University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 7 – Solutions

Posted November 4, 2013 **Due Wednesday, November 20, 2013**

1. **Infinitesimal coordinate transformations.** Under a coordinate transformation, $x^\mu \to \tilde{x}^\mu(x)$, the metric tensor transforms as[1]

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{\partial}_\mu x^\rho \tilde{\partial}_\nu x^\sigma g_{\rho\sigma}(x) \ .$$

Show that for an infinitesimal transformation $\tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$, this takes the form

$$\delta g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -\left(\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu\right).$$

(This will be useful for the following problem.)

$$\tilde{x}^\mu = x^\mu + \epsilon^\mu(x), \quad \tilde{\partial}_\mu x^\rho = \delta^\rho_\mu - \tilde{\partial}_\mu \epsilon^\rho = \delta^\rho_\mu - \tilde{\partial}_\mu x^\alpha \partial_\alpha \epsilon^\rho = \delta^\rho_\mu - \partial_\mu \epsilon^\rho + \mathcal{O}(\epsilon^2)$$

In the following the $+\mathcal{O}(\epsilon^2)$ is understood, that is, we treat ϵ^2 as zero.

$$\begin{aligned}
\delta g_{\mu\nu} &\equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = \tilde{\partial}_\mu x^\rho|_{x-\epsilon} \tilde{\partial}_\nu x^\sigma|_{x-\epsilon} g_{\rho\sigma}(x-\epsilon) - g_{\mu\nu}(x) \\
&= \left(\delta^\rho_\mu - \partial_\mu \epsilon^\rho\right)\left(\delta^\sigma_\nu - \partial_\nu \epsilon^\sigma\right)\left(g_{\rho\sigma}(x) - \epsilon^\alpha \partial_\alpha g_{\rho\sigma}(x)\right) - g_{\mu\nu}(x) \\
&= -(\partial_\mu \epsilon^\rho) g_{\rho\nu} - (\partial_\nu \epsilon^\sigma) g_{\mu\sigma} - \epsilon^\alpha \partial_\alpha g_{\mu\nu} \\
&= -\left(\partial_\mu(\epsilon^\rho g_{\rho\nu}) - \epsilon^\rho(\partial_\mu g_{\rho\nu}) + \partial_\nu(\epsilon^\sigma g_{\mu\sigma}) - \epsilon^\sigma(\partial_\nu g_{\mu\sigma}) + \epsilon^\alpha \partial_\alpha g_{\mu\nu}\right) \\
&= -\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu - \epsilon_\alpha g^{\rho\alpha}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)\right) \\
&= -\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu - 2\epsilon_\alpha \Gamma^\alpha_{\mu\nu}\right) \\
&= -\left(\partial_\mu \epsilon_\nu - \epsilon_\alpha \Gamma^\alpha_{\mu\nu} + \partial_\nu \epsilon_\mu - \epsilon_\alpha \Gamma^\alpha_{\nu\mu}\right) \\
&= -\left(\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu\right).
\end{aligned} \tag{1}$$

2. **Conservation of the improved stress tensor.** Our improved stress tensor is defined as

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

where S is an action for a matter field (such as a scalar field ϕ or a Maxwell field A_μ, or a particle trajectory). We would like to show that $T^{\mu\nu}$ defined this way is covariantly conserved when evaluated on solutions of the equations of motion of the matter fields.

For simplicity consider a scalar field ϕ. Its EoM is $0 = \frac{\delta S}{\delta \phi}$.

[1] Why is this relation true? The invariant distance between two nearby points doesn't care about what coordinates you use:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = \tilde{g}_{\rho\sigma}(\tilde{x}) d\tilde{x}^\rho d\tilde{x}^\sigma.$$

(This equation is on page 70 of Zee's book, by the way.) Using the chain rule then gives exactly the relation above.

(a) Show that under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, the action changes as

$$\delta_\epsilon S = -\int \mathrm{d}^4x \left(\frac{1}{2}\sqrt{g} T^{\mu\nu} \left(\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \right) + \frac{\delta S}{\delta \phi} \epsilon^\mu \partial_\mu \phi \right).$$

Using the chain rule,

$$\delta_\epsilon S = \int \mathrm{d}^4x \left(\underbrace{\frac{\delta S}{\delta g_{\mu\nu}(x)}}_{=\frac{\sqrt{g}}{2}T^{\mu\nu}} \underbrace{\delta g_{\mu\nu}(x)}_{=-\nabla_\mu\epsilon_\nu - \nabla_\nu \epsilon_\mu} + \frac{\delta S}{\delta \phi(x)} \underbrace{\delta \phi(x)}_{=-\epsilon^\mu \partial_\mu \phi} \right)$$

Note the sign in the second term: a scalar field transforms as $\tilde{\phi}(\tilde{x}) = \phi(x)$, which means

$$\delta\phi(x) = \tilde{\phi}(x) - \phi(x) = \phi(x - \epsilon) - \phi(x) = -\epsilon^\mu \partial_\mu \phi(x).$$

(b) By using the invariance of the action under coordinate transformations, show that the equation of motion for ϕ implies

$$\nabla_\mu T^{\mu\nu} = 0.$$

Coordinate invariance of the action implies that the preceding expression is zero; the scalar EoM imply that $0 = \frac{\delta S}{\delta \phi(x)}$ on its own, so

$$0 = \delta_\epsilon S = -\int \mathrm{d}^4x \frac{1}{2}\sqrt{g} T^{\mu\nu} \left(\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \right) \overset{IBP}{=} + \int \mathrm{d}^4x \sqrt{g} \nabla_\mu T^{\mu\nu} \epsilon_\nu(x)$$

for all ϵ means that $0 = \nabla_\mu T^{\mu\nu}$.

3. **Not so much GR in $D = 1 + 1$.**

Show that in two spacetime dimensions, the left hand side of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

vanishes identically. This means that the metric does not have any dynamics in $D = 1 + 1$, and the Einstein equations impose the constraint $T_{\mu\nu} = 0$ on the matter fields.

Recall from the previous problem set that in two dimensions, the Riemann tensor is

$$R_{ijkl} = \frac{1}{2} R \left(g_{ik} g_{jl} - g_{il} g_{jk} \right)$$

which means the Ricci tensor is

$$R_{ik} = g^{jl} R_{ijkl} = \frac{1}{2} R \left(g_{ij} \underbrace{\delta^l_l}_{=n=2} - g_{ij} \right) = \frac{1}{2} R g_{ik}$$

and the Ricci scalar is

$$R = g^{ik}R_{ik} = \frac{1}{2}R2 = R$$

So the einstein tensor is

$$G_{ik} = R_{ik} - \frac{1}{2}Rg_{ik} = \frac{1}{2}Rg_{ik} - \frac{1}{2}Rg_{ik} = 0.$$