University of California at San Diego – Department of Physics – TA: Shauna Kravec

Quantum Mechanics C (Physics 212A) Fall 2015 Worksheet 3

Announcements

- The 212A web site is:

  http://physics.ucsd.edu/∼mcgreevy/f15/ .

  Please check it regularly! It contains relevant course information!

Problems

1. **Purity**

   Define again the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{\mathbf{i}\phi}|1\rangle)$ as well as $\rho_\beta = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

   (a) Write the density matrix ρ_ψ associated with $|\psi\rangle$

   (b) Show that for both the states $\langle Z\rangle = 0$

   (c) Define the *purity* of a state as Tr $[\rho^2]$. Prove that this equal to 1 if ρ is pure. Compute it for both ρ_ψ and ρ_β.

   (d) Compute $\langle X\rangle$ with the above density matrices.

2. **With All Your ρ's Combined**

   Quantum states are said to form a convex set with extremal points corresponding to pure states. What does that mean?

   (a) Prove that if ρ_1 and ρ_2 are density matrices then $\sigma \equiv q\rho_1 + (1-q)\rho_2$ is as well where q is a probability.

   This implies that for every pair of points in the set the straight line that connects them is also contained in the object. For example a disk is a convex set but an annulus is not.

   (b) A point is said to be extremal if it can't be written as a (non-trivial) linear combination of other states.

   Prove the claim that pure states are extremal in this set.

   (c) Define $S(\rho) = -\text{Tr}\,(\rho\log\rho)$ to be the von Neumann entropy. I'd like to prove something about $S(\sigma)$. Assume that ρ_1 and ρ_2 have mutually orthogonal support.[1]

   Show that in this case $S(\sigma) = qS(\rho_1) + (1-q)S(\rho_2) - (q\log q + (1-q)\log(1-q))$

[1] Eigenvectors of ρ_1 with non-zero eigenvalue are all orthogonal to the similarly defined eigenvectors of ρ_2

3. **C-NOT Evil**

Consider the following operator

$$U_{CNOT} \equiv |0_A\rangle\langle 0_A| \otimes \mathbb{1}_B + |1_A\rangle\langle 1_A| \otimes \sigma_B^x \tag{1}$$

Notice that (1) *does not* factorize as $U = U_A \otimes U_B$. Therefore we might be able to create entanglement with it.

(a) Write a matrix representation of (1) in the computational basis

(b) What are the eigenvectors of (1)? Are any entangled?

(c) Construct an input state $|\text{In}\rangle$ which has no entanglement but where $|\text{Out}\rangle \equiv U_{CNOT}|\text{In}\rangle$ is *maximally* entangled.