x)|k\rangle = \langle y|T^\dagger(-x)|k\rangle \tag{2}$$
   What does 2 imply about $\phi_k(y)$?
   (c) Recall plane waves have momentum related to their de Broglie wavelength: $p = \hbar k$
   Rewrite $T(x)$ with the implied expression for the operator $P$. This is why we say momentum *generates* translations!
   We can use a clever trick to get $P$ on its own: $\frac{\partial}{\partial x}T(x) = -\frac{i}{\hbar}P\ T(x)$
   Write an expression for $\langle x'|P|x\rangle$ Hint: Derivative of a delta function
   Recall that by some integration by parts:
   $$\int \mathrm{d}x\ \delta'(x-y)f(x) = 0 - \int \mathrm{d}x\ \delta(x-y)f'(x) = -f'(y) \tag{3}$$
   For $f$ which vanishes at $\pm\infty$
   Use (3) to derive the familiar expression for $P$ by considering:
   $$P\psi(x) = \langle x|P|\psi\rangle \tag{4}$$

2. **Building Bloch's Theorem**

Consider a 1D Hamiltonian with a periodic potential $V(x) = V(x + na)$ for $n \in \mathbb{Z}$ and $a$ the lattice spacing.

(a) Define the operator $T^n$ by $T^n|x\rangle = |x + na\rangle$. Show this is a symmetry.

(b) Assuming $H$ has no shared degeneracy with $T$, show that any eigenfunctions of this system can be chosen to obey

$$\psi_k(x - a) = e^{-\mathbf{i}ka}\psi_k(x) \tag{5}$$

Recall that $T|k\rangle = e^{-\mathbf{i}ka}|k\rangle$ and $\langle x|k\rangle \equiv \psi_k(x)$.

(c) Infer from (5) that one can then write $\psi_k(x) = e^{\mathbf{i}kx}u_k(x)$ where $u_k(x) = u_k(x + a)$

Note that $k$ is different from our usual momentum. It's a *crystal momentum*!

(d) Show explicitly that for $P = -\mathbf{i}\partial_x$ that $P\psi_k(x) \neq k\psi_k(x)$

(e) Show that $\frac{-\pi}{a} \leq k \leq \frac{\pi}{a}$. What is $k + \frac{2\pi}{a}$?

3. **A Theorem of Kramer**

Most symmetries are unitary. Some are *anti*-unitary. Time reversal is one of the latter. Denote this operator with $\mathcal{T}$.

Something one might expect classically is that $\mathcal{T}x\mathcal{T}^{-1} = x$ but $\mathcal{T}p\mathcal{T}^{-1} = -p$. It makes things run backwards.

A similar story is true for angular and spin momentum. $\mathcal{T}S\mathcal{T}^{-1} = -S$

(a) Consider the action of $\mathcal{T}$ on a spin-$\frac{1}{2}$: $\mathcal{T}|0\rangle$ where $S_z|0\rangle = \frac{1}{2}|0\rangle$.
Show that $\mathcal{T} = -\mathbf{i}YK$ is a suitable representation for $\mathcal{T}$ where $K$ implements complex conjugation [1] and $Y$ is the Pauli matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.
What is $\mathcal{T}^2$ in this case?

(b) Consider a system whose Hamiltonian $H$ is time reversal symmetric. Show that if $|\psi\rangle$ is an eigenstate then $\mathcal{T}|\psi\rangle$ is as well.
Does this change the energy of the state?

(c) Imagine this is a spin-$\frac{1}{2}$ system such that $|\psi\rangle$ is an eigenstate of $S_z$ as well.
How are $|\psi\rangle$ and $\mathcal{T}|\psi\rangle$ related? Can they be the same?

[1] This is what makes it anti-unitary.