University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 5

Posted October 23, 2013 **Due Monday, November 4, 2013**

1. **A constant vector field.** Consider the vector field $W \equiv \partial_x$ in the flat plane with metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. Write the components of W in polar coordinates $W = W_r \partial_r + W_\varphi \partial_\varphi$ and compute their partial derivatives. Then compute the (metric-compatible) covariant derivative in polar coordinates.

2. **Vector fields on the 2-sphere.** [from Ooguri] A two-dimensional sphere S^2 of unit radius can be embedded in the three-dimensional euclidean space $\mathbb{R}^3$ by the equation
$$x^2 + y^2 + z^2 = 1 \ .$$
For coordinates on the sphere we can use (θ, φ) defined by
$$x = \sin\theta\cos\varphi, \ y = \sin\theta\sin\varphi, \ z = \cos\theta,$$
except at the north and south poles $\theta = 0, \pi$ where the value of φ is ambiguous.
An infinitesimal rotation of $\mathbb{R}^3$ around its origin induces a tangent vector field on S^2, which is said to *generate* the rotation[1]. Show that there are three linearly-independent vector fields[2] of this type and compute their commutators $[\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j]$.

3. **E&M in curved space.** Consider EM fields $A_\mu(x)$ in a curved spacetime with a general metric $g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$.
   (a) Write an action functional $S[A_\mu, g_{\mu\nu}]$ which is general-coordinate invariant *and gauge invariant* and which reduces to the Maxwell action if we evaluate it in Minkowski spacetime $S[A_\mu, \eta_{\mu\nu}]$.
   (b) Vary this action with respect to A_μ to find the equations of motion governing electrodynamics in curved space.

4. **The badness cancels.** Using the coordinate transformation property of the Christoffel connection $\Gamma^\rho_{\mu\nu}$, verify that
$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho$$
transforms as a rank-2 covariant tensor if ω is a one-form.

[1] More precisely, consider the result of acting with a rotation by an infinitesimal angle θ on an arbitrary smooth function:
$$f(x) \mapsto f(Rx) = f(x + \theta A x) = f(x) + (\theta A x)^i \partial_i f(x) + \ldots$$
– the vector field $(Ax)^i \partial_i$ generates the rotation.

[2] A vector field v on M is linearly dependent on some others $\{v_\alpha\}$ if there exist *constants* a^α s.t. $v = a^\alpha v_\alpha$ everywhere in M.