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General Relativity (225A) Fall 2013
Assignment 6 – Solutions

Posted November 4, 2013 **Due Wednesday, November 13, 2013**

Note that this homework has been shortened relative to the previous version; the last two problems are postponed until next week. I also added an optional problem.

1. **Consistency check.** (This problem is optional, since I added it late.)

   Before we discussed the complications of parallel transport and used this to make covariant derivatives, we managed to construct a covariant *divergence* by thinking about actions. Since it arose from the variation of a covariant action with respect to a scalar quantity, we found that for any vector field v^μ,

$$\nabla_\mu v^\mu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}v^\mu\right)$$

   was a scalar quantity. Show that this agrees with what you get by contracting the indices on the (metric-compatible) covariant derivative:

$$\nabla_\mu v^\mu = \delta^\mu_\nu \nabla_\mu v^\nu .$$

   This quantity is

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}v^\mu\right) = \partial_\mu v^\mu + (\frac{1}{2}\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g})v^\mu \ .$$

   Using

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{ab}\delta g_{ab} \implies \partial_\mu\sqrt{g} = \frac{1}{2}\sqrt{g}g^{ab}\delta g_{ab,\mu},$$

   this is

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}v^\mu\right) = \partial_\mu v^\mu + \frac{1}{2}g^{ab}g_{ab,\mu}v^\mu \ .$$

   On the other hand, the metric-compatible covariant derivative gives

$$\nabla_\mu v^\mu = \delta^\mu_\nu\left(\partial_\mu v^\nu + \Gamma^\nu_{\mu\rho}v^\rho\right) = \partial_\mu v^\mu + \Gamma^\mu_{\mu\rho}v^\rho$$

   and

$$\Gamma^\nu_{\mu\rho} = \frac{1}{2}g^{\nu\sigma}\left(g_{\sigma\rho,\mu} + g_{\mu\sigma,\rho} - g_{\mu\rho,\sigma}\right)$$

   so

$$\Gamma^\mu_{\mu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\sigma\rho,\mu} + g_{\mu\sigma,\rho} - g_{\mu\rho,\sigma}\right) = +\frac{1}{2}g^{\mu\sigma}g_{\mu\sigma,\rho} \ .$$

   So these are indeed the same thing.

2. **Gauss-Bonnet theorem in action.** Consider the round 2-sphere of radius r_0, whose metric is:

$$\mathrm{d}s^2 = r_0^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right).$$

   (a) Compute all the non-vanishing components of the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$, where the indices μ, ν, ρ, σ run over θ, φ.

   The nonzero Christoffel symbols are only

$$\Gamma^\theta_{\varphi\varphi} = -\cos\theta\sin\theta, \quad \Gamma^\varphi_{\varphi\theta} = \cot\theta = \Gamma^\varphi_{\theta\varphi}.$$

   The nonzero Riemann tensor components are

$$R_{\varphi\theta\varphi}{}^\theta = \sin^2\theta = -R_{\varphi\varphi\theta}{}^\theta, \quad R_{\theta\theta\varphi}{}^\varphi = -1 = -R_{\theta\varphi\theta}{}^\varphi.$$

   (b) Show that the surface integral of the scalar curvature R

$$\int_{S^2} \sqrt{g}\mathrm{d}\varphi\mathrm{d}\theta R$$

   over the whole 2-sphere is independent of r_0. Obtain the numerical value of this integral.

   The Ricci scalar has nonzero components:

$$R_{\theta\theta} = 1, \quad R_{\varphi\varphi} = -\sin^2\theta.$$

   The scalar curvature is $R = \frac{2}{r_0^2}$, and is *not* independent of the radius. The measure is $\sqrt{g} = \sqrt{r_0^2 \cdot r_0^2\sin^2\theta} = r_0^2|\sin\theta|$. The integral is

$$\int_{S^2} \sqrt{g}\mathrm{d}\varphi\mathrm{d}\theta R = \int_0^{2\pi}\mathrm{d}\varphi\int_0^\pi \sin\theta\mathrm{d}\theta r_0^2\frac{2}{r_0^2} = 2\pi\underbrace{\int_0^\pi \mathrm{d}\cos\theta}_{=2}\cdot 2 = 4\pi\cdot 2\,.$$

   The Gauss-Bonnet theorem guarantees that this quantity is $\int_{\Sigma_g} R = 4\pi(2-2g)$ for *any* metric on a closed (no boundaries) surface Σ_g of genus g (that is, a surface with g-handles). [1]

3. **The Riemann tensor in $d = 2$ dimensions.** [Wald chapter 3 problem 3b, 4a.]

   (a) (This part is optional.) In d dimensions, a 4-index tensor has d^4 components; using the symmetries of the Riemann tensor, show that it has only $d^2(d^2-1)/12$ independent components.

   The relevant symmetries are $R_{cdab} = R_{abcd} = -R_{bacd} = -R_{abdc}$ and $R_{[abc]d} = 0$. The first equation says we can think of R as a symmetric matrix in the bi-indices ab

[1] Note that the version of this theorem that you'll find on Wikipedia involves an *extrinsic* curvature, which requires that you embed your manifold in flat space; that different equation has a different factor of two.

and cd. A general such object will have $d^2(d^2+1)/2$ components. But the second relation says that each of these bi-indices only runs over the $X = d(d-1)/2$ values where ab satisfy $a < b$. So before imposing the last condition we have $X(X+1)/2$ components. How many independent conditions is $R_{[abc]d} = 0$ then? It only further constrains the tensor if all the indices are different, which only happens in $d \geq 4$. In that case it is $\binom{d}{4} = \frac{d(d-1)(d-2)(d-3)}{4!}$ conditions, giving

$$\begin{aligned} & X(X+1)/2|_{X=d(d-1)/2} - \frac{d(d-1)(d-2)(d-3)}{4!} \\ = & \frac{d(d-1)}{24}\left(3(d^2-d+2)-(d-2)(d-3)\right) \\ = & \frac{d(d-1)}{24} 2d(d+1) = \frac{d^2(d^2-1)}{12}. \end{aligned} \tag{1}$$

(b) Show that in two dimensions, the Riemann tensor takes the form

$$R_{abcd} = R g_{a[c}g_{d]b} \equiv \frac{1}{2}R\left(g_{ac}g_{db} - g_{ad}g_{cb}\right).$$

One way to do this is to use the previous part of the problem to show that $g_{a[c}g_{d]b}$ spans the vector space of tensors having the symmetries of the Riemann tensor.

First, let's check that the stated form has the right symmetry properties:

$$\boxed{R_{abcd} = -R_{bacd}:} \qquad g_{b[c}g_{d]a} = \frac{1}{2}\left(g_{bc}g_{da} - g_{bd}g_{ca}\right) = -g_{a[c}g_{d]b} \quad \checkmark$$

$$\boxed{R_{abcd} = -R_{abdc}:} \qquad g_{a[d}g_{c]b} = -\frac{1}{2}g_{a[c}g_{d]b} \quad \checkmark$$

$$\boxed{R^d_{[abc]} = 0:} \qquad \begin{pmatrix} (g_{ac}g_{db} - g_{ad}g_{cb}) & +(g_{ba}g_{dc} - g_{bd}g_{ac}) & +(g_{cb}g_{da} - g_{cd}g_{ba}) \\ -(g_{bc}g_{da} - g_{bd}g_{ca}) & -(g_{ca}g_{db} - g_{cd}g_{ab}) & -(g_{ab}g_{dc} - g_{ad}g_{bc}) \end{pmatrix} = 0 \quad \checkmark$$

(like-colored terms eat each other). To see that in 2d R_{abcd} has only one independent term, notice that we can determine all the nonzero terms from *e.g.* R_{0101}:

$$0 = R_{11cd} = R_{00cd} = R_{ab11} = R_{cd00}$$

and

$$R_{0101} = -R_{1001} = +R_{1010} = -R_{0110}.$$

This is to say that $d^2(d^2-1)/12|_{d=2} = \frac{4\cdot 3}{12} = 1$. It remains to check the normalization. The Ricci tensor and scalar are

$$R_{ac} = R_{abcd}g^{bd}. \qquad R = R_{ac}g^{ac} = R_{abcd}g^{bd}g^{ac}$$

$$R \stackrel{?}{=} \frac{1}{2}R\left(d^2-d\right) = \frac{1}{2}R\left(2^2-2\right) = \frac{2}{2}R = R\ .\checkmark$$

(c) Verify that the general expression for curvature in two dimensions is consistent with the result of the previous problem.

$$R_{\theta\varphi\theta\varphi} = +r_0^2 \sin^2\theta \stackrel{?}{=} \frac{1}{2} R \left(g_{\theta\theta} g_{\varphi\varphi} - \underbrace{g_{\theta\varphi} g_{\theta\varphi}}_{=0} \right) = \frac{1}{2} \cdot \frac{2}{r_0^2} \cdot r_0^2 \cdot r_0^2 \sin^2\theta = r_0^2 \sin^2\theta \quad \checkmark$$

and the other components agree by the symmetries of R_{abcd}.

4. **Geodesics in FRW.**

Consider a particle in an FRW (Friedmann-Robertson-Walker) spacetime:

$$\mathrm{d}s^2_{FRW} = -\mathrm{d}t^2 + a^2(t)\mathrm{d}s_3^2 \tag{2}$$

where $\mathrm{d}s_3^2 = \mathrm{d}x^i \mathrm{d}x^i$ is Euclidean 3-space, and $a(t)$ is some given function of t. Show that energy of the particle (the momentum conjugate to t) is not conserved along the particle trajectory. Find three quantities which are conserved.

The action for a particle propagating in this spacetime is

$$S[x] = \int \mathrm{d}\mathfrak{s} \sqrt{-g_{\mu\nu}(x(\mathfrak{s}))\dot{x}^\mu \dot{x}^\nu} = \int \mathrm{d}\mathfrak{s} \sqrt{\dot{t}^2 - a^2(t)\dot{\vec{x}}^2}$$

with $\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}\mathfrak{s}}$. The energy is the momentum conjugate to $t(\mathfrak{s})$,

$$E = \frac{\partial L}{\partial \dot{t}} = \frac{-g_{tt}\dot{t}}{e}$$

where $e \equiv \sqrt{\dot{x}^2}$ = constant, ind of $\mathfrak{s}$ if we choose $\mathfrak{s}$ to be affine. The equations of motion for t are

$$0 = \frac{\delta S}{\delta t(\mathfrak{s})} = -\dot{E} - 2a\partial_t a \dot{\vec{x}}^2$$

which says that E is not constant in $\mathfrak{s}$. The EoM for $\vec{x}$ are

$$0 = \frac{\delta S}{\delta x^i(\mathfrak{s})} = \dot{p}_i$$

where the spatial momenta are

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = \frac{g_{ij}\dot{x}^j}{e} = \frac{a^2 \dot{x}^j \delta_{ij}}{e},$$

so these are conserved, as we expect since the action does not depend on $\vec{x}$. (Three angular momenta are also conserved.)

5. **Riemann normal coordinates.** [from Ooguri]

A more formal definition of Riemann normal coordinates $(\xi^1, ...\xi^n)$ in a neighborhood of a point $p \in M$ than we gave in lecture is as follows. Pick a tangent vector $\xi \in T_pM$ and find an affine geodesic $x_\xi(s)$ with the initial condition

$$x_\xi(s=0) = p, \quad \frac{\mathrm{d}x^\mu_\xi}{\mathrm{d}s}(s=0) = \xi^\mu \ .$$

Then define the *exponential map*, $\exp : T_pM \to M$ as

$$\exp(\xi) \equiv x_\xi(s=1) \in M \quad .$$

If the manifold M is geodesically complete, the map exp is defined for any tangent vector ξ. Otherwise, we may have to limit ourselves to a subset $\mathcal{V}_p \subset T_pM$ on which $\exp(\xi)$ is nonsingular. Since $T_pM \simeq \mathbb{R}^n$, its subspace $\mathcal{V}$ is an open subset of $\mathbb{R}^n$. We can then use a set of basis vectors $\xi \in \mathcal{V}_p$ to produce coordinates in the neighborhood $\exp(\mathcal{V}_p)$ of p which is the image of $\mathcal{V}_p$.

(a) Show that, in the normal coordinates, the Christoffel connection $\Gamma^\rho_{\mu\nu}$ vanishes at p, although its derivatives may not vanish.

Since $\exp(t\xi) = x_{t\xi}(s=1) = x_\xi(s=t)$ straight lines in $\mathcal{V}_p$ – those for which

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}t^2} = 0$$

– are mapped to *geodesics* in M, *i.e.* curves in M such that

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}t^2} + \Gamma^\mu_{\nu\rho}|_p \frac{\mathrm{d}\xi^\nu}{\mathrm{d}t}\frac{\mathrm{d}\xi^\rho}{\mathrm{d}t} = 0.$$

By comparing the previous two displayed equations, we see that in Riemann normal coordinates at p

$$\Gamma^\mu_{\nu\rho}|_p = 0.$$

This argument doesn't say anything about derivatives of Γ, and rightly so, since derivatives of Γ are covariantized to the Riemann tensor, and cannot set the components of a tensor to zero by a (nonsingular) coordinate change.

Another point worth noting (emphasized to me by Mike Gartner) is that only at p do we know that $\Gamma^\mu_{\nu\rho}|_p \frac{\mathrm{d}\xi^\nu}{\mathrm{d}t}\frac{\mathrm{d}\xi^\rho}{\mathrm{d}t} = 0$ for *all* $\frac{\mathrm{d}\xi^\nu}{\mathrm{d}t}$ (from which we conclude that $\Gamma|_p = 0$). At other points this equation is true only for the tangent to the geodesic that we took there from p.

(b) Prove the Bianchi identity at p using the normal coordinates. Since the identity is independent of coordinates (*i.e.* is a tensor equation), this is sufficient to prove the identity in general.

See the lecture notes, section 6.0.6.

6. **Zero curvature means flat.** [from Ooguri]

In this problem, we would like to show that if the Riemann curvature $R_{\mu\nu\rho}{}^\sigma$ vanishes in some neighborhood $\mathcal{U}$ of a point p, then we can find coordinates such that the metric tensor $g_{\mu\nu}$ takes the form $\eta_{\mu\nu} = \text{diag}(-1, +1, ..., +1)_{\mu\nu}$ (or $\delta_{\mu\nu} = \text{diag}(+1, ..., +1)_{\mu\nu}$ in the case of Euclidean signature metric), not only at the point p (which is always possible), but everywhere in the neighborhood U. In other words: $R_{\mu\nu\rho\sigma} = 0$ means that the space is flat. This shows that the curvature tensor $R_{\mu\nu\rho\sigma}$ contains all the local information about the curvature of spacetime. Prove this statement in the following steps.

(a) Show that we can find n linearly independent cotangent vectors $\omega_\mu^{(i)}(x)(i = 1...n)$ which are *covariantly constant*, *i.e.* $\nabla_\mu \omega_\nu^{(i)} = 0, \forall i$.

In a space $\mathcal{U}$ with $R_{\mu\nu\rho\sigma} = 0$, parallel transport is path independent, in the following sense. Starting with any vector v at a point p, pick two curves $\gamma_{1,2}$ connecting p to another (infinitesimally close) point q. Then the change in v after parallel transport along γ_i is $\Delta_i v$, and the difference is

$$\Delta_1 v^\mu - \Delta_2 v^\mu = R_{\alpha\beta\gamma}{}^\mu (\text{area})^{\alpha\beta} v^\gamma = 0$$

where $(\text{area})^{\alpha\beta}$ is an (infinitesimal) area element whose boundary is the difference of the two curves. The same argument applies to parallel transport of 1-forms because we can just raise their indices and $\nabla_a g^{ab} = 0$. So pick any basis of $T_p^\star M$, $\{\omega^{(i)}\}_{i=1}^n$ for some $p \in \mathcal{U}$ and define n co-vector fields on U by parallel tranporting the $\omega^{(i)}$ along any paths you like – whichever you choose you get the same answer. Because these one-forms are defined by parallel transport, they are covariantly constant: the deviation from parallel transport in *any* direction ϵ^μ is

$$0 = \epsilon^\mu \nabla_\mu \omega_\nu^{(i)}, \quad \forall i$$

but ϵ^μ is arbitrary, so $\nabla_\mu \omega_\nu^{(i)}$.

Notice also that since parallel transport preserves angles (*i.e.* $g^{\mu\nu}\omega_\mu^i \omega_\nu^j = \delta^{ij}$ (or η_{ij} in Minkowski signature) is constant) these one-forms remain orthonormal at every point and so are certainly independent.

(b) The result of (a) in particular means that the one-forms satisfy $\partial_{[\mu}\omega_{\nu]}^{(i)} = 0$. Show that we can find functions $f^{(i)}(x)$ such that $\omega_\mu^{(i)} = \partial_\mu f^{(i)}(x)$[2].

$$0 = \nabla_{[\mu}\omega_{\nu]}^{(i)} = \partial_{[\mu}\omega_{\nu]}^{(i)} - \underbrace{\Gamma^\rho_{[\mu\nu]}}_{\text{torsion-sfree}} \omega_\rho^{(i)}.$$

This says that the exterior derivatives vanish: $\mathrm{d}\omega^{(i)} = 0$ ('ω is a closed form'). Since $\mathcal{U}$ is simply connected (if not just take a smaller neighborhood which is), any closed form is exact, $\omega^{(i)} = \mathrm{d}f^{(i)}$. Explicitly, for any $x \in \mathcal{U}$, pick a path $\gamma_x : [0, 1] \to \mathcal{U}$ with $\gamma_x(0) = p, \gamma_x(1) = x$ and let

$$f^{(i)}(x) = \int_{\gamma_x} \omega^{(i)}.$$

[2]You may assume that the neighborhood U has the topology of a ball.

(This does not depend on the choice of path because ω is closed, $\mathrm{d}\omega = 0$. The difference in the resulting f from a different choice of path $\tilde{\gamma}_x$ is

$$\int_{\gamma_x} \omega^{(i)} - \int_{\tilde{\gamma}_x} \omega^{(i)} = \int_{\gamma_x - \tilde{\gamma}_x} \omega^{(i)} \overset{\text{Stokes}}{=} \int_B \mathrm{d}\omega^{(i)} = 0$$

where B is a two-dimensional region whose boundary is the *closed* curve $\gamma_x - \tilde{\gamma}_x$. Notice that here we required the coordinate neighborhood to be simply connected so that any closed curve is the boundary of some region; this is not the case on *e.g.* the surface of a donut.) Therefore

$$\partial_{x^\mu} f^{(i)}(x) = \partial_{x^\mu} \left(\int_0^1 \omega_\nu^{(i)} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \right) = \omega_\mu^{(i)}(x).$$

(c) Show that, if we use $y^i = f^{(i)}(x), i = 1..n$ as coordinates, the metric is expressed as a constant tensor in the neighborhood U. Therefore, by an appropriate linear transformation in y^i, the metric can be put in the indicated form (η or δ).

Notice that this is a good idea since $\det \partial_\mu y^i = \det \omega_\mu^{(i)} \neq 0$ by linear independence of our choice of basis vectors for $T_p^\star M$. We want to show that $\partial_{y^i} \tilde{g}_{kl}(y) = 0$ for points in $\mathcal{U}$. Once we show this we are done, because then we can make a linear transformation on the ys to put it in the standard form with ± 1s on the diagonal[3]. The metric in the y coordinates is

$$\tilde{g}_{ij}(y) = \partial_{y^i} x^\mu \partial_{y^j} x^\nu g_{\mu\nu}.$$

First note that $0 = \nabla_\nu \omega_\mu^{(i)} = \partial_\nu \omega_\mu^{(i)} - \Gamma^\rho_{\nu\mu} \omega_\rho^{(i)}$ and the chain rule says

$$\delta_i^j = (\partial_{y^i} x^\mu) \partial_{x^\mu} y^i = (\partial_{y^i} x^\mu) \omega_\mu^{(i)}.$$

Differentiating this again, we have

$$0 = \partial_\nu \delta_i^j = (\partial_\nu \partial_i x^\mu) \omega_\mu^{(j)} + (\partial_i x^\mu)\, \partial_\nu \omega_\mu^{(j)} = \partial_\nu y^k (\partial_k \partial_i x^\mu) \partial_\mu y^j + \partial_i x^\mu \Gamma^\rho_{\nu\mu} \omega_\rho^{(j)}$$

where in the second step we used the fact that the ωs are covariantly constant. Using the fact that these Jacobian matrices (of which $\omega_\mu^{(j)}$ is one!) are invertible (*e.g.* $\partial_i x^\nu \partial_\nu y^j = \delta_i^j$) we learn that

$$\partial_k \partial_i x^\mu = -\partial_k x^\nu \partial_i x^\alpha \underbrace{\partial_j x^\mu \Gamma^\rho_{\nu\alpha} \omega_\rho^{(j)}}_{=\partial_j x^\mu \partial_\rho y^j = \delta^\mu_\rho} = -\partial_k x^\nu \partial_i x^\alpha \Gamma^\mu_{\nu\alpha}. \tag{3}$$

Also

$$g_{\sigma\alpha} \Gamma^\alpha_{\mu\nu} = \frac{1}{2} \left(g_{\alpha\mu,\nu} + g_{\nu\sigma,\alpha} - g_{\mu\nu,\alpha} \right).$$

[3]More explicitly, the resulting constant metric is a symmetric matrix which can be diagonalized by an orthogonal transformation (i.e. constant rotation); after this rotation its eigenvalues can be scaled to 1 by a rescaling of the rotated coordinates.

So:

$$
\begin{aligned}
\partial_k \tilde{g}_{ij} &= \partial_k \left(\partial_{y^i} x^\mu \partial_{y^j} x^\nu g_{\mu\nu}\right) \\
&= \left((\partial_k \partial_i x^\mu)\partial_j x^\nu + \partial_i x^\mu (\partial_k \partial_j x^\nu)\right) g_{\mu\nu} + \partial_i x^\mu \partial_j x^\nu \underbrace{\partial_k g_{\mu\nu}}_{=\partial_k x^\sigma g_{\mu\nu,\sigma}} \\
&\overset{(3)}{=} \left(-\partial_i x^\alpha \partial_l x^\mu \partial_i x^\sigma \Gamma^\rho_{\alpha\sigma} \omega^{(l)}_\rho \partial_j x^\nu + \partial_i x^\mu \left(-\partial_k x^\alpha \partial_l x^\nu \partial_j x^\sigma \Gamma^\rho_{\alpha\sigma} \omega^{(l)}_\rho\right)\right) g_{\mu\nu} \\
&\quad + \partial_i x^\mu \partial_j x^\nu \partial_k x^\sigma g_{\mu\nu,\sigma} \\
&= \partial_i x^\mu \partial_j x^\nu \partial_k x^\alpha \left(-\partial_l x^\sigma \Gamma^\rho_{\alpha\mu} \omega^{(l)}_\rho g_{\sigma\nu} - \partial_l x^\sigma \Gamma^\rho_{\alpha\nu} \omega^{(l)}_\rho g_{\mu\sigma} g_{\mu\nu,\alpha}\right) \\
&\overset{(3)}{=} \partial_i x^\mu \partial_j x^\nu \partial_k x^\alpha \left(-g_{\sigma\nu} \Gamma^\sigma_{\alpha\mu} - g_{\mu\sigma} \Gamma^\sigma_{\alpha\nu} + g_{\mu\nu,\alpha}\right) \\
&= \partial_i x^\mu \partial_j x^\nu \partial_k x^\alpha \left(-\frac{1}{2}\left(g_{\nu\alpha,\mu} + g_{\mu\nu,\alpha} - g_{\alpha\mu,\nu} + (\mu \leftrightarrow \nu)\right) + g_{\mu\nu,\alpha}\right) \\
&= \partial_i x^\mu \partial_j x^\nu \partial_k x^\alpha \left(-\frac{1}{2}\left(g_{\nu\alpha,\mu} + g_{\mu\nu,\alpha} - g_{\alpha\mu,\nu} + g_{\alpha\mu,\nu} + g_{\nu\mu,\alpha} - g_{\alpha\nu,\mu}\right) + g_{\mu\nu,\alpha}\right) \\
&= \partial_i x^\mu \partial_j x^\nu \partial_k x^\alpha \left(-\frac{1}{2}\left(g_{\mu\nu,\alpha} g_{\nu\mu,\alpha}\right) + g_{\mu\nu,\alpha}\right) = 0.
\end{aligned}
\tag{4}
$$

In fact, we could have avoided the preceding horror simply by using the fact noted above that parallel transport preserves angles. This says that the inner products between the our basis cotangent vectors (which we've chosen to be orthonormal at x_0) are

$$g^{\mu\nu}\omega^i_\mu \omega^j_\nu = \delta^{ij}$$

(or η_{ij} in Minkowski signature). But this is the inverse metric in the new coordinates! If the inverse metric is constant then so is the metric, and in fact with our choice of orthonormal basis vectors, that constant is the canonical metric.