University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 1

Posted September 29, 2013 **Due Monday, Oct 7, 2013**

1. Show that
$$\epsilon_{lmn} R_{li} R_{mj} R_{nk} = \det R \epsilon_{ijk}$$
for any 3×3 matrix R.
(Recall that the determinant is a scalar function of square matrices which is odd under interchange of rows and columns and has $\det \mathbb{1} = 1$. And ϵ_{ijk} is the completely antisymmetric collection of numbers with $\epsilon_{123} = 1$.)

2. Show that the Maxwell equations (in Minkowski space)
$$\epsilon_{ijk} \partial_j E_k + \frac{1}{c} \partial_t B_i = 0, \quad \partial_i H_i = 0$$
$$\epsilon_{ijk} \partial_j B_k - \frac{1}{c} \partial_t E_i = \frac{4\pi}{c} J_i, \quad \partial_i E_i = 4\pi\rho.$$
are invariant under the following set of transformations:
$$\begin{aligned} x^i &\mapsto \tilde{x}^i \equiv R_{ij} x^j, \qquad R \in O(3) \\ E_i(x) &\mapsto \tilde{E}_i(\tilde{x}) = R_{ij} E_j(x) \\ B_i(x) &\mapsto \tilde{B}_i(\tilde{x}) = \det R R_{ij} B_j(x) \\ \rho(x) &\mapsto \tilde{\rho}(\tilde{x}) = \rho(x), \quad J_i(x) \mapsto \tilde{J}_i(\tilde{x}) = R_{ij} J_j(x). \end{aligned} \tag{1}$$
That is, $\vec{E}$ is a polar vector and $\vec{B}$ is an axial vector. Recall that an $O(3)$ matrix satisfies $R^T R = \mathbb{1}$.

3. Prove the identity
$$\epsilon_{ijk} \epsilon_{lmn} = \det \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix} .$$
Use this identity to show that
$$\epsilon_{ijk} \epsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}.$$

4. **Lorentz contraction exercise** [from Brandenberger]

   (a) Suppose frame S' moves with velocity v relative to frame S. A projectile in frame S' is fired with velocity v' at an angle θ' with respect to the forward direction of motion ($\vec{v}$). What is this angle θ measured in S? What if the projectile is a photon?

(b) An observer A at rest relative to the fixed distant stars sees an isotropic distribution of stars in a galaxy which occupies some region of her sky. The number of stars seen within an element of solid angle $\mathrm{d}\Omega$ is

$$P\mathrm{d}\Omega = \frac{N}{4\pi}\mathrm{d}\Omega$$

where N is the total number of stars that A can see. Another observer B moves uniformly along the z axis relative to A with velocity v. Letting θ' and φ' be respectively the polar (with respect to $\hat{z}$) and azimuthal angle in the inertial frame of B, what is the distribution function $P'(\theta', \varphi')$ such that $P'(\theta', \varphi')\mathrm{d}\Omega'$ is the number of stars seen by B in the solid angle $\mathrm{d}\Omega' = \sin\theta'\mathrm{d}\theta'\mathrm{d}\varphi'$.

(c) Check that when integrating the distribution function over the sphere in the coordinates of B you obtain N! Discuss the behavior of the distribution P' in the limiting cases when the velocity v goes to 0 or to 1.

5. Show that the half of the Maxwell equations

$$0 = \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}$$

is invariant under the general coordinate transformation,

$$x^\mu \mapsto \tilde{x}^\mu = f^\mu(x), \quad F_{\mu\nu}(x) \mapsto \tilde{F}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}F_{\rho\sigma}(x)$$

for an arbitrary $f^\mu(x)$ with non-zero Jacobian.

The following problems are optional.

6. **Eötvös**

   What is the optimal latitude at which to perform the Eötvös experiment?

7. **Poincaré group**

   Show that the Poincaré group satisfies all the properties of a group. (That is: it has an identity, it is closed under the group law, it is associative, and every element has an inverse in the group.)