University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 9 – Solutions

Posted November 22, 2013 **Due Monday, December 9, 2013**

This will be the last problem set.

1. **Energy loss to gravitational waves.**

Calculate the rate at which a binary star system loses energy to gravitational radiation. Assume two stars of equal mass M with an instantaneous separation $2r$.

The part of this I didn't do in lecture is computing the quadrupole moment of the binary star and plugging it into the $\partial_t^3 Q^2$ formula for the radiated power that we found. We put the origin of coordinates at the center of rotation, which occurs in the xy plane, so the mass density is

$$T^{00} = \rho = M\left(\delta^{(3)}(\vec{x} - \vec{x}_1(t)) + \delta^{(3)}(\vec{x} - \vec{x}_2(t))\right)$$

where the stars are at

$$\vec{x}_1(t) = (R\cos\omega t, R\sin\omega t, 0), \quad \text{and} \quad \vec{x}_2(t) = \left(R\cos\left(\omega t + \pi\right), R\sin\left(\omega t + \pi\right), 0\right)$$

The traceful quadrupole moment is

$$q_{ij} = 3\int \mathrm{d}^3x\ T^{00} x^i x^j\ .$$

Any entry with a 3 is zero because of the $\delta(z)$s. A nonzero one is:

$$q_{11} = 3\int \mathrm{d}^3x M x^2\left(\delta() + \delta()\right) = 3MR^2\cos^2\omega t \cdot 2.$$

Altogther

$$q_{ij} = 6MR^2\begin{pmatrix}\cos^2\omega t & \cos\omega t\sin\omega t & 0\\ \cos\omega t\sin\omega t & \sin^2\omega t & 0\\ 0 & 0 & 0\end{pmatrix}_{ij};$$

The trace is $q = \sum_i q_{ii} = 6MR^2$. The traceless one is

$$Q_{ij} \equiv q_{ij} - \frac{1}{3}\delta_{ij}q = MR^2\begin{pmatrix}3\cos^2\omega t - 1 & 3\sin 2\omega t & 0\\ 3\sin 2\omega t & 3\sin^2\omega t - 1 & 0\\ 0 & 0 & -1\end{pmatrix}_{ij}.$$

$$\dot{Q}_{ij} = 6\omega M R^2 \begin{pmatrix} -\sin 2\omega t & \cos 2\omega t & 0 \\ \cos\omega t & \sin 2\omega t & 0 \\ 0 & 0 & -1 \end{pmatrix}_{ij}$$

$$\partial_t^3 Q_{ij} = 24\omega^3 M R^2 \begin{pmatrix} +\sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & -1 \end{pmatrix}_{ij}$$

Power radiated at time t through a shell of radius $r \gg R$ surrounding the binary system is

$$P(t) = \frac{G_N}{45} \sum_{ij=1}^{3} \partial_t^3 Q_{ij}^2|_{t'=t-r} = \frac{G_N}{45}\left(\partial_t^3 Q_{11}^2 + 2\partial_t^3 Q_{12}^2 + \partial_t^3 Q_{22}^2\right)$$

$$= \frac{G_N}{45}\left(24 M R^2 \omega^3\right)^2 \left(2\sin^2 2\omega t' + 2\cos^2 2\omega t'\right) = \frac{4\cdot 2\cdot 12\cdot 12}{3\cdot 3\cdot 5} M^2 R^4 \omega^6 G_N$$

which using $\omega^6 = \frac{G_N^3 M^3}{4^3 R^9}$ is

$$P = \frac{2}{5}\frac{M^5 G_N^4}{R^5} .$$

The energy of the binary system is

$$E = 2\left(\frac{1}{2} M\omega^2 R^2 + \frac{G_N M^2}{R}\right)|_{\omega=\sqrt{\frac{G_N M}{4R^3}}} = -\frac{G_N M^2}{4R} = -M^{5/3} G_N^{2/3} 4^{-2/3} \omega^{2/3}$$

– smaller radius means faster orbiting. Assuming that M doesn't change with time, the radiated power is

$$P = -\frac{\mathrm{d}E}{\mathrm{d}t} = +\frac{2}{5}\frac{M^5 G_N^4}{R^5}$$

Using the expression for $E(\omega)$ above and the resulting

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = +\omega^{-1/3}\frac{\mathrm{d}\omega}{\mathrm{d}t}\frac{M^{5/3} G^{2/3}}{4^{2/3}}\frac{2}{3}$$

for the rate of change of the frequency I find

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{6}{5}\frac{M^{7/2} G_N^{7/2}}{R^{11/2}}$$

– notice that the units work out: $2 = [\dot{\omega}] = \frac{7}{2} - 2\frac{7}{2} + \frac{11}{2}\checkmark$.

2. **Coordinate horizons.** [from Brandenberger]

   In this problem we will come to terms with the notion of an event horizon using only special relativity.

   Consider a train moving with constant acceleration a in Minkowski spacetime.

   [Hint 1: a covariant definition of constant acceleration is

   $$a_0^2 = a^\mu a_\mu, \quad a^\mu \equiv \partial_\tau^2 x^\mu \tag{1}$$

   is the proper acceleration and a_0 is a constant.

   Hint 2: show that the solution of (1) is a hyperbola in the xt plane.]

(a) Using only special relativity, determine the frequency with which light emitted on the last car of the train is received by a detector in the front car.

Let the position of the front of the train be $x(t)$. (A more annoying definition of constant acceleration is: relative to some fixed inertial frame $S(0)$, if you instantaneously transform to a comoving (this means $u_t(0) = 0$) frame $S(t)$ then $\dot{u}_t(0) = a$, a constant.) Then $x^\mu = (t, x)^\mu$ has

$$-\mathrm{d}\tau^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 \implies -\partial_t \tau^2 = -1 + \Bigg(\underbrace{\partial_t x}_{\equiv u} \Bigg)^2 \implies \partial_\tau t = \frac{1}{\sqrt{1-u^2}} \equiv \gamma \ .$$

(Notice that the proper velocity $V^\mu = \partial_\tau x^\mu$ satisfies

$$V^\mu V_\mu = -1 \implies 0 = \partial_\tau \left(V^\mu V_\mu \right) = 2 V^\mu a_\mu \ . \tag{2}$$

This shows that the two definitions of constant acceleration agree: in the instantaneous rest from of the particle, the proper acceleration is $(0, a_0)^\mu$.) In terms of $u(t)$, we have $V^\mu = \gamma(1, u)^\mu$ and therefore

$$a^\mu = \gamma^3 \partial_\tau u (u, 1)^\mu,$$

which happily agrees with (2). So the constant-acceleration condition (1) is

$$a_0^2 = a^\mu a_\mu = \gamma^4 (\partial_\tau u)^2 \qquad \implies \pm a_0 = \gamma^2 \partial_\tau u = \gamma^3 \partial_t u$$

$$\implies \frac{\mathrm{d}u}{(1-u^2)^{3/2}} = a_0 \mathrm{d}t$$

(I take the positive root WLOG since we can just replace $x \to -x$) which we can integrate to get

$$\frac{u}{\sqrt{1-u^2}} = a_0 t \ .$$

(I will drop the subscript on a_0 from now on.) To find $x(t)$ we solve for u

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{at}{\sqrt{1+a^2t^2}}$$

and integrate again:

$$x(t) - x(0) = \frac{\sqrt{1+a^2t^2}}{a} - \frac{1}{a}.$$

By integrating instead wrt the proper time we can arrive more elegantly at the hyperbola:

$$\gamma^2 \partial_\tau u = a_0 \implies a_0 \mathrm{d}\tau = \frac{\mathrm{d}u}{1-u^2} = \mathrm{d} \log \sqrt{\frac{1+u}{1-u}}$$

$$\frac{1+u}{1-u} = e^{2a\tau} \implies \boxed{u = \tanh a\tau, \quad \gamma = \cosh a\tau}.$$

So:

$$\partial_\tau t = \gamma = \cosh a\tau \implies t(\tau) = \frac{1}{a}\sinh a\tau.$$

$$\tanh a\tau = u = \partial_t x = \partial_t \tau \partial_\tau x = \gamma^{-1}\partial_\tau x = \frac{1}{\cosh a\tau}\partial_\tau x$$

$$\implies \partial_\tau x = \sinh a\tau \implies x(\tau) - x(0) = \frac{1}{a}\cosh a\tau$$

So indeed

$$(x - x(0))^2 - t^2 = \frac{1}{a^2}\left(\cosh^2 a\tau - \sinh^2 a\tau\right) = \frac{1}{a^2}$$

describes a hyperbola.

Let the length of the train in its rest frame be L, and let the frequency of the light emitted by the back of the train in the instantaneous rest frame be ν_0. Then the time t_0 when light emitted by the back of the train (at $t = 0$) reaches the front satisfies

$$t_0 - L = x(t_0) = \frac{1}{a}\left(\sqrt{1 + a^2 t_0^2} - 1\right)$$

which says

$$t_0 = \frac{L}{2}\frac{2 - La}{1 - La}.$$

The (relativistic) doppler shift is

$$\frac{\omega}{\omega_0} = \sqrt{\frac{1 - u(t_0)}{1 + u(t_0)}}$$

(where ω_0 is the frequency at the source and ω is the received frequency, and you agree with the sign because if $u > 0$ we should have a redshift, $\omega < \omega_0$). The velocity of the front of the train at time t_0 is

$$u(t_0) = \frac{a t_0}{\sqrt{1 + a^2 t_0^2}}\ .$$

(Notice that it is not true that $u(t_0) \overset{?}{=} a t_0$ because of velocity addition in SR.) So we have

$$\omega = \omega_0 \sqrt{\frac{1 - u(t_0)}{1 + u(t_0)}} \overset{\substack{\text{computer algebra}\\ \text{is helpful}}}{=} \omega_0(1 - aL).$$

Since $aL > 0$ (the receiver is moving away from the source), this is a redshift, as long as $aL < 1$. Clearly something bad is happening if $a > \frac{1}{L}$. We can understand what the something is next.

[Actually we could have avoided some pain above by never introducing u and γ. The constant-acceleration condition is

$$a_0^2 = \ddot{t}^2 - \ddot{x}^2$$

(where dot is ∂_τ), and the definition of proper time gives

$$1 = \dot{t}^2 - \dot{x}^2 \ .$$

The latter is clearly true if

$$\dot{t} = \cosh\eta, \dot{x} = \sinh\eta,$$

in which case the former says $\dot{\eta}^2 = a_0^2$, or $\eta = \pm a_0\tau$.]

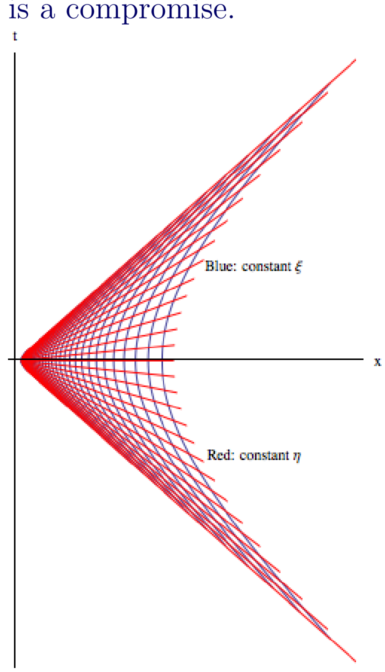
(b) Introduce new coordinates ξ and η in which each point on the train is at constant position ξ and for which η is (proportional to) the proper time of the train. Show that the coordinate transformation is

$$t = a^{-1}e^{a\xi}\sinh(a\eta), \quad x = a^{-1}e^{a\xi}\cosh(a\eta)$$

Above we found that the trajectory of a point on the train satisfies

$$t(\tau) = \frac{1}{a}\sinh a\tau, \quad x(\tau) = \frac{1}{a}\cosh a\tau \quad .$$

We can use the proper time $\tau \propto \eta$ as one of the coordinates on our spacetime. For the other coordinate we could simply choose the acceleration a; alternatively, we could use the initial position in x as the other coordinate. The coordinate ξ is a compromise.



Notice that they only cover part of the Minkowski spacetime.

(c) What is the metric in these new coordinates? Where have you seen this before?
Rindler space, problem 5 of problem set 3. The y and z directions are irrelevant and we ignore them.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = e^{2a\xi}\left(-\mathrm{d}\xi^2 + \mathrm{d}\eta^2\right).$$

(d) Use the Einstein equivalence principle to answer the question in part (a).

The EEP says the train should experience a gravitational field, a uniform one. So the frequency is shifted according to[1]

$$\frac{\Delta E}{E} = \frac{\hbar \Delta\omega}{\hbar\omega} = \frac{\Delta\phi}{mc^2} = \frac{maL}{mc^2} = aL$$

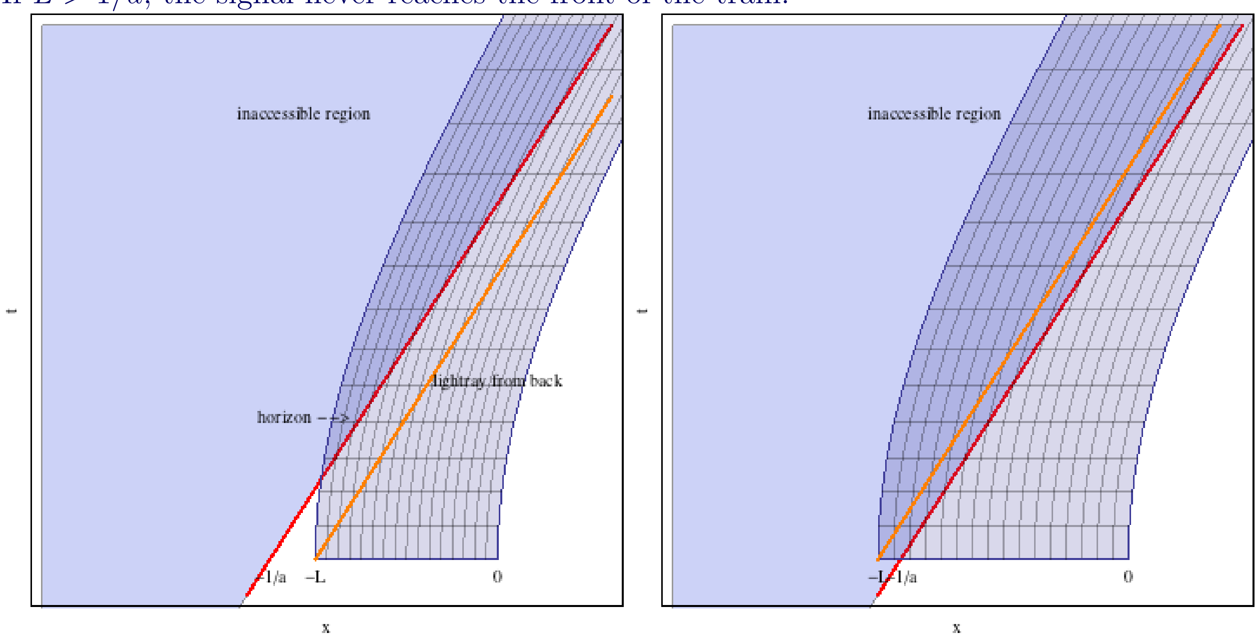
where L is the length of the train. So

$$\Delta\omega = \omega_0 - \omega = (aL)\omega, \quad \omega = (1 - aL)\omega_0.$$

– a redshift, which becomes an infinite redshift if $aL = 1$. What happens there can be understood from the subsequent bit.

(e) Discuss the causal structure of Minkowski spacetime from the point of view of an observer on the train. Specifically, can the observer see all of Minkowski spacetime?

If $L > 1/a$, the signal never reaches the front of the train!



Left: $L < 1/a$. Right: $L > 1/a$. The red line is the Rindler horizon of the front of the train; the blue shaded region is the region behind the horizon, which is

[1] More generally, for source and receiver at fixed (or slowly varying compared to ω) coordinate x_E and X_R respectively, in a static spacetime,

$$\frac{\omega_E}{\omega_R} = \sqrt{\frac{g_{tt}(x_R)}{g_{tt}(x_E)}}.$$

(Why? This is because the coordinate time between crests of the wave is $\mathrm{d}t = \frac{2\pi}{\omega}$ for everyone, but the observers' proper time intervals differ:

$$\mathrm{d}\tau_E^2 = g_{tt}(x_E)\mathrm{d}t^2, \quad \mathrm{d}\tau_R^2 = g_{tt}(x_R)\mathrm{d}t^2,$$

which we can use to relate the time between crests experienced by the each. This explained in Zee on page 303.)

not in the causal past of the front of the train and therefore cannot influence it. The orange line is the lightray sent from the back of the train at $t = 0$. The gray shaded region is the worldsheet of the train.

Notice the following strange thing: the position of the front of the train $x_f(\tau)$ and that of the back of the train $x_b(\tau)$ *both* satisfy $\gamma^2 \partial_\tau \partial_t x = a_0$, and therefore only differ by a constant shift, which must be L, since that what it is when the train starts at rest: $x_f(\tau) = x_b(\tau) + L$. But this means that the length of the train *in the lab frame* is always L! This contradicts a naive expectation based on length contraction. The (surprising to me) resolution is that the train is actually *longer* in the instantaneous rest from of one point on the train (which is *not* the instantaneous rest from of other points on the train). The process of acceleration (imagine, following Daniel Walsh, attaching little rockets to each point on the train) stretches the train out.

I learned from Mike Gartner that a sharp version of this puzzle is discussed in: Bell J S 1993 "How to teach special relativity" in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press) pp 6780. There one is asked to consider a delicate string stretched between two rockets (the front and back of the train), and the question is whether the string breaks. The answer is 'yes', from every point of view.

3. **Sanity check.**

The motion of the Earth about the Sun can be described as the motion of a test particle in the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2G_N M}{r}}\mathrm{d}r^2 + r^2 \mathrm{d}s^2_{S^2}$$

with M the mass of the Sun. Verify that the Newtonian limit of this problem gives the well-known $1/r^2$ force between the Earth and Sun. Calculate to first order in the small quantities the correction terms to the Newtonian force due to GR.

I should have asked a harder problem here. Using time-translation invariance and rotation invariance (conservation of energy and angular momentum) we showed that the geodesic equation implies that

$$\frac{p^2}{2m} + V = E$$

with

$$V(r) = -\frac{\kappa G_N M}{r} + \frac{1}{2}\frac{L^2}{r^2} - \frac{G_N M L^2}{r^3}.$$

and $E = \frac{1}{2}\epsilon^2, \epsilon = g_{tt}\partial_\tau t$. In the NR limit, the special relativistic effects go away. The difference from the Newtonian problem is the $-1/r^3$ term in the potential, which produces an additional short-ranged attractive $-\frac{L^2}{r^4}$ force.

4. **Falling into a black hole**

A massive particle initially at rest at radius $r > 2GM$ starts falling radially into the Schwarzchild metric. Compute the proper times it takes for the particle to reach the Schwarzchild radius at $r = 2GM$ and the singularity at $r = 0$. Compute also the Schwarzschild coordinate time t to reach the horizon.

[Don't forget to take advantage of conservation laws.]

Imprecisely-formulated bonus problem: Suppose the test particle has some finite extent, ϵ. Compute and describe the tidal forces on the particle as a function of its radial position r.

By an SO(3) rotation we can set $\theta = \pi/2$ and it will stay there. By time translation, pick $s = t = 0$ at the initial position $r = R$. Conserved quantities are

$$L = g_{\mu\nu} \left(\partial_\varphi\right)^\mu \dot{x}^\nu = r^2 \sin^2\theta \partial_\varphi = r^2 \partial_\tau \varphi = 0$$

(from the initial data, and it's conserved so it stays zero) and

$$\epsilon = -g_{\mu\nu} \left(\partial_t\right)^\mu \dot{x}^\nu = (1 - r_0/r)\partial_\tau t$$

and

$$-1 = \kappa = g_{\mu\nu}\partial_\tau x^\mu \partial_\tau x^\nu = -(1 - r_0/r)\left(\partial_\tau t\right)^2 + (1 - r_0/r)^{-1}\partial_\tau r^2.$$

First determine the value of ϵ from the initial data ($\partial_\tau r = 0$ at $r = R$):

$$-1 = -(1 - r_0/r)\left(\partial_\tau t\right)^2|_0 \implies \partial_\tau t|_0 = +(1 - r_0/R)^{-1/2}$$

$$\implies \epsilon = \epsilon|_0 = (1 - r_0/R)\partial_\tau t|_0 = (1 - r_0/R)^{1/2}$$

$$\implies -1 = -(1 - r_0/r)\left(\partial_\tau t\right)^2 + (1 - r_0/r)^{-1}\left(\partial_\tau r\right)^2 = (1 - r_0/r)^{-1}\left(-\epsilon^2 + \left(\partial_\tau r\right)^2\right).$$

$$\implies \left(\partial_\tau r\right)^2 = \epsilon^2 - 1 + \frac{r_0}{r} = -\frac{r_0}{R} + \frac{r_0}{r}.$$

Notice that you can't set $\partial_\tau t \stackrel{?}{=} 1$ even when the particle starts at rest; this ratio is measuring the gravitational redshift, which causes the watch carried by a particle held at position r to tick more slowly than a clock at $r = \infty$.

$$\implies \partial_\tau r = -\sqrt{\frac{r_0}{r} - \frac{r_0}{R}}$$

where we take the negative root since the particle is falling in; notice that the argument of the sqrt is positive since $r(\tau > 0) < R$.

$$\mathrm{d}\tau = -\frac{\mathrm{d}r}{\sqrt{r_0}}\sqrt{\frac{rR}{R - r}} = -\mathrm{d}r\sqrt{\frac{R}{r_0}}\frac{r}{\sqrt{Rr - r^2}}$$

The proper time to reach a point $r_\star$ is

$$\tau(r_\star) = -\sqrt{\frac{R}{r_0}}\int_R^{r_\star} \mathrm{d}r \frac{r}{\sqrt{Rr - r^2}}\ .$$

This integral can be done by trig substitution: $r - R/2 \equiv \frac{R}{2}\sin\theta$

$$\sqrt{\frac{r_0}{R}}\tau(r_\star) = \sqrt{Rr_\star - r_\star^2} + \frac{R}{2}\left(\frac{\pi}{2} - \arcsin\left(\frac{2r_\star}{r} - 1\right)\right).$$

In particular, for $r_\star = 0$, we find the proper time to reach the singularity is

$$\tau(0) = \sqrt{\frac{R}{r_0}}\left(0 + \frac{R}{2}\left(\frac{\pi}{2} - \underbrace{\arcsin(-1)}_{=-\pi/2}\right)\right) = \sqrt{\frac{r_0}{R}}\frac{\pi R}{2}$$

and the time to reach the horizon is

$$\tau(r_0) = \sqrt{R(R-r_0)} + \left(\frac{R}{2}\right)^{3/2}\frac{2}{r_0}\left(\frac{\pi}{2} - \arcsin\left(\frac{2r_0}{R} - 1\right)\right)$$

Notice that $R > r_0$ keeps the first term real and the argument of the arcsin safely between -1 and 0. The Schwarzschild coordinate time to get to the horizon can be obtained from

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\mathrm{d}\tau}{\mathrm{d}r}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(-\sqrt{\frac{r_0}{r} - \frac{r_0}{R}}\right)^{-1}\left(\frac{1-r_0/R}{1-r_0/r}\right) = -\frac{r_0^{1/2}}{R^{3/2}}(R-r_0)\frac{\sqrt{Rr-r^2}}{r-r_0}$$

where the red factor is the only important one because it produces a logarithmic diverence in

$$\Delta t = -\int_R^r \frac{\mathrm{d}r}{r-r_0}\cdot(\text{factors regular at } r=r_0) \simeq +\log\frac{r-r_0}{r-R}|_{r=r_0} r_0\left(\frac{R-r_0}{R}\right)^{3/2}.$$

The following problems are optional.

5. **Killing vector fields form a closed algebra.**

Show that, if ξ and η are Killing vector fields, then so is their Lie bracket $[\xi, \eta]$.

The key point here is that the definition of ξ is a Kvf in some spacetime with metric g is

$$\mathcal{L}_\xi g = 0$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. But the Lie derivative wrt the commutator is

$$\mathcal{L}_{[\xi,\eta]}g = L_\xi L_\eta g - L_\eta L_\xi g$$

which certainly vanishes if both $L_\xi g$ and L_η do.

6. **Exact gravitational wave solutions.**

   Show that with the ansatz

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + \sum_{i=1,2} \mathrm{d}y^i \mathrm{d}y^i + F(u, y^i)\mathrm{d}u^2 \tag{3}$$

   the vacuum Einstein equations are solved if

$$R_{\mu\nu} = 0 \quad \Leftrightarrow \quad \sum_i \frac{\partial^2}{\partial y^i \partial y^i} F(u, y^1, y^2) = 0. \tag{4}$$

   Further, show that (4) is solved by

$$F(u, y^i) = \sum_{i,j=1,2} h_{ij}(u) y^i y^j \quad \text{if} \quad \sum_i h_{ii}(u) = 0 \ .$$

7. **Boundary of anti-de Sitter spacetime.**

   Recall the metric of anti-de Sitter space (AdS) in Poincaré coordinates.

$$\mathrm{d}s^2_{AdS} = L^2 \frac{\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}{z^2} \tag{5}$$

   Here the coordinate z is positive. Show that a lightray sent from a point with $z > 0$ with $\dot{z} < 0$ reaches $z = 0$ in a finite coordinate time δt.

   [In this sense, $z = 0$ is a (timelike) *boundary* of the AdS spacetime, and we must specify boundary conditions there for the time evolution to be well-defined.]

8. **Surface gravity and Hawking temperature.**

   Compute the periodicity of y for which this metric is regular at $z = z_m$ (*i.e.* has no conical deficit):

$$ds^2 = \Omega(z)\left(f(z)dy^2 + \frac{dz^2}{f(z)} + ds^2_{\text{other}}\right)$$

   where $f(z)$ is a function with a first-order zero at $z = z_m$
   (*i.e.* $f(z_m) = 0$ and $\partial_z f(z_m) \neq 0$)
   and $\Omega(z)$ is regular and non-vanishing at $z = z_m$.

   If we think of y as imaginary time, this periodicity determines the temperature of the black hole, since equilibrium at finite temperature means periodic Euclidean time.

   a) Specialize your answer to the case of the euclidean Schwarzchild black hole in flat space, for which

$$f(z) = 1 - \frac{2GM}{z}, \ \Omega = 1, \ ds^2_{\text{other}} = z^2 d\vec{x}^2.$$

   b) Specialize your answer to the case of the euclidean AdS black hole (with planar horizon), for which

$$f(z) \equiv 1 - \frac{z^4}{z_m^4}, \ \Omega = \frac{1}{z^2}, \ ds^2_{\text{other}} = d\vec{x}^2.$$

c) Show that you get the same answer by computing the 'surface gravity' κ of the horizon (the locus $z = z_m$), which can be defined by

$$\kappa^2 \equiv \frac{1}{2} \nabla_a \xi^b \nabla_c \xi^d g_{bd} g^{ac}|_{z=z_m}$$

where ξ^a is the tangent vector to the shrinking circle, $\xi = \partial_y$.

9. Derive the energy-momentum conservation equations for an ideal gas in an FRW universe. Give a physical interpretation.

10. **Extremal Reissner-Nordstrom black hole.**

a) Consider Einstein-Maxwell theory in four dimensions, with action

$$S_{EM} = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} \left(\mathcal{R} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

Show that the Einstein equation $0 = \frac{\delta S_{EM}}{\delta g^{\mu\nu}}$ implies that

$$\mathcal{R}_{\mu\nu} = a G_N \left(2 F_{\mu .} F_{\nu}^{.} - \frac{1}{2} g_{\mu\nu} F^2 \right)$$

for some constant a.

b) Consider the ansatz

$$ds^2 = H^{-2}(\rho) \left(-dt^2\right) + H^2(\rho) \left(d\rho^2 + \rho^2 d\Omega_2^2\right),$$

$$F = b dt \wedge d \left(H(\rho)^{-1}\right)$$

where b is some constant. Show that the Einstein equation $0 = \frac{\delta S_{EM}}{\delta g^{\mu\nu}}$ and Maxwell's equation $0 = \frac{\delta S_{EM}}{\delta A_\mu}$ are solved by the ansatz if H is a harmonic function on the $\mathbb{R}^3$ whose metric is

$$\gamma_{ab} dx^a dx^b := d\rho^2 + \rho^2 d\Omega_2^2.$$

Recall that H is harmonic iff $0 = \Box H = \frac{1}{\sqrt{\gamma}} \partial_a (\sqrt{\gamma} \gamma^{ab} \partial_b H)$.

c) Find the form of the harmonic function which gives a spherically symmetric solution; fix the two integration constants by demanding that i) the spacetime is asymptotically flat and ii) the black hole has charge Q, meaning $\int_{S^2 \text{ at fixed } \rho} \star F = Q$.

d) Take the near-horizon limit. Show that the geometry is $AdS_2 \times S^2$. Determine the relationship between the size of the throat and the charge of the hole.

e) If you're feeling brave, add some magnetic charge to the black hole. You will need to change the form of the gauge field to

$$F = b dt \wedge dH(\rho) + G(\rho) \Omega_2$$

where Ω_2 is the area 2-form on the sphere, and G is some function.