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Quantum Mechanics (Physics 212A) Fall 2015
Assignment 1 – Solutions

Due 12:30pm Wednesday, October 7, 2015

1. **Cauchy-Schwarz inequality.** Prove the Cauchy-Schwarz inequality

$$\langle v|w\rangle^2 \leq \| \, |v\rangle \, \|^2 \| \, |w\rangle \, \|^2$$

under the assumptions we made in lecture about our inner product $\langle v|w\rangle$. Hint: apply the positivity condition to the vectors obtained by applying the Gram-Schmidt procedure to the set $\{|v\rangle, |w\rangle\}$.

2. **Triangle inequality.** [optional] Prove the triangle inequality, $\| \, v + w \, \| \leq \| \, v \, \| + \| \, w \, \|$. When is it saturated? Hint: Use the Cauchy-Schwarz inequality and the fact that $\mathrm{Re}\langle v|w\rangle \leq |\langle v|w\rangle|$.

3. Consider a (linear) operator $\mathbf{A} : \mathcal{H} \to \mathcal{H}$ acting on a Hilbert space $\mathcal{H}$. Consider the three statements (a) $\mathbf{A}$ is Hermitian, (b) $\mathbf{A}$ is unitary, (c) $\mathbf{A}^2 = \mathbb{1}$. Show that any two of these statements imply the third, *i.e.* $a + b \implies c, a + c \implies b, c + b \implies a$.

$$a + b \implies c : \quad \mathbb{1} = \mathbf{A}^\dagger \mathbf{A} = \mathbf{A}\mathbf{A} = \mathbf{A}^2.$$

$$a + c \implies b : \quad \mathbb{1} = \mathbf{A}^2 = \mathbf{A}\mathbf{A} = \mathbf{A}^\dagger \mathbf{A}.$$

$$c + b \implies a : \quad \mathbf{A}^\dagger \mathbf{A} = \mathbb{1} \implies \mathbf{A} = \mathbb{1}\mathbf{A} = \left(\mathbf{A}^\dagger \mathbf{A}\right) \mathbf{A} = \mathbf{A}^\dagger \mathbf{A}^2 = \mathbf{A}^\dagger.$$

4. For any linear operators $\mathbf{A}$ and $\mathbf{B}$ which may be composed, show that $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$.

This problem is redundant with problem 6 below (why didn't anyone say anything?), but just for kicks here's a different argument: From the definition of adjoint, if $\mathbf{A}|u\rangle = |v\rangle$, then $\langle u|\mathbf{A}^\dagger = \langle v|$. So: $\mathbf{BA}|u\rangle = \mathbf{B}|v\rangle$, and so $\langle v|\mathbf{B}^\dagger = \langle u|\mathbf{A}^\dagger \mathbf{B}^\dagger$. But this is true for all $|u\rangle$, and we conclude $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$.

5. (a) Show that the commutator $[\mathbf{A}, \mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}$ of two hermitian operators is antihermitian: $([\mathbf{A}, \mathbf{B}])^\dagger = -[\mathbf{A}, \mathbf{B}]$.

   (b) Show that the eigenvalues of an antihermitian operator are pure imaginary.

   Zero is also possible.

6. Show that for any two linear maps which may be composed, $\mathbf{T} : V_1 \to V_2, \mathbf{S} : V_2 \to V_3$,

$$(\mathbf{ST})^\dagger = \mathbf{T}^\dagger \mathbf{S}^\dagger.$$

Let $\{|i\rangle\}, \{|n\rangle\}, \{|a\rangle\}$ denote bases for $V_{1,2,3}$ respectively. Then

$$\begin{aligned} \langle i| (\mathbf{ST})^\dagger |a\rangle &= (\langle a|\mathbf{ST}|i\rangle)^\star \\ &= \sum_n (\langle a|\mathbf{S}|n\rangle\langle n|\mathbf{T}|i\rangle)^\star \\ &= \sum_n \langle n|\mathbf{S}^\dagger|a\rangle\langle i|\mathbf{T}^\dagger|n\rangle \\ &= \sum_n \langle i|\mathbf{T}^\dagger|n\rangle\langle n|\mathbf{S}^\dagger|a\rangle \\ &= \langle i|\mathbf{T}^\dagger\mathbf{S}^\dagger|a\rangle. \end{aligned} \tag{1}$$

7. For $\mathbf{A}$ and $\mathbf{B}$ hermitian operators, show that $\mathbf{AB}$ is hermitian if and only if $\mathbf{A}$ and $\mathbf{B}$ commute.

From either 6 or 4, we know that in general,

$$(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$$

but for hermitian operators, the RHS is $\mathbf{BA}$ which is equal to the LHS only when $0 = \mathbf{AB} - \mathbf{BA} = [\mathbf{A}, \mathbf{B}]$.

8. Consider three normal operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$ satisfying

$$[\mathbf{A}, \mathbf{B}] = 0, [\mathbf{A}, \mathbf{C}] = 0 \quad \text{but} \quad [\mathbf{B}, \mathbf{C}] \neq 0.$$

(a) Show that there must be a degeneracy in the spectrum of $\mathbf{A}$.

(b) The nature of the degeneracy depends on the form of $[\mathbf{B}, \mathbf{C}]$. Suppose that $[\mathbf{B}, \mathbf{C}]$ is a nonzero c-number. Show that under this assumption the degeneracy of each eigenvalue of $\mathbf{A}$ cannot be finite.

Let V_a be the eigenspace of $\mathbf{A}$ with some eigenvalue a. On V_a,

$$\text{tr}_{V_a}[\mathbf{B}, \mathbf{C}] = \text{tr}_{V_a} q \mathbb{1} = q\dim(V_a)$$

On the other hand,

$$\text{tr}_{V_a}[\mathbf{B}, \mathbf{C}] = \text{tr}_{V_a}\mathbf{BC} - \text{tr}_{V_a}\mathbf{CB} = 0$$

by cyclicity of the trace.

9. **Supersymmetry algebra.** Consider an operator $\mathbf{Q}$ satisfying $\mathbf{Q}^2 = 0$.

   Let $\mathbf{A} \equiv \mathbf{Q}\mathbf{Q}^\dagger + \mathbf{Q}^\dagger\mathbf{Q}$.

   Show that any nonzero eigenvalue of $\mathbf{A}$ is degenerate (that is, there is more than one eigenvector with the same eigenvalue), but zero eigenvalues need not be.

   Hint: consider the action of $\left(\mathbf{Q} + \mathbf{Q}^\dagger\right)$ on an eigenstate of $\mathbf{A}$.

   A good example to keep in mind in doing this problem is a two state system with

$$\mathbf{Q} \equiv \boldsymbol{\sigma}^+ \equiv \boldsymbol{\sigma}^x + \mathbf{i}\boldsymbol{\sigma}^y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \mathbb{1}.$$

   First notice that $\mathbf{A} = \left(\mathbf{Q} + \mathbf{Q}^\dagger\right)^2$. Second notice that

$$\mathbf{A}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\dagger\mathbf{Q} = \mathbf{Q}\mathbf{A}$$

   i.e. $[\mathbf{A}, \mathbf{Q}] = 0$, $\mathbf{A}$ and $\mathbf{Q}$ commute. Similarly, $\mathbf{A}$ and $\mathbf{Q}^\dagger$ commute. $\mathbf{Q}$ and $\mathbf{Q}^\dagger$ are not hermitian – in fact, since they square to zero, all their eigenvalues must be zero (act with $\mathbf{Q}$ on the eigenvalue equation for $\mathbf{Q}$). But $\mathbf{Q} + \mathbf{Q}^\dagger$ and $\mathbf{i}\left(\mathbf{Q} - \mathbf{Q}^\dagger\right)$ are hermitian and simultaneously diagonalizable with $\mathbf{A}$. Now suppose we have an $\mathbf{A}$ eigenstate:

$$\mathbf{A}|\alpha\rangle = \alpha|\alpha\rangle.$$

   Consider the state $\mathbf{Q}|\alpha\rangle$. Since $[\mathbf{A}, \mathbf{Q}] = 0$, it is also an $\mathbf{A}$ eigenstate with eigenvalue α. Is it proportional to $|\alpha\rangle$, *i.e.* the same state? (If not, we have shown a degeracy.) If it were,

$$\mathbf{Q}|\alpha\rangle = q|\alpha\rangle$$

   then it would be an eigenvector of $\mathbf{Q}$ with eigenvalue q. But we know that any eigenvalue of $\mathbf{Q}$ is zero. So the only case where there is not a degeneracy is when

$$\mathbf{Q}|\alpha\rangle = 0\ .$$

10. **Normal matrices.**

    An operator (or matrix) $\hat{A}$ is *normal* if it satisfies the condition $[\hat{A}, \hat{A}^\dagger] = 0$.

    (a) Show that real symmetric, hermitian, real orthogonal and unitary operators are normal.

    (b) Show that any operator can be written as $\hat{A} = \hat{H} + \mathbf{i}\hat{G}$ where $\hat{H}, \hat{G}$ are Hermitian. [Hint: consider the combinations $\hat{A} + \hat{A}^\dagger, \hat{A} - \hat{A}^\dagger$.] Show that $\hat{A}$ is normal if and only if $[\hat{H}, \hat{G}] = 0$.

(c) Show that a normal operator $\hat{A}$ admits a spectral representation

$$\hat{A} = \sum_{i=1}^{N} \lambda_i \hat{P}_i$$

for a set of projectors $\hat{P}_i$, and complex numbers λ_i.

11. **The space of linear operators on $\mathcal{H}$ is also a Hilbert space.**

(a) Show that with the inner product

$$(S, T) \equiv \mathrm{tr}(S^\dagger T)$$

(the *Hilbert-Schmidt inner product*) the space of linear operators on a Hilbert space $\mathcal{H}$ is itself a Hilbert space.

(b) What is its dimension? (Hint: make an orthonormal basis for it using an ON basis for $\mathcal{H}$.)

Let $\mathcal{H} = \mathrm{span}\{|j\rangle, j = 1..N\}$, and assume this basis is ON so that $\mathbb{1} = \sum_j |j\rangle\langle j|$. Then any linear operator on $\mathcal{H}$ can be written

$$\mathbf{A} = \mathbb{1}\mathbf{A}\mathbb{1} = \sum_{j,k=1}^{N} |j\rangle\langle j|\mathbf{A}|k\rangle\langle k| = \sum_{jk} A_{jk}|j\rangle\langle k|.$$

This says that the operators

$$\mathbf{E}(jk) \equiv |j\rangle\langle k|$$

span the vector space of operators on $\mathcal{H}$. What are their inner products?

$$(\mathbf{E}(jk), \mathbf{E}(lm)) = \mathrm{tr}\mathbf{E}(jk)^\dagger \mathbf{E}(lm) = \sum_i \langle i| \left(|k\rangle\langle j|\right) \left(|l\rangle\langle m|\right) |i\rangle = \delta_{km}\delta_{lj}.$$

In particular, the norm is

$$\| \mathbf{E}(jk) \|^2 = (\mathbf{E}(jk), \mathbf{E}(jk)) = 1.$$

They are orthonormal. The dimension of the space of operators is therefore $(\dim \mathcal{H})^2$.

(c) Show that the Pauli matrices and the identity (appropriately normalized) are orthonormal with respect to the Hilbert-Schmidt inner product. Do they provide a basis for the space of operators acting on a two-state system?