University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 4 – Solutions

Posted October 16, 2013 **Due Monday, October 28, 2013**

1. **Non-relativistic limit of a perfect fluid**

The stress-energy tensor for a perfect fluid in Minkowski space is

$$T^{\mu\nu} = \left((\epsilon + p)\, u^\mu u^\nu + p\eta^{\mu\nu} \right).$$

Consider the continuity equation $\partial_\mu T^{\mu\nu} = 0$ in the nonrelativistic limit, $\epsilon \gg p$ (recall that ϵ includes the rest mass!). Show that it implies the conservation of mass, and Euler's equation:

$$\rho \left(\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} \right) = -\vec{\nabla} P.$$

(See section 4.2 of Wald for more on this. Note that he uses ρ for ϵ and sets $c = 1$.)

The continuity equation says

$$0 = \partial_\mu T^{\mu\nu} = \left(\partial_\mu(\epsilon + p)\right) u^\mu u^\nu + (\epsilon + p)\left(\left(\partial_\mu u^\mu\right) u^\nu + u^\mu \partial_\mu u^\nu \right) + \left(\partial_\mu p\right) \eta^{\mu\nu}. \quad (1)$$

The vector field u satisfies

$$-1 = u^\nu u_\nu \implies 0 = \partial_\mu (u_\nu u^\nu) = 2\left(\partial_\mu u^\nu\right) u_\nu$$

(in flat space). Projecting the conservation equation onto u_ν gives

$$\begin{aligned} 0 &= -u^\mu \partial_\mu (\epsilon + p) - (\epsilon + p)\partial_\nu u^\nu + 0 + \partial_\mu p u^\mu \\ &= -u^\mu \partial_\mu \epsilon - \partial_\mu u^\mu (\epsilon + p) \end{aligned} \quad (2)$$

To find the content of the conservation law $\perp$ to u^ν add u^ν times (2)

$$0 = u^\nu \left(-(\epsilon + p)\partial_\nu u^\nu + 0 + \partial_\mu p u^\mu \right)$$

to both sides of (1) to get:

$$\begin{aligned} 0 &= +u^\mu u^\nu \partial_\mu p + (\epsilon + p)\, u^\mu \partial_\mu u^\nu + \partial_\mu p \eta^{\mu\nu} \\ &= (\epsilon + p)\, u^\nu \partial_\mu u^\nu + \partial_\mu p \left(\eta^{\mu\nu} + u^\mu u^\nu\right). \end{aligned} \quad (3)$$

Now take the nonrelativistic limit: $\epsilon \gg p$, $u^\mu = (1, \vec{v})^\mu$ in which case $\partial_\mu v^\mu \to -0 + \vec{\nabla} \cdot \vec{v}$. Eqn (2) becomes

$$0 = u^\mu \partial_\mu \epsilon + \partial_\mu u^\mu (\epsilon + p) \simeq +\partial_{x^0} \epsilon + \vec{v} \cdot \vec{\nabla} \epsilon + \vec{\nabla} \cdot v \epsilon \left(1 + \frac{p}{\epsilon}\right) \simeq +\frac{1}{c}\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot (\epsilon \vec{v})$$

which is energy conservation. Eqn (3) becomes

$$0 = (\epsilon + p)\, u^\mu \partial_\mu u^\nu + \partial_\mu p \left(\eta^{\mu\nu} + u^\mu u^\nu\right) \simeq \epsilon \left(\partial_0 + \vec{v} \cdot \vec{\nabla}\right) u^\nu + \left((\partial_0 + \vec{v} \cdot \vec{\nabla}) p\right) u^\nu + \partial_\mu p \eta^{\mu\nu}$$

Now look at the time and space components:

$$\boxed{\nu = 0} : 0 = \epsilon(0) + \frac{1}{c}\partial_t p + \vec{v} \cdot \vec{\nabla} p - \frac{1}{c}\partial_t p = \vec{v} \cdot \vec{\nabla} p$$

$$\boxed{\nu \neq 0} : 0 = \epsilon \left(\frac{1}{c}\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}\right) + \vec{v} \left(\frac{1}{c}\partial_t p + \vec{v} \cdot \vec{\nabla} p\right) + \vec{\nabla} p$$

In the NR limit $v \ll c$, so the important terms are:

$$\boxed{\nu \neq 0} : 0 = \epsilon \left(\frac{1}{c}\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}\right) + \vec{\nabla} p$$

which is the Euler equation.

2. **Stress tensors for fields in Minkowski space**

   (a) Given a (translation-invariant) lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ for a scalar field ϕ, define the energy-momentum tensor as

   $$T^\mu_\nu = -\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial_\nu \phi + \delta^\mu_\nu \mathcal{L}.$$

   Show that the equation of motion for ϕ implies the conservation law $\partial_\mu T^\mu_\nu$.

   The EoM for ϕ is

   $$0 = \frac{\delta S}{\delta \phi(x)} = \frac{\partial \mathcal{L}}{\partial \phi}(x) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}(x)$$

   (see the lecture notes section 3.1), so

   $$\begin{aligned} \partial_\mu T^\mu_\nu &= -\underbrace{\partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)}}_{=-\frac{\partial \mathcal{L}}{\partial \phi}} - \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial_\mu \partial_\nu \phi + \underbrace{\partial_\nu \mathcal{L}}_{=\partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\nu (\partial_\mu \phi) \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}} \\ &= -\frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi - \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial_\mu \partial_\nu \phi + \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} = 0. \end{aligned} \tag{4}$$

   (b) Show that the energy-momentum tensor for the Maxwell field

   $$T^{\mu\nu}_{EM} = \frac{1}{4\pi c}\left(F^{\mu\rho} F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu} F^2\right)$$

   is *traceless*, that is $(T_{EM})^\mu_\mu = 0$.

   $$T^\mu_\mu = \frac{1}{4\pi c}\left(\underbrace{F^{\mu\rho} F_{\mu\rho}}_{=F^2} - \frac{1}{4} 4 F^2\right) = 0.$$

An apology: the notation F^ν_ρ in the pset statement is ambiguous: when raising and lowering indices on an object with multiply (non-symmetrized) indices, you have to keep track of which one was raised or lowered. I should have written $F^\nu{}_\rho$ (which is harder to TeX).

(c) Show that the energy-momentum tensor for the Maxwell field

$$T^{\mu\nu}_{EM} = \frac{1}{4\pi c}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F^2\right)$$

in the presence of an electric current j^μ obeys

$$\partial_\mu T^{\mu\nu}_{EM} = -j^\rho F^\nu{}_\rho.$$

Explain this result in words.

I am setting $c = 1$. Recall that Maxwell's equations with a source are

$$0 = \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}, \quad \partial^\rho F_{\mu\rho} = 4\pi j_\mu.$$

We will need both.

$$\begin{aligned}
\partial_\mu T^{\mu\nu}_{EM} &= \frac{1}{4\pi}\left(\left(\partial_\mu F^{\mu\rho}\right)F^\nu{}_\rho + F^{\mu\rho}\partial_\mu F^\nu_\rho - \frac{2}{4}\eta^{\mu\nu}\left(\partial_\mu F_{\alpha\beta}F_{\gamma\delta}\eta^{\alpha\gamma}\eta^{\beta\delta}\right)\right) \\
&= \frac{1}{4\pi}\left(-\left(\partial^\mu F_{\rho\mu}\right)F^{\nu\rho} + F^{\mu\rho}\left(\partial_\mu F^\nu_\rho\right) - \frac{1}{2}\left(\partial^\nu F_{\mu\rho}\right)F^{\mu\rho}\right) \\
&= -j_\rho F^{\nu\rho} + \frac{1}{4\pi}F^{\mu\rho}\eta^{\nu\sigma}\left(\partial_\mu F_{\sigma\rho} - \frac{1}{2}\partial_\sigma F_{\mu\rho}\right) \qquad (5)
\end{aligned}$$

But by antisymmetry of F: $F^{\mu\rho}(\partial_\mu F_{\sigma\rho}) = \frac{1}{2}F^{\mu\rho}(\partial_\mu F_{\sigma\rho} - \partial_\rho F_{\sigma\mu})$, so

$$\partial_\mu T^{\mu\nu}_{EM} = -j_\rho F^{\nu\rho} + \frac{1}{8\pi}F^{\mu\rho}\eta^{\nu\sigma}\underbrace{\left(\partial_\mu F_{\sigma\rho} + \partial_\rho F_{\mu\sigma} + \partial_\sigma F_{\rho\mu}\right)}_{\propto \epsilon_{\mu\sigma\rho\nu}\epsilon^{\nu\alpha\beta\gamma}\partial_\alpha F_{\beta\gamma}=0} = -j^\rho F^\nu{}_\rho\ .$$

The EM field can do work on the currents and vice versa; therefore the energy and momentum in the EM field can turn into energy of the charges and vice versa, and is therefore not conserved. The RHS of the $\nu = 0$ component is minus the work done on the particles by the fields; the RHS of the $\nu = i$ component is minus the force (change in momentum per unit time) in the i direction exerted on the particles by the Maxwell field.

(d) **Optional:** show that tracelessness of T^μ_μ implies conservation of the *dilatation current* $D^\mu = x^\nu T^\mu_\nu$. Convince yourself that the associated conserved charge $\int_{\text{space}} D^0$ is the generator of scale transformations.

$$\partial_\mu D^\mu = T^\mu_\mu + x^\nu\partial_\mu T^\mu_\nu = T^\mu_\mu.$$

$$\int_{\text{space}} D^0 = \int_{\text{space}}\left(tT^0_0 + x^i T^0_i\right) = t\mathcal{H} + x^i\mathcal{P} = -\mathbf{i}\left(t\partial_t + x^i\frac{\partial}{\partial x^i}\right)$$

which actions on functions of t, x by : $\left(t\partial_t + x\partial_x\right)t^\alpha x^\beta = (\alpha+\beta)t^\alpha x^\beta$ and exponentiates to $e^{at\partial_t}f(t) = f(e^a t)$ (it shifts $\log t$ by a).

3. **Polyakov form of the worldline action**

(a) Consider the following action for a particle trajectory $x^\mu(t)$:

$$S_?[x] = -m \int \mathrm{d}t \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} g_{\mu\nu}(x) \ .$$

(Here $g_{\mu\nu}$ is some given metric. You may set $g_{\mu\nu} = \eta_{\mu\nu}$ if you like.) Convince yourself that the parameter t is meaningful, that is: reparametrizing t changes $S_?$.

Reparemetrizing by $\tilde{t} = \tilde{t}(t)$ changes the measure by one factor of $J \equiv \partial_t \tilde{t}$, but multiplies $\dot{x}^2$ by J^{-2}.

Now consider instead the following action

$$S[x,e] = -\int \mathrm{d}\mathfrak{s} \left(\frac{1}{e(\mathfrak{s})} \frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}}\frac{\mathrm{d}x^\nu}{\mathrm{d}\mathfrak{s}} g_{\mu\nu}(x) - m^2 e(\mathfrak{s}) \right) \ .$$

The dynamical variables are $x^\mu(\mathfrak{s})$ (positions of a particle) and $e(\mathfrak{s})$; e is called an *einbein*[1]:

$$\mathrm{d}s^2_{1d} = e^2(\mathfrak{s})\mathrm{d}\mathfrak{s}^2.$$

(b) Show that $S[x,e]$ is reparametrization invariant if we demand that $\mathrm{d}s^2_{1d}$ is an invariant line element.

Notice that the second term is just $\int \sqrt{g}$ our usual way of making a coordinate-invariant integration measure. In the first term the $\frac{\partial}{\partial \mathfrak{s}}$s each transform like e, so $\frac{1}{e}\dot{x}^2$ transforms like e.

(c) Derive the equations of motion for e and x^μ. Compare with other reparametrization-invariant actions for a particle.

$$0 = \frac{\delta S}{\delta e(\mathfrak{s})} = -e^{-2}\dot{x}^2 + m^2 \implies \underline{e} = m\sqrt{\dot{x}^2}$$

$$S[x,\underline{e}] = -m \int \mathrm{d}\mathfrak{s}\sqrt{\dot{x}^2}$$

is the reparam-invariant action we discussed before, so obviously the EoM agree. Even the values of the action agree when we set e equal to the solution of its EoM.

(d) Take the limit $m \to 0$ to find the equations of motion for a massless particle.

In this limit, e is a Lagrange multiplier which enforces that the tangent vector is lightlike:

$$0 = \frac{\delta S_{m=0}}{\delta e(\mathfrak{s})} \propto \dot{x}^2 \ .$$

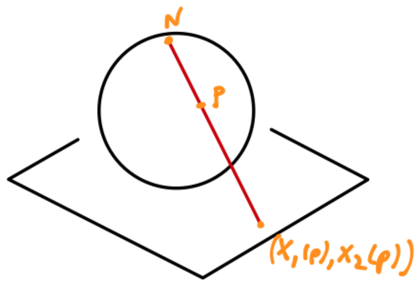
The EoM for x is still the geodesic equation. Now we can't choose $\mathfrak{s}$ to be the proper time, but we *have to* demand that the lengths of the tangent vectors are independent of $\mathfrak{s}$ – they are all zero. So any parametrization satisfying the lightlike-constraint is affine.

[1] that's German for 'the square root of the metric in one dimension'

4. Show that S^2 using stereographic projections (aka Poincaré maps)for the coordinate charts (see the figure)

$$x_N : S^2 - \{\text{north pole}\} \to \mathbb{R}^2 \qquad x_S : S^2 - \{\text{south pole}\} \to \mathbb{R}^2$$

is a differentiable manifold of dimension two. More precisely:
(a) Write the Poincaré maps explicitly in terms of the embedding in $\mathbb{R}^3$ ($\{(x_1, x_2, x_3) \in \mathbb{R}^3 | \sum_i x_i^2 = 1\} \to \mathbb{R}^2$).



If we place the center of the unit sphere at the origin, the plane in the picture is $\{x_3 = -1\}$. The line L_N in the picture passes through $i(N) = (0,0,1)$ (the embedding of the north pole) and $i(p) = (x, y, z)$, the embedding of the point of interest p. The intersection of L_N with $z = -1$ determines $x_N(p) = (x_1^N, x_2^N, -1)$. We can parametrize it as $\vec{L}_N(t)$ so $\vec{L}_N(0) = i(N) = (0,0,1)$ and $\vec{L}_N(1) = (x, y, z)$, which gives

$$\vec{L}_N(t) \equiv (x_1^N(t), x_2^N(t), x_3^N(t)) = (tx, ty, 1 - t + tz).$$

The value of $t = t_N$ where $-1 = x_3^N(t_N) = 1 - (1-z)t_N t_N$ is then $t_N = \frac{2}{1-z}$. Notice that this is a nice function as long as $z \neq 1$, and that's why we must exclude the north pole (where $(x, y, z) = (0, 0, 1)$) from this patch. The coordinates of the stereographic projection from N are then

$$x_N(p) = (x_1^N(t_N), x_2^N(t_N)) = (t_N x, t_N y) = \frac{2}{1-z}(x, y).$$

For the south pole, the plane is at $x_3 = +1$ and the line in question is

$$L_S(t) = (tx, ty, (1-t)(-1) + tz)$$

so the value of t where $x_3(t_S) = +1$ is $t_S = \frac{-2}{1+z}$ and

$$x_s(p) = (x_1^N(t_S), x_2^N(t_S)) = \frac{-2}{1+z}(x, y).$$

(b) Show that the transition function $x_S \circ x_N^{-1} : \mathbb{R}_N^2 \to \mathbb{R}_S^2$ is differentiable on the overlap of the two coordinate patches (everything but the poles).)

The overlap $\mathcal{U}_N \cap \mathcal{U}_S$ where both coord systems are defined is everywhere but the north and south poles. Notice that $\frac{x_1^N}{x_1^S} = \frac{z+1}{z-1} = \frac{x_2^N}{x_2^S}$. The transition functions are nice and diagonal

$$\begin{pmatrix} x_1^N \\ x_2^N \end{pmatrix} = \frac{z+1}{z-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^S \\ x_2^S \end{pmatrix} \equiv M(x_1^S, x_2^S) \begin{pmatrix} x_1^S \\ x_2^S \end{pmatrix}$$

– notice that we must think of z here as a function of the coordinates $x_{1,2}^S$. Explicitly it is findable by

$$r^2 \equiv \left(x_1^S\right)^2 + \left(x_2^S\right)^2 = 4\frac{x^2+y^2}{(1+z)^2} = 4\frac{1-z^2}{(1+z)^2}$$
$$\implies z = \frac{-2r^2 \pm \sqrt{4r^4 - 4(r^2-4)(r^2+4)}}{2(r^2+4)} = -\frac{r^2 \mp 4}{r^2+4}.$$

The bottom root is just $z = -1$ which is not what we want – $-1 < z < 1$ on the overlap of patches. So the top root it is:

$$z = \frac{4-r^2}{4+r^2} = \frac{4 - \left(x_1^S\right)^2 - \left(x_2^S\right)^2}{4 + \left(x_1^S\right)^2 + \left(x_2^S\right)^2}.$$

And $\frac{z+1}{z-1} = \frac{-4}{r^2} = \frac{-4}{\left(x_1^S\right)^2 + \left(x_2^S\right)^2}$ and therefore the transition map is

$$\begin{pmatrix} x_1^N \\ x_2^N \end{pmatrix} = \frac{-4}{\left(x_1^S\right)^2 + \left(x_2^S\right)^2} \begin{pmatrix} x_1^S \\ x_2^S \end{pmatrix}$$

The inverse map is obtainable noticing that $\left(x_1^N\right)^2 + \left(x_2^N\right)^2 = \frac{16}{\left(x_1^S\right)^2 + \left(x_2^S\right)^2}$, so it has exactly the same form, but with $N \leftrightarrow S$. The transition function

$$\phi = x_N \circ x_S^{-1} : x_S(\mathcal{U}_S \cap \mathcal{U}_N) \to x_N(\mathcal{U}_S \cap \mathcal{U}_N)$$

is then perfectly smooth away from the points where $0 = \left(x_1^N\right)^2 + \left(x_2^N\right)^2$ or $0 = \left(x_1^S\right)^2 + \left(x_2^S\right)^2$ which are just S and N.

5. Verify explicitly that if ω_μ is a one-form (cotangent vector), then $\partial_\mu \omega_\nu - \partial_\nu \omega_\mu$ transforms as a rank-2 covariant tensor.

   Under the coordinate change $x^\mu \to \tilde{x}^a(x)$, $\omega_\mu(x) \to \tilde{\omega}_a(\tilde{x}) = J_a^\mu \omega_\mu(x)$, and $\partial_\mu \to \tilde{\partial}_a = J_a^\mu \partial_\mu$ with $J_a^\mu \equiv \frac{\partial x^\mu}{\partial \tilde{x}^a} = \tilde{\partial}_a x^\mu$.

$$\begin{aligned}
(\mathrm{d}\omega)_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \quad & \to \quad \tilde{\partial}_a \tilde{\omega}_b - \tilde{\partial}_b \tilde{\omega}_a \\
& = \quad \tilde{\partial}_a \left(\tilde{\partial}_b x^\mu \omega_\mu \right) - (a \leftrightarrow b) \\
& = \quad \underbrace{\left(\tilde{\partial}_a \tilde{\partial}_b x^\mu - \tilde{\partial}_b \tilde{\partial}_a x^\mu \right)}_{=0} \omega_\mu + \tilde{\partial}_b x^\mu \tilde{\partial}_a \omega_\mu - \tilde{\partial}_a x^\mu \tilde{\partial}_b \omega_\mu \\
& \overset{\text{chain rule}}{=} \quad \tilde{\partial}_b x^\mu \tilde{\partial}_a x^\nu \partial_\nu \omega_\mu - \tilde{\partial}_a x^\mu \tilde{\partial}_b x^\nu \partial_\nu \omega_\mu \\
& \overset{\text{relabel dummy indices}}{=} \quad \tilde{\partial}_b x^\nu \tilde{\partial}_a x^\mu \partial_\mu \omega_\nu - \tilde{\partial}_a x^\mu \tilde{\partial}_b x^\nu \partial_\nu \omega_\mu \\
& \overset{\text{factorize}}{=} \quad \tilde{\partial}_b x^\mu \tilde{\partial}_a x^\nu \left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu \right) \qquad (6)
\end{aligned}$$

6. **Lie brackets.** The *commutator* or *Lie bracket* $[u, v]$ of two vector fields u, v on M is defined as follows, by its action on any function on M:

$$[u, v](f) = u(v(f)) - v(u(f)) \ .$$

(a) Show that its components in a coordinate basis are given by

$$[u, v]^\mu = u^\nu \partial_\nu v^\mu - v^\nu \partial_\nu u^\mu \ .$$

$$\begin{aligned}
[u,v](f) &= u^\mu \partial_\mu (v^\nu \partial_\nu (f)) - v^\mu \partial_\mu (u^\nu \partial_\nu (f)) \\
&\overset{\text{product rule}}{=} u^\mu (\partial_\mu v^\nu) \partial_\nu f - v^\mu (\partial_\mu u^\nu) \partial_\nu f + u^\mu v^\nu \partial_\mu \partial_\nu f - v^\mu u^\nu \partial_\mu \partial_\nu f \\
&\overset{\text{rename dummy indices}}{=} \left(u^\mu (\partial_\mu v^\nu) \partial_\nu - v^\mu (\partial_\mu u^\nu) \partial_\nu \right) f + u^\mu v^\nu \partial_\mu \partial_\nu f - v^\nu u^\mu \partial_\mu \partial_\nu f \\
&= \left(u^\mu (\partial_\mu v^\nu) \partial_\nu - v^\mu (\partial_\mu u^\nu) \partial_\nu \right) f \qquad (7)
\end{aligned}$$

So the components of the bracket are as given above.

(b) Using the fact that u^μ, v^μ transform as contravariant vectors, show explicitly that $[u, v]^\mu$ also transforms this way.

This is very similar to the problem about the transformation of $\mathrm{d}\omega$ above. I will do this one in a more geometric way, by studying the vector itself, rather than the components :

$$\begin{aligned}
u^\mu (\partial_\mu v^\nu) \partial_\nu - v^\mu (\partial_\mu u^\nu) \partial_\nu &\rightarrow \tilde{u}^a (\tilde{\partial}_a \tilde{v}^b) \tilde{\partial}_b - (u \leftrightarrow v) \\
(J^a_\mu \equiv \frac{\partial \tilde{x}^a}{\partial x^\mu}) \quad &= J^a_\mu u^\mu ((J^{-1})^\sigma_a \partial_\sigma J^b_\nu v^\nu)(J^{-1})^\rho_b \partial_\rho - (u \leftrightarrow v) \\
(J^a_\mu (J^{-1})^\sigma_a = \delta^\sigma_\mu) \quad &= u^\sigma (\partial_\sigma v^\rho) \partial_\rho - (u \leftrightarrow v) \\
&\overset{\text{rename dummy indices}}{=} u^\mu (\partial_\mu v^\nu) \partial_\nu - v^\mu (\partial_\mu u^\nu) \partial_\nu . \qquad (8)
\end{aligned}$$

(c) (Optional extra bit) Convince yourself from the general definition of Lie derivative given in lecture that $\mathcal{L}_u v = [u, v]$.

See the footnote in section 5.3.1 of the lecture notes.