University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 4

Posted October 16, 2013 **Due Monday, October 28, 2013**

1. **Non-relativistic limit of a perfect fluid**

   The stress-energy tensor for a perfect fluid in Minkowski space is

   $$T^{\mu\nu} = ((\epsilon + p) u^\mu u^\nu + p\eta^{\mu\nu}) .$$

   Consider the continuity equation $\partial_\mu T^{\mu\nu} = 0$ in the nonrelativistic limit, $\epsilon \gg p$ (recall that ϵ includes the rest mass!). Show that it implies the conservation of mass, and Euler's equation:

   $$\rho \left(\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} \right) = -\vec{\nabla} P.$$

   (See section 4.2 of Wald for more on this. Note that he uses ρ for ϵ and sets $c = 1$.)

2. **Stress tensors for fields in Minkowski space**

   (a) Given a (translation-invariant) lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ for a scalar field ϕ, define the energy-momentum tensor as

   $$T^\mu_\nu = -\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi + \delta^\mu_\nu \mathcal{L}.$$

   Show that the equation of motion for ϕ implies the conservation law $\partial_\mu T^\mu_\nu$.

   (b) Show that the energy-momentum tensor for the Maxwell field

   $$T^{\mu\nu}_{EM} = \frac{1}{4\pi c} \left(F^{\mu\rho} F^\nu_\rho - \frac{1}{4} \eta^{\mu\nu} F^2 \right)$$

   is *traceless*, that is $(T_{EM})^\mu_\mu = 0$.

   (c) Show that the energy-momentum tensor for the Maxwell field

   $$T^{\mu\nu}_{EM} = \frac{1}{4\pi c} \left(F^{\mu\rho} F^\nu_\rho - \frac{1}{4} \eta^{\mu\nu} F^2 \right)$$

   in the presence of an electric current j^μ obeys

   $$\partial_\mu T^{\mu\nu}_{EM} = -j^\rho F^\nu_\rho .$$

   Explain this result in words.

(d) **Optional:** show that tracelessness of T^μ_μ implies conservation of the *dilatation current* $D^\mu = x^\nu T^\mu_\nu$. Convince yourself that the associated conserved charge $\int_{\text{space}} D^0$ is the generator of scale transformations.

3. **Polyakov form of the worldline action**

(a) Consider the following action for a particle trajectory $x^\mu(t)$:

$$S_?[x] = -m \int \mathrm{d}t \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} g_{\mu\nu}(x) \ .$$

(Here $g_{\mu\nu}$ is some given metric. You may set $g_{\mu\nu} = \eta_{\mu\nu}$ if you like.) Convince yourself that the parameter t is meaningful, that is: reparametrizing t changes $S_?$.

Now consider instead the following action

$$S[x, e] = - \int \mathrm{d}\mathfrak{s} \left(\frac{1}{e(\mathfrak{s})} \frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}} \frac{\mathrm{d}x^\nu}{\mathrm{d}\mathfrak{s}} g_{\mu\nu}(x) - m^2 e(\mathfrak{s}) \right) \ .$$

The dynamical variables are $x^\mu(\mathfrak{s})$ (positions of a particle) and $e(\mathfrak{s})$; e is called an *einbein*[1]:

$$\mathrm{d}s_{1d}^2 = e^2(\mathfrak{s}) \mathrm{d}\mathfrak{s}^2.$$

(b) Show that $S[x, e]$ is reparametrization invariant if we demand that $\mathrm{d}s_{1d}^2$ is an invariant line element.

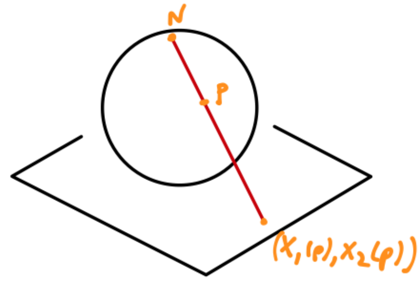
(c) Derive the equations of motion for e and x^μ. Compare with other reparametrization-invariant actions for a particle.

(d) Take the limit $m \to 0$ to find the equations of motion for a massless particle.

4. Show that S^2 using stereographic projections (aka Poincaré maps)for the coordinate charts (see the figure)

$$x_N : S^2 - \{\text{north pole}\} \to \mathbb{R}^2 \qquad x_S : S^2 - \{\text{south pole}\} \to \mathbb{R}^2$$

is a differentiable manifold of dimension two. More precisely:
(a) Write the Poincaré maps explicitly in terms of the embedding in $\mathbb{R}^3$ ($\{(x_1, x_2, x_3) \in \mathbb{R}^3 | \sum_i x_i^2 = 1\} \to \mathbb{R}^2$).



[1] that's German for 'the square root of the metric in one dimension'

(b) Show that the transition function $x_S \circ x_N^{-1} : \mathbb{R}^2_N \to \mathbb{R}^2_S$ is differentiable on the overlap of the two coordinate patches (everything but the poles).)

5. Verify explicitly that if ω_μ is a one-form (cotangent vector), then $\partial_\mu \omega_\nu - \partial_\nu \omega_\mu$ transforms as a rank-2 covariant tensor.

6. **Lie brackets.** The *commutator* or *Lie bracket* $[u, v]$ of two vector fields u, v on M is defined as follows, by its action on any function on M:

$$[u, v](f) = u(v(f)) - v(u(f)) \ .$$

   (a) Show that its components in a coordinate basis are given by

$$[u, v]^\mu = u^\nu \partial_\nu v^\mu - v^\nu \partial_\nu u^\mu \ .$$

   (b) Using the fact that u^μ, v^μ transform as contravariant vectors, show explicitly that $[u, v]^\mu$ also transforms this way.

   (c) (Optional extra bit) Convince yourself from the general definition of Lie derivative given in lecture that $\mathcal{L}_u v = [u, v]$.