University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 3 – Solutions

Posted October 13, 2013 **Due Monday, October 22, 2013**

1. **It's not a tensor!** [from Brandenberger]

   This problem is a simple game. Identify which of the following equations could be valid tensor equations; for the ones that can't be, say why not. Here I mean tensors *e.g.* under the Lorentz group (or maybe some more general transformations) where we must distinguish between covariant (lower) and contravariant (upper) indices.

   (a) $R^a_{man} = T_{mn}$

   (b) $\text{☺}_a \omega_{bc} = \text{♃}_{ab}$

   (c) $\Diamond_{a\aleph\aleph} = \text{♄}_a$

   (d) $\text{♫}_{ab} + \text{☼}_{ac} = \Upsilon_{bc}$

   (e) $\natural_a(\sharp_b + \flat_b) = \text{☻}_{ab}$

   (f) $\text{☹}_a \bigstar_b = \text{☽}_{ab}$

   a, c, e, f could be OK. All the rest have uncontracted indices which don't match between terms or between the two sides of the putative equation. This has the consequence that the equation will turn into a completely different equation after a transformation. c is questionable in that the repeated indices are both lower; this is OK if the group in question preserves a trivial quadratic form (δ_{ij}), otherwise no.

2. **Relativistic charged particle**

   Consider the action of a (relativistic) charged particle coupled to a background gauge field, A_μ, living in Minkowski space. It is governed by the action

$$S[x^\mu] = \int d\mathfrak{s} \ \left(-mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - \frac{e}{c} A_\mu(x(\mathfrak{s}))\dot{x}^\mu(\mathfrak{s}).\right)$$

   where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\mathfrak{s}}$.

   (a) **General covariance in one dimension.**
   Suppose we reparametrize the worldline according to

$$\mathfrak{s} \mapsto \tilde{\mathfrak{s}}(\mathfrak{s}).$$

   Use the chain rule to show that this change of coordinates preserves S.

Under the worldline reparametrization, the time-derivative transforms as

$$\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}} \mapsto \frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{\mathfrak{s}}} = \frac{\mathrm{d}\mathfrak{s}}{\mathrm{d}\tilde{\mathfrak{s}}}\frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}} \equiv J\frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}}.$$

(Notice that the coordinates x^μ transform as *scalars* under this transformation.) So the two terms in the Lagrangian each pick up a factor of J:

$$\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \mapsto \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{\mathfrak{s}}}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tilde{\mathfrak{s}}}} = |J|\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}}\frac{\mathrm{d}x^\nu}{\mathrm{d}\mathfrak{s}}}$$

and

$$A_\mu(x(\mathfrak{s}))\dot{x}^\mu(\mathfrak{s}) \mapsto A_\mu(x(\tilde{\mathfrak{s}}))\frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{\mathfrak{s}}} = JA_\mu(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}}.$$

This factor of J is cancelled by the transformation of the worldline measure

$$\mathrm{d}\mathfrak{s} \mapsto \mathrm{d}\tilde{\mathfrak{s}} = \frac{1}{|J|}\mathrm{d}\mathfrak{s}.$$

(Notice that if J is negative, so that the new parameter goes backwards in time, the reversal of the order of integration accounts for the extra sign.)

(b) Vary with respect to $x^\mu(\mathfrak{s})$ and find the Lorentz force law.

The variation of the kinetic term (the one with $\sqrt{\dot{x}^2}$) is as before – it gives $\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}p_\mu$. Varying the minimal-coupling term gives

$$\frac{\delta}{\delta x^\mu(\mathfrak{s})}\int \mathrm{d}\mathfrak{s}'\ \dot{x}^\nu A_\nu\left(x(\mathfrak{s}')\right) = \int \mathrm{d}\mathfrak{s}'\left(A_\nu\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}'}\left(\frac{\delta x^\nu(\mathfrak{s}')}{\delta x^\mu(\mathfrak{s})}\right) + \dot{x}^\nu(\mathfrak{s}')\frac{\delta A_\nu(x(\mathfrak{s}'))}{\delta x^\mu(\mathfrak{s})}\right)$$

Using the chain rule

$$\delta A_\nu = \frac{\partial A_\nu}{\partial x^\rho}\delta x^\rho$$

we end up with the promised Lorentz force:

$$\frac{\delta\int A}{\delta x_0^\mu(\mathfrak{s})} = -\frac{e}{c}F_{\mu\nu}\frac{\mathrm{d}x_0^\nu}{\mathrm{d}\mathfrak{s}}$$

It looks more familiar if we choose $\mathfrak{s} = t$, *i.e.* use time as the worldline parameter: $x_0^\mu = (c\mathfrak{s}, \vec{x}(\mathfrak{s}))^\mu$. Then (for any v) the $\mu = i$ component of the previous expression reduces to

$$e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}\ ,$$

the Lorentz force.

(c) Vary with respect to $A_\mu(x)$ and find the source for Maxwell's equations produced by a trajectory of this particle.

The variation of the action the charges with respect to A produces the RHS of Maxwell's equations $\frac{1}{4\pi c}\partial_\nu F^{\mu\nu} = j^\mu$, the source term. In this case, this is:

$$j^\mu(x) = \frac{\delta S_{\text{charges}}}{\delta A_\mu(x)} = \frac{\delta}{\delta A_\mu(x)}\int \mathrm{d}^4y\int \mathrm{d}\mathfrak{s}\,\delta^4(y - x_0(\mathfrak{s}))e\frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}}$$

In the second equation here, we rewrote the $\int A$ term as an integral over all space with a delta function in it; we also observed that the particle kinetic term doesn't depend on A and so doesn't contribute here.

$$j^\mu(x) = e \int_{-\infty}^{\infty} \mathrm{d}\mathfrak{s}\, \delta^{(4)}(x - x_0(\mathfrak{s})) \frac{\mathrm{d}x_0^\mu}{\mathrm{d}\mathfrak{s}}(\mathfrak{s}). \tag{1}$$

3. Show that the current obtained from $j_\mu = \frac{\delta S}{\delta A}$ is conserved as long as the worldlines do not end.

$$j_\mu(x) = \frac{e}{c} \int_{wl} \mathrm{d}\mathfrak{s} \frac{\mathrm{d}x_0^\mu(\mathfrak{s})}{\mathrm{d}\mathfrak{s}} \delta^4(x - x_0(\mathfrak{s}))$$

$$\partial^\mu j_\mu = \frac{e}{c} \int_{wl} \mathrm{d}\mathfrak{s} \frac{\mathrm{d}x_0^\mu(\mathfrak{s})}{\mathrm{d}\mathfrak{s}} \underbrace{\partial^\mu \delta^4(x - x_0(\mathfrak{s}))}_{= -\frac{\partial}{\partial x_0(\mathfrak{s})} \delta^4(x - x_0(\mathfrak{s}))}$$

Notice that this is a partial derivative, not a functional derivative. So, by the chain rule (what else do we do around here?)

$$\partial^\mu j_\mu = \frac{e}{c} \int_{wl} \mathrm{d}\mathfrak{s} \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}} \delta^4(x - x_0(\mathfrak{s}))$$

Now we use Stokes' theorem:

$$\int_{wl} \mathrm{d}\mathfrak{s} \frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}} X = X|_{\text{boundaries of the worldline}}$$

which is nothing if the worldlines don't end.

4. **Non-inertial frames**

   (a) Using the chain rule, rewrite the $D = 2 + 1$ Minkowski line element

   $$\mathrm{d}s_M^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2$$

   in polar coordinates:

   $$x = r\cos\theta, y = r\sin\theta, \tilde{t} = t.$$

   Using $\mathrm{d}\tilde{t} = \mathrm{d}t, \mathrm{d}x = \mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta, \mathrm{d}y = \mathrm{d}r\sin\theta + r\cos\theta\mathrm{d}\theta$, we find

   $$\mathrm{d}s_M^2 = -\mathrm{d}\tilde{t}^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2.$$

   (b) Rewrite $\mathrm{d}s_M^2$ in a rotating frame, where the new coordinates are

   $$\tilde{x} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} x \equiv Rx, \quad \tilde{t} = t.$$

The new differentials are :

$$\mathrm{d}\tilde{x} = R\mathrm{d}x + \partial_\theta R x \omega \mathrm{d}t.$$

$$\begin{aligned} \mathrm{d}s_M^2 &= -\mathrm{d}t^2 \left(1 - \omega^2 \left(\partial_\theta R \vec{x}\right) \cdot \left(\partial_\theta R \vec{x}\right)\right) \mathrm{d}\tilde{x}^2 + 2\omega \mathrm{d}t \vec{x} R^T \partial_\theta R \vec{x} \\ &= -\mathrm{d}t^2 \left(1 - \omega^2 \left(\tilde{x}^2 + \tilde{y}^2\right)\right) \mathrm{d}\tilde{\vec{x}}^2 + \mathrm{d}\tilde{x}^2 + 2\omega \left(\tilde{y}\mathrm{d}\tilde{x} - \tilde{x}\mathrm{d}\tilde{y}\right) \mathrm{d}t. \end{aligned} \tag{2}$$

(c) Redo the previous part in polar coordinates; that is, let the new coordinates be $(\tilde{t}, r, \theta)$ with the relations from part 4a but with $\theta = \omega t + \theta_0$ where θ_0 is the old azimuthal coordinate.

$$\mathrm{d}s_M^2 = -\mathrm{d}\tilde{t}^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta - \omega \mathrm{d}t\right)^2.$$

(d) Consider the action for a relativistic massive particle in $(D = 2+1)$ Minkowski space

$$S = mc \int \mathrm{d}\tau$$

where τ is the proper time along the worldline, $\mathrm{d}s^2 = -c^2 \mathrm{d}\tau^2$. Using the action principle, derive the centripetal force experienced by a particle using uniformly rotating coordinates, $\theta = \omega t$.

We can demonstrate the centripetal force easily by considering orbits where θ is constant; because $\theta_0 \to \theta_0 + \epsilon$ is a symmetry, we can make this choice consistently with the time evolution. The action for such configurations is

$$S = mc \int \mathrm{d}t \sqrt{1 - \dot{r}^2 - \omega^2 r^2}$$

and its variation with respect to r is

$$0 = \frac{\delta S}{\delta r(s)} = mc \int \mathrm{d}t \frac{-\dot{r}\frac{\mathrm{d}}{\mathrm{d}t}\delta(t-s) - \omega^2 r \delta(t-s)}{\sqrt{1 - \dot{r}^2 - \omega^2 r^2}} = mc \left(\frac{\mathrm{d}}{\mathrm{d}s} \frac{\dot{r}}{\sqrt{1 - \dot{r}^2 - \omega^2 r^2}} - \frac{\omega^2 r(s)}{\sqrt{1 - \dot{r}^2 - \omega^2 r^2}} \right)$$

In the non-relativistic limit, we can approximate $\sqrt{1 - \dot{r}^2 - \omega^2 r^2} \simeq 1+$ small, and this says

$$\ddot{r} = +\omega^2 r,$$

an acceleration away from the origin of rotation.

5. **Is it flat?**

Show that the two dimensional space whose metric is

$$\mathrm{d}s^2 = \mathrm{d}v^2 - v^2 \mathrm{d}u^2$$

(it is called 'Rindler space') is just two-dimensional Minkowski space

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$$

in disguise. Do this by finding the appropriate change of coordinates $x(u,v), t(u,v)$.

Notice the similarity to polar coordinates in the plane: $x = r\cos\theta, y = r\sin\theta$ ($r^2 = x^2 + y^2$). The extra minus sign for the time direction is formally accomplished by replacing $\theta = \mathbf{i}u$, which gives

$$x = v\cosh u, t = v\sinh u \quad ; \tag{3}$$

notice that these satisfy $v^2 = x^2 - t^2$. This gives the differentials $\mathrm{d}x = \mathrm{d}v\cosh u + v\sinh u\mathrm{d}u, \mathrm{d}t = \mathrm{d}v\sinh u + v\cosh u\mathrm{d}u$ which implies

$$\mathrm{d}v^2 - v^2\mathrm{d}u^2 = -\mathrm{d}t^2 + \mathrm{d}x^2.$$

Above we kind of guessed; to do this more systematically, we demand

$$\mathrm{d}t = \frac{\partial t}{\partial u}\mathrm{d}u + \frac{\partial t}{\partial v}\mathrm{d}v, \quad \mathrm{d}x = \frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v$$

plug this in to $\mathrm{d}s^2$ and equate the coefficients of each term $\mathrm{d}u^2, \mathrm{d}v^2, \mathrm{d}u\mathrm{d}v$ to get

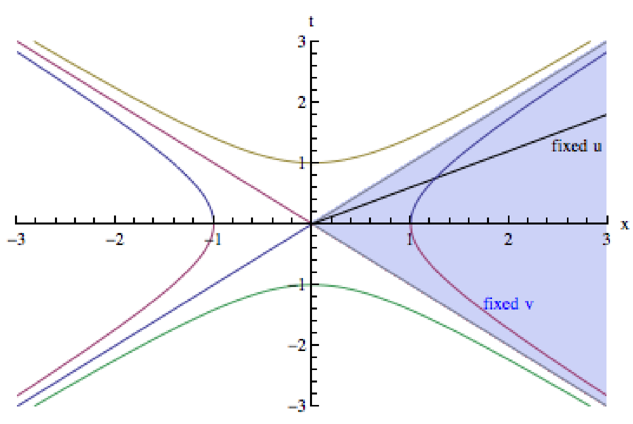
$$-\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\right)^2 = -v^2, \quad -\left(\frac{\partial t}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2 = 1, \quad -\frac{\partial t}{\partial u}\frac{\partial t}{\partial v} + \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = 0$$

The fact that the middle equation has no v dependence on the RHS suggests the separation of variables (ansatz) $t = vT(u), x = vX(u)$, in which case the middle equation becomes

$$-T'^2 + X'^2 = 1$$

which we recognize as demanding to be solved by the hyperbolic trig functions above. Notice that the change of coordinates in (3) actually only covers half of the Minkowski space. As u, v each run from $-\infty$ to ∞, x is always less than t. More directly, for $v > 0$ and $u \in (-\infty, \infty)$, we cover just the region to the right of the lightcone of the origin (shaded in the plot below – $\{x > 0, t \in (-x, x)\}$).

Notice also that curves of constant v (with $x > 0$) are the trajectories of particles experiencing uniform acceleration; their worldlines are parametrized by the boost parameter u (I previously called it Υ, the rapidity). The coordinate v is itself a Lorentz-invariant distance, just like the radial coordinate r in polar coordinates is a rotation-invariant distance.



The shaded region is the region which is causally accessible to the uniformly accelerating observer.