University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 2 – Solutions

Posted October 3, 2013 **Due Monday, October 15, 2013**

1. **Special relativity reminders**

   This problem is optional. Do it if you have to think about it or if you find it a useful reminder.

   (a) Derive the special relativity addition law for coordinate velocities $\frac{dx^i}{dt}$ using the Lorentz transformation $\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu$.

   The infinitesimal coordinate lengths are related by

$$dx^\mu = \Lambda^\mu_\nu dx^\nu$$

   where Λ is a boost; wlog take the relative velocity β between the frames in the x direction. And for simplicity let's take $v = dx/dt$ in this direction, too.

$$\tilde{v} = \frac{d\tilde{x}}{d\tilde{t}} = \frac{\gamma dx + \gamma\beta dt}{\gamma dt + \gamma\beta dx} = \frac{\frac{dx}{dt} + \beta}{1 + \beta\frac{dx}{dt}} = \frac{v + \beta}{1 + \beta v}.$$

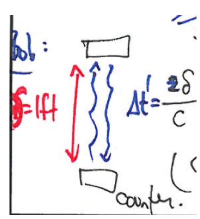
   In this way of doing it we didn't need to worry about the fact that along the trajectory there is a relation between dx and dt; if we wanted to do it by

$$\tilde{v} = \frac{d}{d\tilde{t}}\Lambda^x_\nu dx^\nu = \left(\Lambda^{-1}\right)^\rho_0 \frac{d}{dx^\rho}\Lambda^x_\nu dx^\nu$$

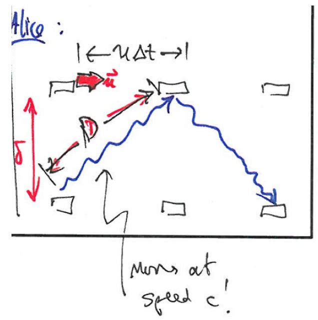
   we would have to worry about this.

   (b) Beginning with the assumption that the speed of light is constant, derive the time dilation effect as follows. Build a clock by bouncing a light ray between mirrors. Watch the clock tick in a frame which is boosted transverse to the motion of the light.

   Compare Bob's experience in the rest frame of the clock (call the distance between the mirrors δ; the time for light to go from one mirror to the other is $\Delta t' = \delta/c$):



with that of Alice who sees the clock move to the right at speed u:



it ticks more slowly!

$D > \delta$

$D^2 = \delta^2 + u^2 \Delta t^2 . \quad D = c\Delta t$

$\to \Delta t^2 (c^2 - u^2) = \delta^2$

$\Delta t = \frac{\delta}{c} \frac{1}{\sqrt{1 - u^2/c^2}} = \Delta t' \gamma$

$> \Delta t'.$

$\boxed{\text{time dilation}}$

2. **Electric-magnetic (Poincaré) duality**

   The vacuum Maxwell equations

$$0 = \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}, 0 = \partial_\mu F^{\mu\nu}$$

   are invariant under the exchange of the electric and magnetic fields $(\vec{E}, \vec{B}) \mapsto (-\vec{B}, \vec{E})$. This is called 'electric-magnetic duality'. [1]

   Show that the duality transformation can be written in a covariant manner as

$$F_{\mu\nu} \mapsto (\star F)_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\cdot\cdot} F^{\cdot\cdot}.$$

   If I identify $F_{i0} = E_i$ and $F_{ij} = \epsilon_{ijk} B^k$ (so $B^i = \frac{1}{2}\epsilon^{ijk} F_{jk}$) as I did in lecture, and raise spatial indices with δ^{ij}, so $E^i = \delta^{ij} E_j$, then I should have an extra minus sign (relative to the original problem statement):

$$(\star F)_{ij} = \frac{1}{2}\epsilon_{ij\rho\sigma} F^{\rho\sigma} = \frac{1}{2}\epsilon_{ijk0} \left(F^{k0} - F^{0k} \right) = -\epsilon_{ijk} F^{k0} = +\epsilon_{ijk} \left(F_{k0} \right) = +\epsilon_{ijk} E_k$$

   (so $B_k \to \star B_k = +E_k$) and

$$(\star F)_{i0} = \frac{1}{2}\epsilon_{i0\rho\sigma} F^{\rho\sigma} = \frac{1}{2}\epsilon_{i0jk} F^{jk} = -\frac{1}{2}\epsilon_{ijk} F^{jk} = -\frac{1}{2}\epsilon_{ijk}\epsilon^{jkl} B_l = -\frac{1}{2}\epsilon_{ijk}\epsilon^{ljk} B_l = -\frac{1}{2} 2 B_i = -B_i$$

   (so $E_k \to \star E_k = +B_k$). Notice that $(E, B) \to -(E, B)$ is also a symmetry of Max's equations (namely charge conjugation) so the overall sign of the transformation shouldn't trouble us too much – both are symmetries.

   Verify that the (vacuum) Maxwell equations are invariant under this replacement.

   The Bianchi identity becomes

$$0 \;=\; \epsilon^{\mu\nu\rho\sigma} \partial_\nu \left(\star F\right)_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \partial_\nu \epsilon_{\rho\sigma\alpha\beta} F^{\alpha\beta}$$

[1] In the presence of electric charges, the duality transformation is only a symmetry if we allow for magnetic monopole charge; the duality transformation then must exchange electric charges and magnetic charges.

$$
\begin{aligned}
&= \frac{1}{2} \underbrace{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\rho\sigma\alpha\beta}}_{=-2\left(\delta^\mu_\alpha\delta^\nu_\beta-(\mu\leftrightarrow\nu)\right)} \partial_\nu F^{\alpha\beta} \\
&= -2\frac{1}{2}\partial_\nu\left(F^{\alpha\beta}-F^{\beta\alpha}\right) = 2\partial_\nu F^{\alpha\beta} \qquad (1)
\end{aligned}
$$

which is the vacuum Maxwell equation. The vacuum Maxwell equation becomes

$$0 = \partial_\nu \left(\star F\right)^{\mu\nu} = \frac{1}{2}\partial_\nu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

which is the Bianchi identity.

Check that doing the duality map twice gives -1: $\star\left(\star F\right) = -F$.

$$\star\left(\star F\right)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\cdot\cdot}\left(\star F\right)^{\cdot\cdot} = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\alpha\beta}F_{\alpha\beta}$$

But now we can use the identity

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\alpha\beta} = a\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\right)$$

where a a constant that I determine by tracing both sides with δ^μ_α:

$$\underbrace{\epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\mu\beta}}_{=-3!\delta^\beta_\nu} = a(4-1)\delta^\beta_\nu$$

so $a=-2$. Therefore:

$$\star\left(\star F\right)_{\mu\nu} = -2\frac{1}{4}\left(F_{\mu\nu}-F_{\nu\mu}\right) = -F_{\mu\nu}.$$

3. Show that $\vec{E}^2-\vec{B}^2$ and $\vec{E}\cdot\vec{B}$ are respectively scalar and pseudo-scalar under the Lorentz group by expressing them in terms of $F_{\mu\nu}$.

$$\frac{1}{2}F_{\mu\nu}F_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma} = \left(\vec{E}^2-\vec{B}^2\right)$$

is a scalar since all tensor indices are contracted with the $\eta_{\mu\nu}$ Minkowski metric.

$$\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{1}{4}\left(\star F\right)_{\mu\nu}F^{\mu\nu} = \vec{E}\cdot\vec{B}$$

is a pseudo scalar since some of the indices are contracted with the Levi-Civita symbol.

4. **Lorentz transformations of the EM fields**

This problem is to convince yourself that the formula we gave for the Lorentz transformation of the Maxwell field strength:

$$F_{\mu\nu}(x)\mapsto\tilde{F}_{\mu\nu}(\tilde{x})\ ,\quad F_{\mu\nu}(x) = \Lambda^\rho_\mu\Lambda^\sigma_\nu\tilde{F}_{\rho\sigma}(\tilde{x}) \qquad (2)$$

is correct, *i.e.* it is a Lorentz tensor.

(a) Consider the case where Λ is a Lorentz boost in some direction. What transformation laws does this imply for the components of the electric field along the boost $E_\parallel$? And the component transverse to the boost $E_\perp$?

This is a problem in unpacking, but an important one. Take the boost along x, so that it takes the form of eqn (15) of the lecture notes. Then y and z indices don't acquire matrices. For example, $E_y = F_{y0}$ becomes

$$\tilde{F}_{y0} = \Lambda_y^\mu \Lambda_0^\nu \; F_{\mu\nu} = \underbrace{\Lambda_y^y}_{=1} \left(\Lambda_0^0 F_{y0} + \Lambda_{0^x} F_{yx} \right) = \gamma E_y - \gamma v B_z$$

which is the same (restoring a factor of c) as

$$\vec{E}_\perp \mapsto \gamma \left(\vec{E}_\perp + \frac{\vec{v}}{c} \times \vec{B}_\perp \right).$$

The longitudinal bit of E is $E_x = F_{x0}$ which becomes

$$\tilde{F}_{x0} = \Lambda_x^\mu \Lambda_0^\nu \; F_{\mu\nu} = \Lambda_x^x \Lambda_0^0 F_{x0} + \Lambda_x^0 \Lambda_0^x F_{0x} = \gamma^2 \left(1 - \frac{v^2}{c^2} \right) F_{x0} = F_{x0}$$

– it doesn't change. The results for $\vec{B}$ are of the same form with $\vec{E} \to \vec{B}, \vec{B} \to -\vec{E}$.

$$\tilde{\vec{B}}_\perp = \gamma \left(\vec{B}_\perp - \frac{\vec{v}}{c} \times \vec{E}_\perp \right) \;\; , \quad \tilde{\vec{B}}_\parallel = \vec{B}_\parallel .$$

In fact, we can deduce this immediately from the duality transformation.

(b) Describe a charge configuration which creates a constant electric field in some region of space.

Parallel plates with opposite uniform charge!

(c) For the electric field case, boost the charge configuration along the direction of the field and transverse to the direction of the field, and find the new fields that they create. You may assume the experimental fact that electric charge is a scalar quantity. Convince yourself that they agree with the answers from the fancy formula (2), in the case where there is no magnetic field.

This is really two problems. If we boost a capacitor along the plates, the charge density gets contracted along the boost, so it goes up by a factor of γ.

$$E_\perp = 4\pi\sigma \implies E_\perp \mapsto \gamma E_\perp .$$

If we boost transverse to the plates, the plates are closer together in the new frame, but the charge is the same, so:

$$E_\parallel \mapsto E_\parallel .$$

(d) Admit to yourself that magnetic charge could exist. Use this fact and the idea in problem 2 to figure out how $B_\parallel$ and $B_\perp$ transform in the absence of an electric field.

Just imagine a magnetic capacitor – rather, just parallel sheets of uniform magnetic charge. This produces a uniform magnetic field in between. Never mind what the plates are made of. You can get the $B_\parallel$ answer pretty easily by boosting a solenoid, but $B_\perp$ is harder to get that way.

(e) Consider current in a neutral wire made from negative charge per unit length λ ('electrons') moving at velocity v_{drift} superposed with a static positive charge density. What field does this create? By boosting to the rest frame of the negative charges, find the transformation rule for $E_\perp, B_\perp$ when both are nonzero.

In the rest from of the 'electrons', the linear charge density of the negative charges is dilated to $\tilde{\lambda}_- = -\lambda/\gamma$ ($\gamma = \frac{1}{\sqrt{1-v_{\text{drift}}^2/c^2}}$) while that of the positive charges is contracted to $\tilde{\lambda}_- = +\lambda\gamma$, since they now move with velocity $-v_{\text{drift}}$. The magnetic field There is therefore a net charge density, which is the sum of the two

$$\tilde{\lambda}_t = \tilde{\lambda}_+ + \tilde{\lambda}_- = \lambda\left(\gamma - \gamma^{-1}\right) = \gamma\lambda\frac{v^2}{c^2} > 0.$$

This produces an electric field pointing away from the wire

$$\tilde{\vec{E}} = \hat{r}\frac{2\lambda_t'}{r}.$$

This is related to $\vec{B} = \hat{\varphi}\frac{2\lambda}{c^2 r}$ by $\tilde{\vec{E}} = \gamma\frac{\vec{v}}{c}\times\vec{B}$. The magnetic field changes too: the current density is $\tilde{\lambda}_- 0 + \tilde{\lambda}_+(-v_{\text{drift}}) = +\gamma\lambda v_{\text{drift}}$, so the new magnetic field is $\gamma B_\perp$. Combining this with the EM dual case of a magnetic current, we learn that

$$\tilde{\vec{E}}_\perp = \gamma\left(\vec{E}_\perp + \frac{\vec{v}}{c}\times\vec{B}_\perp\right)\tilde{\vec{B}}_\perp = \gamma\left(\vec{B}_\perp - \frac{\vec{v}}{c}\times\vec{E}_\perp\right)$$

as we found above.

5. **Easy.** Consider the worldline of a particle which sits at the origin for all time. In lecture we showed that a tangent vector to this curve has negative proper length-squared,

$$\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} < 0\ ,$$

where τ is an arbitrary parameter along the worldline. That is, the tangent vector is timelike. How do I know that a Lorentz boost cannot take this vector to one which is spacelike?

The proper length is Lorentz invariant.

6. **Light-cone accounting**

(a) Show that any timelike vector u^μ for which $u^0 > 0$ and $u^\mu u_\mu = -1$ is the four-velocity of some worldline.

We may consider

$$-1 = u^\mu u_\mu = -(u^0)^2 + \vec{u} \cdot \vec{u}$$

as a differential equation for a path $x^\mu(\mathfrak{s})$ whose instantaneous velocity is $\frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}}$. More explicitly, we can choose $\mathfrak{s} = t$ so the proper length along a segment of the path is

$$\mathrm{d}\tau = \sqrt{-\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} = \sqrt{1 - \vec{v}^2}\mathrm{d}t, \quad \vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}.$$

Then the four-velocity of this worldline is

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-v^2}}(1, \vec{v})^\mu$$

which indeed has

$$u^\mu u_\mu = \frac{1}{1 - \vec{v}^2}\left(-1 + \vec{v}^2\right) = -1$$

and $u^0 = \frac{1}{\sqrt{1-v^2}} > 0$. So the relevant path has

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{v} = \frac{\vec{u}}{u^0}$$

which we could regard as a differential equation for $\vec{x}(t)$, treating u^μ as a constant. OK hold on that sounded really complicated. What I mean is: you can choose the path to be a straight line with constant 4-velocity u^μ.

(b) Use the previous result to show that for any timelike vector v^μ there is a Lorentz frame in which v^μ has zero spatial components.

Our goal is to find a Lorentz transformation which sets to zero the spatial components of v. WLOG assume $v^0 > 0$, since we can reverse it's sign by an 'improper' Lorentz transformation. Let $u^\alpha = \frac{v^\alpha}{\sqrt{v^\alpha v_\alpha}}$. This u satisfies the assumptions of the previous result. Our statement now is just that every particle has a rest frame.
This answer is not as satisfying as I thought it would be. Here is a more direct argument. First, WLOG, we can rotate the spatial components of v so that in only points in the x direction (recall that a rotation is a special case of a Lorentz transformation). So now the problem we are trying to solve is to find a $D = 1+1$ boost by velocity u such that

$$\Lambda v = \begin{pmatrix} \gamma & \gamma u \\ \gamma u & \gamma \end{pmatrix} \begin{pmatrix} v^0 \\ v_x \end{pmatrix} = \begin{pmatrix} w^0 \\ 0 \end{pmatrix} \tag{3}$$

given that

$$0 > v^2 = -(v^0)^2 + \vec{v}^2 = -(v^0)^2 + v_x^2 = -(w^0)^2. \tag{4}$$

Let's just solve the bottom equation of (3) for u: it says

$$\gamma v^0 + \gamma u v_x = 0$$

which says

$$u = -\frac{v^0}{v_x}$$

which is smaller in magnitude than the speed of light by the assumption that v is timelike (4).

(c) Show that the sum of any two orthogonal[2] spacelike vectors is spacelike.

V, W spacelike means their norms are positive: $V^2 \equiv \eta_{\mu\nu} V^\mu V^\nu > 0$, $W^2 > 0$. V, W ortthogonal means $V \cdot W \equiv \eta_{\mu\nu} V^\mu W^\nu = 0$. The sum $V + W$ has norm

$$(V+W)^2 = \underbrace{\eta_{\mu\nu} V^\mu V^\mu}_{>0} + 2 \underbrace{\eta_{\mu\nu} V^\mu W^\nu}_{=0} + \underbrace{\eta_{\mu\nu} W^\mu W^\mu}_{>0} > 0.$$

(d) Show that a (nonzero) timelike vector and a (nonzero) null vector cannot be orthogonal.

A timelike vector T has $T^2 < 0$, and a null vector has $N^2 = 0$. Their inner product $\eta_{\mu\nu} T^\mu N^\nu$ is Lorentz invariant. By part (b) there's a Lorentz frame where T has zero spatial components; N is still a nonzero null vector in that frame. In that frame

$$\eta_{\mu\nu} T^\mu N^\nu = -T^0 N^0 \ .$$

If T and N orthogonal, then, we must have either $T^0 = 0$ (in which case T is zero which it's not) or $N^0 = 0$. But if $N^0 = 0$ then clearly N can't be null, since

$$0 \overset{N \text{ is Null}}{=} N^2 = \eta_{\mu\nu} N^\mu N^\nu = -(N^0)^2 + \vec{N}^2 \leq \vec{N}^2$$

[2] Note that the assumption of orthogonal is necessary: otherwise $V = (v^0, \vec{v})$ and $\tilde{V} \equiv (v^0, -\vec{v})$ is a counterexample.