

General Relativity (225A) Fall 2013 Assignment 2

Posted October 3, 2013

Due Monday, October 15, 2013

1. Special relativity reminders

This problem is optional. Do it if you have to think about it or if you find it a useful reminder.

- (a) Derive the special relativity addition law for coordinate velocities $\frac{dx^i}{dt}$ using the Lorentz transformation $\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu$.
- (b) Beginning with the assumption that the speed of light is constant, derive the time dilation effect as follows. Build a clock by bouncing a light ray between mirrors. Watch the clock tick in a frame which is boosted transverse to the motion of the light.

2. Electric-magnetic (Poincaré) duality

The vacuum Maxwell equations

$$0 = \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}, 0 = \partial_\mu F^{\mu\nu}$$

are invariant under the exchange of the electric and magnetic fields $(\vec{E}, \vec{B}) \mapsto (\vec{B}, -\vec{E})$. This is called ‘electric-magnetic duality’.¹

Show that the duality transformation can be written in a covariant manner as

$$F_{\mu\nu} \mapsto (\star F)_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}.$$

Verify that the (vacuum) Maxwell equations are invariant under this replacement.

Check that doing the duality map twice gives -1 : $\star(\star F) = -F$.

- 3. Show that $\vec{E}^2 - \vec{B}^2$ and $\vec{E} \cdot \vec{B}$ are respectively scalar and pseudo-scalar under the Lorentz group by expressing them in terms of $F_{\mu\nu}$.

4. Lorentz transformations of the EM fields

This problem is to convince yourself that the formula we gave for the Lorentz transformation of the Maxwell field strength:

$$F_{\mu\nu}(x) \mapsto \tilde{F}_{\mu\nu}(\tilde{x}), \quad F_{\mu\nu}(x) = \Lambda_\mu^\rho \Lambda_\nu^\sigma \tilde{F}_{\rho\sigma}(\tilde{x}) \quad (1)$$

is correct, *i.e.* it is a Lorentz tensor.

¹In the presence of electric charges, the duality transformation is only a symmetry if we allow for magnetic monopole charge; the duality transformation then must exchange electric charges and magnetic charges.

- (a) Consider the case where Λ is a Lorentz boost in some direction. What transformation laws does this imply for the components of the electric field along the boost E_{\parallel} ? And the component transverse to the boost E_{\perp} ?
 - (b) Describe a charge configuration which creates a constant electric field in some region of space.
 - (c) For the electric field case, boost the charge configuration along the direction of the field and transverse to the direction of the field, and find the new fields that they create. You may assume the experimental fact that electric charge is a scalar quantity. Convince yourself that they agree with the answers from the fancy formula (1), in the case where there is no magnetic field.
 - (d) Admit to yourself that magnetic charge could exist. Use this fact and the idea in problem 2 to figure out how B_{\parallel} and B_{\perp} transform in the absence of an electric field.
 - (e) Consider current in a neutral wire made from negative charge per unit length λ ('electrons') moving at velocity v_{drift} superposed with a static positive charge density. What field does this create? By boosting to the rest frame of the negative charges, find the transformation rule for E_{\perp}, B_{\perp} when both are nonzero.
5. **Easy.** Consider the worldline of a particle which sits at the origin for all time. In lecture we showed that a tangent vector to this curve has negative proper length-squared,

$$\eta_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} < 0 ,$$

where τ is an arbitrary parameter along the worldline. That is, the tangent vector is timelike. How do I know that a Lorentz boost cannot take this vector to one which is spacelike?

6. Light-cone accounting

- (a) Show that any timelike vector u^{μ} for which $u^0 > 0$ and $u^{\mu}u_{\mu} = -1$ is the four-velocity of some worldline.
- (b) Use the previous result to show that for any timelike vector v^{μ} there is a Lorentz frame in which v^{μ} has zero spatial components.
- (c) Show that the sum of any two **orthogonal**² spacelike vectors is spacelike.
- (d) Show that a (nonzero) timelike vector and a (nonzero) null vector cannot be orthogonal.

²Note that the assumption of orthogonal is necessary: otherwise $V = (v^0, \vec{v})$ and $\tilde{V} \equiv (v^0, -\vec{v})$ is a counterexample.