University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics (Physics 212A) Fall 2015
Assignment 10

Due 12:30pm Wednesday, December 9, 2015

December 2 is the final lecture. This problem set may be handed in at my office (MH5222) or Shauna's (MH4206). Just put it under the door if no one is there.

1. **Bases for rotation generators.**

   Find the transformation which relates the $(\check{x}, \check{y}, \check{z})$ basis where the rotation generators are $-\mathbf{i}\hbar\vec{\mathcal{J}}$ with

   $$\mathcal{J}_1 = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ -1 & & 0 \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}.$$

   to the $|j, m\rangle$ basis for spin $j = 1$. Hint: diagonalize $\mathbf{J}_3$.

2. **Uncertainty and angular momentum.** [from Commins]

   (a) As usual, a simultaneous eigenstate of $\mathbf{J}^2$ and $\mathbf{J}_3$ is denoted $|j, m\rangle$. Show that the expectation values for $\mathbf{J}_1$ and $\mathbf{J}_2$ are zero in such a state.

   (b) Show that if any operator commutes with two components of an angular momentum operator then it also commutes with the third component.

   (c) Show that
   $$(\Delta J_x)^2 + (\Delta J_y)^2 \geq \hbar |\langle \mathbf{J}_z \rangle|. \tag{1}$$

   (d) Earlier, we showed that
   $$\Delta J_x \Delta J_y \geq \frac{\hbar}{2} |\langle \mathbf{J}_z \rangle|. \tag{2}$$
   Show that for a state $|j, m\rangle$ the inequalities (1) and (2) are both saturated if and only if $m = -j$ or $m = j$.

3. **Relation between angular momentum and harmonic oscillator algebras.**

   You'll have noticed some similarities between our analysis of the SHO and the angular momentum algebra. Here is a precise connection between them. Consider two independent SHOs, with destruction operators $\mathbf{a}_r, r = 1, 2$, so that

   $$[\mathbf{a}_r, \mathbf{a}_s] = [\mathbf{a}_r^\dagger, \mathbf{a}_s^\dagger] = 0, \quad [\mathbf{a}_r, \mathbf{a}_s^\dagger] = \delta_{rs}.$$

Let

$$\mathbf{S} \equiv \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{a}_2^\dagger \mathbf{a}_2\right) \quad \mathbf{J}_1 \equiv \frac{1}{2}\left(\mathbf{a}_2^\dagger \mathbf{a}_1 + \mathbf{a}_1^\dagger \mathbf{a}_2\right)$$
$$\mathbf{J}_2 \equiv \frac{\mathbf{i}}{2}\left(\mathbf{a}_2^\dagger \mathbf{a}_1 - \mathbf{a}_1^\dagger \mathbf{a}_2\right) \quad \mathbf{J}_3 \equiv \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$$

(a) Show that the the $\mathbf{J}_i$ satisfy the $\mathfrak{su}(2)$ algebra (set $\hbar = 1$), and that $\sum_i \mathbf{J}_i^2 = \mathbf{S}(\mathbf{S}+1)$. Conclude that $[\mathbf{S}, \vec{\mathbf{J}}] = 0$.

(b) Show that

$$|j, m\rangle = \left(\mathbf{a}_1^\dagger\right)^{j+m} \left(\mathbf{a}_2^\dagger\right)^{j-m} |0\rangle \frac{1}{\sqrt{(j+m)!(j-m)!}}$$

form the (normalized, orthogonal) basis for the standard representation of $\{\mathbf{J}^2, \mathbf{J}_3\}$ discussed in lecture. Here $|0\rangle$ is the SHO groundstate $\mathbf{a}_1|0\rangle = 0 = \mathbf{a}_2|0\rangle$.

(c) (Simple but useful, I think.) Draw a picture of these states: label the axes by the eigenvalues of the two number operators. Circle the states with the same j.

(d) [optional] Now regard the two SHOs as describing the coordinates of one particle in a two-dimensional rotation-invariant potential, so $\mathbf{a}_r \sim \mathbf{Q}_r + \mathbf{i}\mathbf{P}_r$. What transformations do these operators generate?

4. **Addition of angular momentum example.**

What is $\underline{1} \otimes \underline{1}$? Find the matrix of Clebsch-Gordan coefficients by the method described in lecture.

5. **Spherical harmonics and rotation matrices.**

Consider a particle free to move on the unit sphere (imagine a particle in $d = 3$ with a central potential $V(r)$ with a very deep minimum at $r = 1$). A basis is labelled by polar coordinates $|\theta, \varphi\rangle \equiv |\check{n}\rangle$ where $\check{n}$ is a unit vector. A resolution of the identity in this (position) basis is $\mathbb{1} = \int d\Omega \, |\theta, \varphi\rangle\langle\theta, \varphi|$ with $\int d\Omega ... \equiv \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta$.

(a) We can make the state $|\check{n}\rangle$ by starting with $|\check{z}\rangle$ and acting with an appropriate rotation:

$$|\check{n}\rangle = R_{\check{n}}|\check{z}\rangle.$$

Show that there is an ambiguity in choosing $R_{\check{n}}$ since $R_{\check{n}}$ and $R_{\check{n}}R(\check{z}, \gamma)$ will produce the same state.

(b) Another basis for this Hilbert space is the one $|\ell, m\rangle$ which diagonalizes $\mathbf{L}^2$ and $\mathbf{L}_z$; the identity is $\mathbb{1} = \sum_{\ell \in \mathbb{Z}_{\geq 0}, m=-\ell,..\ell} |\ell, m\rangle\langle\ell, m|$. The position basis components of $|\ell, m\rangle$ are the spherical harmonics:

$$\langle\theta, \varphi|\ell, m\rangle = Y_{\ell,m}(\theta, \varphi).$$

Starting with this expression, show that

$$Y_{\ell,m}(\theta,\varphi) = \sum_{m'} \langle \check{z}|\ell, m'\rangle \left(\mathcal{D}^{(\ell)}_{mm'}(R_{\check{n}})\right)^\star$$

where $\mathcal{D}^{(\ell)}_{mm'}(R_{\check{n}}) \equiv \langle \ell, m|R_{\check{n}}|\ell, m'\rangle$.

(c) Show that the freedom to redefine $R_{\check{n}}$ in part 5a implies that

$$\langle \check{z}|\ell, m\rangle = 0 \text{ unless } m = 0.$$

(d) Conclude that

$$Y_{\ell,m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}} e^{\mathbf{i}m\varphi} d^{(\ell)}_{m,0}(\theta)$$

where d was defined in lecture.