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General Relativity (225A) Fall 2013
Assignment 8 – Solutions

Posted November 13, 2013 **Due Monday, December 2, 2013**

In the first two problems here we will think about solutions of Einstein's equations with nonzero cosmological constant, positive and negative.

1. **AdS as a solution of Einstein's equations.**

   Consider the anti-de Sitter (AdS) metric (in so-called Poincaré coordinates)

$$ds^2_{AdS} = L^2 \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} \tag{1}$$

   where $\eta_{\mu\nu}$ is the Minkowski metric in *e.g.* $D = 2+1$ dimensions. Show that this metric satisfies Einstein's equations with negative cosmological constant Λ. More specifically:

   (a) Consider the action

$$S[g] = \int d^4x \sqrt{g} \left(R - 2\Lambda\right)$$

   with Λ a (negative) constant[1]. What are the equations of motion for the metric?

$$0 = \frac{\delta S}{\delta g_{\mu\nu}(x)} = -G_{\mu\nu} - \Lambda g_{\mu\nu} \ .$$

   We can interpret the second term here as the 'matter' stress tensor $T_{\mu\nu} = -\Lambda g_{\mu\nu}$. Notice that I have chosen units where $16\pi G_N = 1$.

   (b) Compute the Einstein tensor for the metric (1).

   The nonzero Christoffel symbols are

$$\Gamma^z_{ii} = \frac{1}{z} = -\Gamma^z_{tt} = -\Gamma^z_{zz} \quad (i = x, y, \text{ no sum}).$$

   The nonzero components of the Riemann tensor (in these coordinates) are all $\pm\frac{1}{z^2}$. The nonzero parts of the Ricci tensor are

$$R_{\mu\mu} = (-1)^\mu \frac{3}{z^2} \quad (\text{no sum}) \ .$$

[1] A way to remember the correct sign (in red) here: one way to get Λ is to put a scalar field at the minimum of its potential every where in spacetime: $\Lambda = V(\phi_0)$, but $V(\phi)$ appears with a minus sign in the action.

The Ricci scalar is $-\frac{12}{L^2}$. Therefore we see that AdS space is a so-called *Einstein space*, for which the Einstein tensor is proportional to the metric itself:

$$G_{\alpha\beta} = a g_{\alpha\beta}$$

with $a = \frac{3}{L^2}$ for $3+1$-dimensional AdS.

(c) Find the relation between Λ and L for which Einstein's equation is solved.

The equation of motion says

$$G_{\alpha\beta} = \Lambda g_{\alpha\beta}$$

so we see that we must set

$$\Lambda = -a = -\frac{3}{L^2}$$

for AdS_4.

(d) Optional bonus problem: determine the relation between L and Λ for AdS_D for arbitrary D.

To find the pattern for general D it is best to use tetrad methods. Then you will find that the spin connection coefficients are ${\omega_{\hat{z}}}^{\hat{i}} = \frac{1}{z}\mathrm{d}x^i\eta^{ii}$ (no sum on i) and ${\Omega_{\hat{z}}}^{\hat{i}} = \frac{1}{z^2}\mathrm{d}x^i \wedge \mathrm{d}z$ so the nonzero components of ${R_{\mu\nu\rho}}^{\sigma} = e^b_\rho e^\sigma_a {\Omega_{\mu\nu b}}^a$ are

$${R_{\mu\nu\mu}}^{\nu} = \eta_{\mu\mu}\eta_{\nu\nu}\frac{1}{z^2}, \mu \neq \nu$$

(notice that the indices in this expression run over t, x^i and z). Then we get all the numerical factors from the traces:

$$R_{\mu\nu} = {R_{\mu\rho\nu}}^{\rho} = (D-1)\frac{1}{z^2}\eta_{\mu\nu}$$

and

$$R = g^{\mu\nu}\eta_{\mu\nu} = \frac{z^2}{L^2}\eta^{\mu\nu}\frac{D-1}{z^2}\eta_{\mu\nu} = \frac{D(D-1)}{L^2}.$$

Finally, the Einstein tensor is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{z^2}\eta_{\mu\nu}\left((D-1) - \frac{1}{2}D(D-1)\right) = \frac{1}{z^2}\eta_{\mu\nu}\left((D-1) - \frac{1}{2}D(D-1)\right)$$

$$= -\frac{1}{z^2}\eta_{\mu\nu}\frac{(D-2)(D-1)}{2}$$

so the general relation between the cosmological constant Λ and the so-called AdS radius L is

$$-\frac{(D-2)(D-1)}{2} = L^2\Lambda.$$

(e) Show that the AdS metric can also be written as

$$\mathrm{d}s^2_{\mathrm{AdS}} = \mathrm{d}y^2 + e^{-\frac{2y}{L}}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \qquad (2)$$

by the change of variables $z = Le^{\frac{y}{L}}$. (This form will be useful for comparison with the next problem.)

Plug in z and $\mathrm{d}z = \frac{z}{L}\mathrm{d}y$.

2. **de Sitter space.**

Consider the ansatz

$$\mathrm{d}s^2_{\mathrm{FRW}} = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}x^i \mathrm{d}x^j \delta_{ij}$$

where $a(t)$ is a function we will determine.

Remarks: (1) this is the form of the FRW metric from a previous problem set, and (2) this metric is quite similar to the expression for the AdS metric in (2); specifically, they are related by the replacement $t \to \mathbf{i}y, y \to -\mathbf{i}t$ which exchanges a spacelike coordinate for a timelike coordinate (a 'Wick rotation'), if we set set $a = e^{\frac{y}{L}}$.

(a) Compute the Einstein tensor for this metric.

The nonzero components are

$$G_{tt} = 3\left(\frac{\dot{a}}{a}\right)^2, \quad G_{ii} = -\dot{a}^2 - 2a\ddot{a} \ .$$

(b) Consider the Einstein's equations with the stress tensor resulting from a *positive* cosmological constant, and find the resulting differential equation for the 'scale factor' $a(t)$.

$$G_{\mu\nu} = 16\pi G_N \Lambda g_{\mu\nu}$$

gives

$$tt: \qquad 3\left(\frac{\dot{a}}{a}\right)^2 = 16\pi G_N \Lambda$$

and

$$ii: \qquad -\dot{a}^2 - 2a\ddot{a} = 16\pi G_N \Lambda a^2 \quad .$$

(c) Solve this equation to find the form of the FRW metric which results when the stress tensor is dominated by a positive cosmological constant, as it is presently in our universe. (Note that if we chose some other matter on the RHS we would find a different behavior of $a(t)$.)

[Hint 1: consider the tt component of the Einstein equation first.

Hint 2: The second remark above is a good hint about the form of the solution.]

The tt equation is solved by

$$\frac{\dot{a}}{a} = \pm H_0$$

with $H_0 \equiv 16\pi G_N \Lambda/3$. The general solution of this (linear) equation is $a(t) = A_+ e^{H_0 t} + A_- e^{-H_0 t}$. However, the ii equation is not linear; it is solved by *either* an expanding solution $a(t) = a(0)e^{H_0 t}$ or a crunching solution $a(t) = a(0)e^{-H_0 t}$. (The two are related by replacing $t \to -t$, which preserves the form of the Einstein equations.) The expanding solution looks like:

$$\mathrm{d}s^2_{\mathrm{FRW}} = -\mathrm{d}t^2 + a_0 e^{-2H_0 t} \mathrm{d}x^i \mathrm{d}x^j \delta_{ij}$$

3. **Killing vector fields imply conservation laws.**

   (a) Show that Killing's equation
   $$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \tag{3}$$
   is equivalent to $\mathcal{L}_\xi g = 0$, where g is the metric tensor, and $\mathcal{L}$ is the Lie derivative[2]. If we take $v_1 = \partial_\mu, v_2 = \partial_\nu$ to be coordinate vector fields, we have $[k, v_1] = -k^\alpha_{,\mu}\partial_\alpha$ and the equation in the footnote says
   $$g_{\mu\nu,\alpha}\xi^\alpha = -\xi^\alpha_{,\mu} g_{\alpha\nu} + \xi^\alpha_{,\nu} g_{\alpha\mu}$$
   It's easy to see how to make this manifestly covariant by choosing Riemann normal coords at the point p, of interest where $g_{\mu\nu,\alpha} = 0$, so the equation becomes
   $$0 = \xi_{\nu,\mu} + \xi_{\mu,\nu}.$$
   But in RNC, we have $, =;$ at p, so the equation becomes
   $$0 = \xi_{\nu;\mu} + \xi_{\mu;\nu}.$$

   (b) If ξ is a Killing vector field (*i.e.* satisfies (3)) and and $T^{\mu\nu}$ is the (covariantly conserved) energy momentum tensor, show that $J^\mu \equiv T^{\mu\nu}\xi_\nu$ is a conserved current, $J^\mu_{;\mu} = 0$.

   (c) Given a time coordinate on your spacetime (and hence a notion of constant-time slices) use ξ to construct a quantity which is time-independent.
   $$Q(t) = \int_{\text{fixed t}} \mathrm{d}^3x \sqrt{g} J^t(x)$$
   This is independent of t by the argument in section 4 of the lecture notes, with appropriate factors of $\sqrt{g}$ inserted, as appropriate for a covariantly conserved current: $0 = \nabla_\mu J^\mu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}J^\mu\right)$.

4. **Conformal coupling of a scalar field.** (This problem is optional.)

   The stress tensor for a scalar field with action (in arbitrary number of dimensions n) is
   $$S_0[\phi, g] = -\frac{1}{2}\int \mathrm{d}^n x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \phi^2\right) = \int \mathrm{d}^4x\sqrt{g}\mathcal{L}$$
   is
   $$(T_0)_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S_0[\phi, g]}{\delta g^{\mu\nu}(x)} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\mathcal{L}g_{\mu\nu}.$$

[2] Reminder: The Lie derivative is a derivation, and therefore for any two vector fields, $v_{1,2}$,
$$\mathcal{L}_\xi\left(g(v_1, v_2)\right) = \left(\mathcal{L}_\xi g\right)(v_1, v_2) + g(\underbrace{\mathcal{L}_\xi v_1}_{=[\xi, v_1]}, v_2) + g(v_1, \mathcal{L}_\xi v_2).$$

(a) Compute the trace, $(T_0)^\mu_\mu$.

Recall that we associated vanishing trace of the stress tensor with scale invariance. Here we will show that it is possible to choose the constant γ in

$$S = S_0 + \gamma \int \mathrm{d}^n x \sqrt{g} R \phi^2$$

(where R is the Ricci scalar curvature) so that this action is scale invariant.[3]

(b) Show that for a certain value of γ the action S is invariant under the local scale transformation

$$\mathrm{d}s^2 \to \mathrm{d}\tilde{s}^2 = \Omega^2(x)\mathrm{d}s^2$$

(also called a *Weyl transformation* or *local conformal transformation* (since it preserves angles)) if we also make the replacement

$$\phi(x) \to \Omega^{\frac{2-n}{2}}(x)\phi(x) \ .$$

Find the right γ.

(c) Find the improved stress tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$$

and show that it is traceless.

[3]Hint: you may find it useful to use Appendix D of Wald. In particular, equation D.8 tells us the behavior of the Ricci tensor under a position-dependent rescaling of the metric. Specifically, if $\mathrm{d}\tilde{s}^2 = \Omega^2(x)\mathrm{d}s^2$ then

$$\begin{aligned} \tilde{R}_{\mu\nu} &= R_{\mu\nu} - (n-2)\nabla_\mu \nabla_\nu \log\Omega - g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \nabla_\sigma \log\Omega \\ &+ (n-2)\left(\nabla_\mu \log\Omega\right)\nabla_\nu \log\Omega - (n-2) g_{\mu\nu} g^{\rho\sigma}\left(\nabla_\rho \log\Omega\right)\nabla_\sigma \log\Omega. \end{aligned} \tag{4}$$