University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics (Physics 212A) Fall 2015
Assignment 9

Due 12:30pm Wednesday, December 2, 2015

1. **Projective representations** [optional]

Consider a group G, with elements $g_1, g_2, g_1 g_2$ et cetera. (For simplicity suppose G has a finite number of elements.) Suppose G has a projective representation on a Hilbert space $\mathcal{H}$ by unitaries $\mathbf{U}(g)$:

$$\mathbf{U}(g_1)\mathbf{U}(g_2) = \omega(g_1, g_2)\mathbf{U}(g_1 g_2) \ .$$

We will ask two questions:

(a) What constraints does associativity

$$\left(\mathbf{U}(g_1)\mathbf{U}(g_2)\right)\mathbf{U}(g_3) = \mathbf{U}(g_1)\left(\mathbf{U}(g_2)\mathbf{U}(g_3)\right)$$

impose on the phases $\omega(g_1, g_2)$?

(b) When can we get rid of the phases $\omega(g_1, g_2)$ by redefining the phases of $\mathbf{U}(g)$:

$$\mathbf{U}(g) \to u(g)\mathbf{U}(g) \quad ?$$

More specifically, identify the resulting equivalence relation among the phases $u(g_1, g_2)$.

Cultural remark: The space of solutions of the condition 1a modulo the equivalence relation 1b defines the Borel cohomology group $H^2(\mathsf{G}, \mathsf{U}(1))$, which classifies projective representations of G.

(c) Now we'll study an example. Take $\mathsf{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$ with elements $1\!\!1, g_x, g_z, g_x g_z$ and multiplication rules: $g_x^2 = 1\!\!1, g_z^2 = 1, g_x g_z = g_z g_x$. Show that on a two-dimensional Hilbert space, the Pauli operators

$$\mathbf{U}(g_x) = \boldsymbol{\sigma}^x, \quad \mathbf{U}(g_z) = \boldsymbol{\sigma}^z$$

furnish a projective representation of G. Find $\mathbf{U}(g_x g_z)$ and $\mathbf{U}(g_z g_x)$. Notice that although g_x and g_z commute, these two objects are not equal.