University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 1 – Solutions

Posted September 29, 2013 **Due Monday, Oct 7, 2013**

1. Show that

$$\epsilon_{lmn} R_{li} R_{mj} R_{nk} = \det R \epsilon_{ijk}$$

for any 3×3 matrix R.

(Recall that the determinant is a scalar function of square matrices which is odd under interchange of rows and columns and has $\det \mathbb{1} = 1$. And ϵ_{ijk} is the completely antisymmetric collection of numbers with $\epsilon_{123} = 1$.)

Since ϵ is completely antisymmetric, this is only one independent equation, which we may take to be $ijk = 123$:

$$\epsilon_{lmn} R_{l1} R_{m2} R_{n3} = \det R \epsilon_{123} = \det R. \tag{1}$$

This is often taken as the *definition* of the det. In terms of the definition in terms of minors, the RHS is

$$\begin{aligned} \det R &= R_{11}(R_{22}R_{33} - R_{23}R_{32}) - R_{12}(R_{21}R_{33} - R_{31}R_{23}) + R_{13}(R_{21}R_{32} - R_{22}R_{31}) \\ &= \sum_{ijk} R_{i1} R_{j2} R_{k3} \begin{cases} +1 \text{ if } ijk \text{ is an even perm. of } 123 \\ -1 \text{ if } ijk \text{ is an odd perm. of } 123 \end{cases} \\ &= \epsilon_{ijk} R_{i1} R_{m2} R_{n3} \end{aligned} \tag{2}$$

This sort of unpacking is a good exercise but gets tedious in higher dimensions. It's useful to notice that given (1), and summing over the $3! = 6$ nonzero terms, the requested equation is equivalent to

$$\frac{1}{3!} \epsilon_{lmn} \epsilon ijk R_{li} R_{mj} R_{nk} = \det R \ .$$

The LHS of this equation satisfies all of the defining properties of the determinant.

2. Show that the Maxwell equations (in Minkowski space)

$$\epsilon_{ijk} \partial_j E_k + \frac{1}{c} \partial_t B_i = 0, \quad \partial_i H_i = 0$$

$$\epsilon_{ijk} \partial_j B_k - \frac{1}{c} \partial_t E_i = \frac{4\pi}{c} J_i, \quad \partial_i E_i = 4\pi\rho.$$

are invariant under the following set of transformations:

$$x^i \ \mapsto \ \tilde{x}^i \equiv R_{ij} x^j, \qquad R \in O(3)$$

$$\begin{array}{rcl} E_i(x) & \mapsto & \tilde{E}_i(\tilde{x}) = R_{ij} E_j(x) \\ B_i(x) & \mapsto & \tilde{B}_i(\tilde{x}) = \det R R_{ij} B_j(x) \\ \rho(x) & \mapsto & \tilde{\rho}(\tilde{x}) = \rho(x), \quad J_i(x) \mapsto \tilde{J}_i(\tilde{x}) = R_{ij} J_j(x). \end{array} \tag{3}$$

That is, $\vec{E}$ is a polar vector and $\vec{B}$ is an axial vector. Recall that an $O(3)$ matrix satisfies $R^T R = 1\!\!1$.

Under the given transformation, the chain rule implies that

$$\frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial \tilde{x}^i} = (R^{-1})_{ji} \partial_j = R_{ij} \partial_j \ .$$

The LHS of Faraday maps according to

$$\epsilon_{ijk} \partial_j E_k + \frac{1}{c} \partial_t B_i \mapsto \epsilon_{ijk} R_{jl} \partial_l R_{km} E_m + \frac{1}{c} \partial_t R_{ij} B_j \det R$$

Multiply the BHS by R_{in} (it's invertible so this is WLOG) to get

$$\begin{array}{rrcl} & 0 & = & \epsilon_{ijk} R_{in} R_{jl} \partial_l R_{km} E_m + \frac{1}{c} \partial_t R_{in} R_{ij} B_j \det R \\ & & = & \epsilon_{ijk} R_{in} R_{jl} R_{km} \partial_l E_m + \frac{1}{c} \partial_t \left(R^T\right)_{ni} R_{ij} B_j \det R \\ & & = & \det R \epsilon_{nlm} \partial_l E_m + \frac{1}{c} \partial_t \delta_{nj} B_j \det R \\ \Leftrightarrow & 0 & = & \epsilon_{nlm} \partial_l E_m + \frac{1}{c} \partial_t B_j \end{array} \tag{4}$$

(in the second step we used the fact that the rotation matrices are constant in space, and in the third step we used the result of the first problem) which is the original equation. The LHS of Ampere maps according to

$$\epsilon_{ijk} \partial_j B_k - \frac{1}{c} \partial_t E_i \mapsto \det R \epsilon_{ijk} R_{jl} \partial_l R_{km} B_m - \frac{1}{c} \partial_t R_{ij} E_j \ .$$

Now the RHS is a nonzero vector $j_i \mapsto R_i^j j_j$. Again multiply the BHS by R_{in} to get

$$\begin{array}{rcl} \frac{4\pi}{c} j_n & = & \det R \epsilon_{ijk} R_{in} R_{jl} \partial_l R_{km} B_m - \frac{1}{c} \partial_t R_{in} R_{ij} E_j \\ & = & \det R \epsilon_{ijk} R_{in} R_{jl} R_{km} \partial_l B_m - \frac{1}{c} \partial_t \left(R^T\right)_{ni} R_{ij} E_j \\ & = & (\det R)^2 \epsilon_{nlm} \partial_l E_m - \frac{1}{c} \partial_t \delta_{nj} E_j \\ & = & \epsilon_{nlm} \partial_l B_m - \frac{1}{c} \partial_t E_j \end{array} \tag{5}$$

which is the original equation. At the last step we used the fact that $R^T R = 1\!\!1 \implies \det R^2 = 1$. The equations involving the divergence are simpler:

$$4\pi\rho = \vec{\partial} \cdot \vec{E} \mapsto 4\pi\rho = \tilde{\vec{\partial}} \cdot \tilde{\vec{E}} = \partial_k R_{ik} R_{ij} E_j = \partial_k \delta_{kj} E_j = \partial_k E_k$$

The LHS is a scalar under a rotation (though not a boost). The no-monopoles equation becomes

$$0 = \vec{\partial} \cdot \vec{B} \mapsto 0 = \tilde{\vec{\partial}} \cdot \tilde{\vec{E}} = \det R \partial_k R_{ik} R_{ij} B_j = \det R \partial_k \delta_{kj} \tilde{E}_j = \det R \partial_k B_k$$

The magnetic charge density would have to be a pseudoscalar.

3. Prove the identity

$$\epsilon_{ijk}\epsilon_{lmn} = \det \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix} . \tag{6}$$

Use this identity to show that

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

(6) is really only one equation: unless ijk are all different the LHS vanishes, and any such ijk can be obtained from $ijk = 123$ by antisymmetry. The same holds for lmn. By the defining properties of the det, the RHS has the same properties: if any of ijk are the same, the matrix on the RHS has linearly-dependent rows and therefore has zero det; if any of lmn are the same, the columns are linearly dependent. So we can set $ijk = 123, lmn = 123$, and the equation says

$$\epsilon_{123}\epsilon_{123} = 1 = \det \mathbb{1}$$

which is the other defining property of the det.
Here is a neat trick which I learned from Elizabeth Wicks: Use the identity $\det A \det B = \det AB$ with

$$A = \begin{pmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{pmatrix}, \quad B = \begin{pmatrix} \delta_{\ell 1} & \delta_{m1} & \delta_{n1} \\ \delta_{\ell 2} & \delta_{m2} & \delta_{n2} \\ \delta_{\ell 3} & \delta_{m3} & \delta_{n3} \end{pmatrix},$$

for which we know from problem 1 that

$$\epsilon_{ijk}\epsilon_{\ell mn} = \det A \det B = \det \begin{pmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{pmatrix} \begin{pmatrix} \delta_{\ell 1} & \delta_{m1} & \delta_{n1} \\ \delta_{\ell 2} & \delta_{m2} & \delta_{n2} \\ \delta_{\ell 3} & \delta_{m3} & \delta_{n3} \end{pmatrix} = \begin{pmatrix} \delta_{ia}\delta_{al} & \delta_{ia}\delta_{am} & \delta_{ia}\delta_{an} \\ \delta_{ja}\delta_{al} & \delta_{ja}\delta_{am} & \delta_{ja}\delta_{an} \\ \delta_{ka}\delta_{al} & \delta_{ka}\delta_{am} & \delta_{ka}\delta_{an} \end{pmatrix}$$

which is the RHS of (6).
To get the other relation, set $k = n$ and sum. This gives

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmk} &= \delta_{il}\left(\delta_{jm}\delta_{kk} - \delta_{jk}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kk} - \delta_{jk}\delta_{kl}\right) + \delta_{ik}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) \\ &= \delta_{il}\left((3-1)\delta_{jm}\right) - \delta_{im}\left((3-1)\delta_{lj}\right) + \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} \\ &= (3-1-1)\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} . \end{aligned} \tag{7}$$

I guess there is a less explicit route using the the symmetries and evaluating special cases, but sometimes it's good to just unpack the damn thing.

4. **Lorentz contraction exercise** [from Brandenberger]

In retrospect, this problem should be called: "What do you see when you look out the windows of a fast-moving spaceship?"

(a) Suppose frame S' moves with velocity v relative to frame S. A projectile in frame S' is fired with velocity v' at an angle θ' with respect to the forward direction of motion ($\vec{v}$). What is this angle θ measured in S? What if the projectile is a photon?

An (un-normalized) velocity 4-vector for the projectile in frame S' is

$$v' \propto \begin{pmatrix} \mathrm{d}t' \\ \mathrm{d}x_1' \\ \mathrm{d}x_2' \\ \mathrm{d}x_3' \end{pmatrix} \propto \begin{pmatrix} 1 \\ v'\cos\theta' \\ v'\sin\theta' \\ 0 \end{pmatrix}$$

(The normalization doesn't matter because we can determine angles by taking ratios of components.) This is related to the velocity 4-vector for the projectile seen in frame S by the Lorentz boost :

$$v = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} v' = \begin{pmatrix} \gamma\,(1 + v'\cos\theta' v) \\ \gamma\,(v + v'\cos\theta') \\ v'\sin\theta' \\ 0 \end{pmatrix}$$

$\gamma = \frac{1}{\sqrt{1-v^2}}$.

$$\tan\theta = \frac{\mathrm{d}x_2/\mathrm{d}t}{\mathrm{d}x_1/\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{v'\sin\theta'}{\gamma(v + v'\cos\theta')}.$$

If the projectile is a photon, $v' = 1$. In this case

$$\tan\theta = \frac{\sin\theta'}{\gamma\,(v + \cos\theta')}.$$

This formula leads to the notion of the *apparent velocity* of an object, which can actually be larger than c and which is observed in the study of high-energy astrophysical objects like active galactic nuclei.

(b) An observer A at rest relative to the fixed distant stars sees an isotropic distribution of stars in a galaxy which occupies some region of her sky. The number of stars seen within an element of solid angle $\mathrm{d}\Omega$ is

$$P\mathrm{d}\Omega = \frac{N}{4\pi}\mathrm{d}\Omega$$

where N is the total number of stars that A can see. Another observer B moves uniformly along the z axis relative to A with velocity v. Letting θ' and φ' be respectively the polar (with respect to $\hat{z}$) and azimuthal angle in the inertial frame of B, what is the distribution function $P'(\theta', \varphi')$ such that $P'(\theta', \varphi')\mathrm{d}\Omega'$ is the number of stars seen by B in the solid angle $\mathrm{d}\Omega' = \sin\theta'\mathrm{d}\theta'\mathrm{d}\varphi'$.

$\varphi' = \varphi$, since the z axes are aligned. From the previous part of the problem, the relevant projectiles are photons, so the angles are related by

$$\tan\theta = \frac{\sin\theta'}{\gamma\,(v + \cos\theta')}. \tag{8}$$

Since the total number of stars is the same, and the limits of integration are the same ($\theta = 0 \implies \theta' = 0, \theta = \pi \implies \theta' = \pi$), we have

$$N = \int_0^{2\pi} \mathrm{d}\varphi' \int_0^{\pi} \mathrm{d}\theta' \sin\theta' \cdot P'(\theta', \varphi') = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta \cdot P(\theta, \varphi).$$

A more refined and also true statement is that the number of stars in any solid angle is the same in the two frames:

$$P'(\theta', \varphi') \sin\theta' \mathrm{d}\theta' = P(\theta, \varphi) \sin\theta \mathrm{d}\theta = \frac{N}{4\pi} \sin\theta \mathrm{d}\theta.$$

Here we used the fact that the azimuthal angles are the same, and in the last step we used the 'isotropic' datum. Now we have to have a little chain rule party to find

$$P'(\theta', \varphi') = \frac{N}{4\pi} \frac{\mathrm{d}\theta}{\mathrm{d}\theta'} \frac{\sin\theta}{\sin\theta'}$$

as a function of θ'. One way to do this is to solve (8) for $\cos\theta$ and differentiate with respect to θ. Notice that we need to take a square root at some point and continuity and correctness demands that we choose $\cos\theta = \pm\frac{1}{\sqrt{1+\tan^2\theta}}$ with the $-$ for $\pi/2 < \theta < \pi$ and the $+$ for $0 < \theta < \pi/2$. But, remarkably, this relation can be simplified to

$$\cos\theta = \frac{v + \cos\theta'}{1 + v\cos\theta'}.$$

Now this is a relation of the form $y = \frac{v+x}{1+vx}$ and we want

$$\frac{\sin\theta \mathrm{d}\theta}{\sin\theta' \mathrm{d}\theta'} = \frac{\mathrm{d}\cos\theta}{\mathrm{d}\cos\theta'} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - v^2}{(1 + v\cos\theta')^2}$$

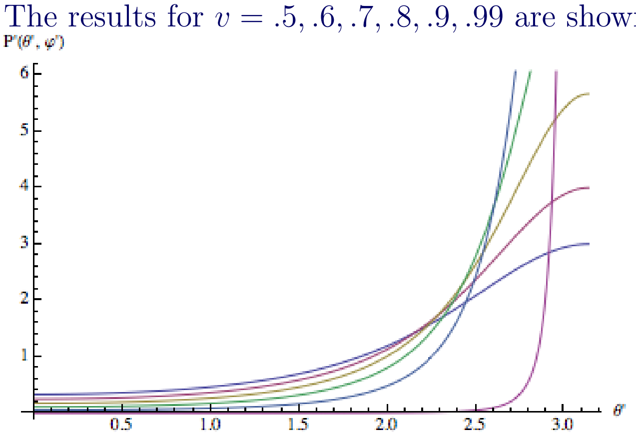
This leads to

$$P'(\theta', \varphi') = \frac{N}{4\pi} \frac{1}{\gamma^2} \frac{1}{(1 + v\cos\theta')^2}$$

An uglier but also correct expression is:

$$P'(\theta', \varphi') = \frac{N}{4\pi} \left| \left(1 + \left(\frac{\sin^2\theta'}{\gamma^2(v + \cos\theta')^2}\right)\right)^{-3/2} \frac{v\cos\theta' + 1}{\gamma^2(v + \cos\theta')^3} \right| .$$

The results for $v = .5, .6, .7, .8, .9, .99$ are shown in the figure:



Some comments:

(1) We can't just transform the spatial bits using Lorentz contraction (like $\mathrm{d}z = \gamma \mathrm{d}z'$) for the following reason: unless we also account for the transformation in the time coordinates, observers in the two frames won't both see the light rays moving at the speed of light!
The light that the observer at rest sees at a given moment was at a spherical shell of radius ct a time t ago:

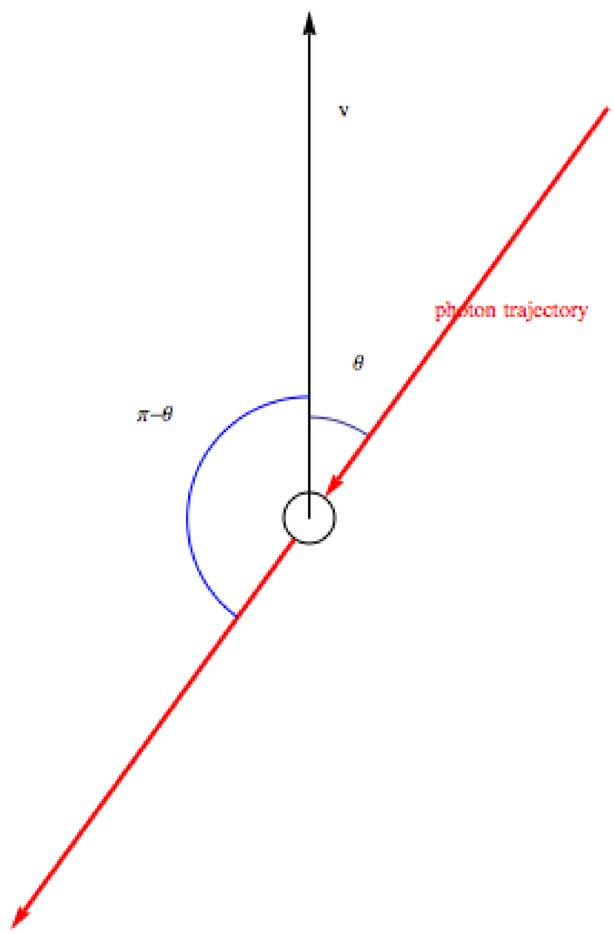
$$\{x, y, z | x^2 + y^2 + z^2 = c^2 t^2\}.$$

The light seen simultaneously by the moving observer was not at the same set of points. (2) Note that the light is also Doppler shifted, so that the light at $\theta \in (0, \pi/2)$ is blueshifted and at $\theta \in (\pi/2, \pi)$ is redshifted.

(c) Check that when integrating the distribution function over the sphere in the coordinates of B you obtain N! Discuss the behavior of the distribution P' in the limiting cases when the velocity v goes to 0 or to 1.
When $v \to 0$, $P' \to \frac{N}{4\pi}$.
When $v \to 1$,

$$P' \to \frac{N}{4\pi} \lim_{\gamma \to \infty} \left(\frac{1}{\gamma^2 \left(1 + \cos\theta'\right)^2} \right)$$

which is zero if $\cos\theta' \neq -1$, that is: the stars all bunch up at $\theta' = \pi$!
But there's one more thing to be said here: which direction is this? The following picture



(that is: the angle between the photon's velocity and *the forward direction of the observer* is actually $\pi - \theta$) and the identity $\cos(\pi - \theta) = -\cos(\theta)$ shows that the stars all look like they are in the *forward direction.* The fact that the light all smushes down to the forward axis can be rationalized as follows: if we're moving at speed c, only if we're heading right at a photon will we run into it – and no matter how fast we go we'll for sure run into the photons that we're going to collide with head-on.

5. Show that the half of the Maxwell equations

$$0 = \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} \tag{9}$$

is invariant under the general coordinate transformation,

$$x^\mu \mapsto \tilde{x}^\mu = f^\mu(x), \quad F_{\mu\nu}(x) \mapsto \tilde{F}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}F_{\rho\sigma}(x)$$

for an arbitrary $f^\mu(x)$ with non-zero Jacobian.

Let $J^\mu_\nu \equiv \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$. By the chain rule, the derivative transforms as

$$\partial_\nu = J^\mu_\nu \tilde{\partial}_\mu.$$

Notice that this is just like our discussion of Lorentz invariance of Maxwell's equations – for the half with the ϵ tensor (the ones which would involve magnetic charge on the LHS were it present), we didn't need any condition on Λ, and only to make the other half (which involve the metric $\eta_{\mu\nu}$) Lorentz invariant did we need a constraint on Λ^ν_μ. So (9) is

$$0 = \epsilon^{\mu\nu\rho\sigma} J^{\nu'}_{\nu} \tilde{\partial}_{\nu'} \left(J^{\rho'}_{\rho} J^{\sigma'}_{\sigma} \tilde{F}_{\rho'\sigma'} \right)$$

where $J^\mu_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$. We should worry about the terms where the derivative hits the J matrices, which depend on x. It is tempting to claim that

$$\frac{\partial}{\partial \tilde{x}^\nu} J^\rho_\sigma = \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} \frac{\partial}{\partial x^\alpha} \frac{\partial x^\rho}{\partial \tilde{x}^\sigma} \stackrel{?}{=} \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} \frac{\partial}{\partial \tilde{x}^\sigma} \delta^\rho_\alpha = 0 \ .$$

However[1] : it is not true in general that partial derivatives *in different coordinate systems* commute with each other. For example, in polar coordinates in the plane,

$$\partial_y \left(\frac{\partial x}{\partial r} \right) = \partial_y \cos\theta = \partial_y \frac{x}{\sqrt{x^2+y^2}} \neq 0.$$

The issue is that when we take a partial derivative ∂_μ, we are holding fixed the other coordinates *in the same coordinate system*. However, it is true that the resulting terms are symmetric in $\rho \leftrightarrow \sigma$, and cancel when contracted with $\epsilon^{\mu\nu\rho\sigma}$, so we can ignore them and we have

$$0 = \epsilon^{\mu\nu\rho\sigma} J^{\nu'}_{\nu} J^{\rho'}_{\rho} J^{\sigma'}_{\sigma} \tilde{\partial}_{\nu'} \tilde{F}_{\rho'\sigma'} \tag{10}$$

The Jacobian of the transformation $x \to \tilde{x}(x)$ is the determinant of the matrix J^μ_ν. So we can use the 4d version of the relation you showed in problem 1, which is

$$\epsilon^{\mu\nu\rho\sigma} J^{\mu'}_{\mu} J^{\nu'}_{\nu} J^{\rho'}_{\rho} J^{\rho'}_{\rho} = \det J \epsilon^{\mu'\nu'\rho'\sigma'}$$

So multiply the BHS of (10) by $J^{\mu'}_{\mu}$ (it's invertible by the assumption that the Jacobian is nonzero, so this is without loss of generality (WLOG)) to get:

$$0 = \epsilon^{\mu\nu\rho\sigma} J^{\nu'}_{\mu} J^{\nu'}_{\nu} J^{\rho'}_{\rho} J^{\sigma'}_{\sigma} \tilde{\partial}_{\nu'} \tilde{F}_{\rho'\sigma'} = \det J \epsilon^{\mu'\nu'\rho'\sigma'} \tilde{\partial}_{\nu'} \tilde{F}_{\rho'\sigma'}$$

which up to the nonzero factor $\det J$ and the erasure of primes on the dummy variables is the original form of the equation.

The following problems are optional.

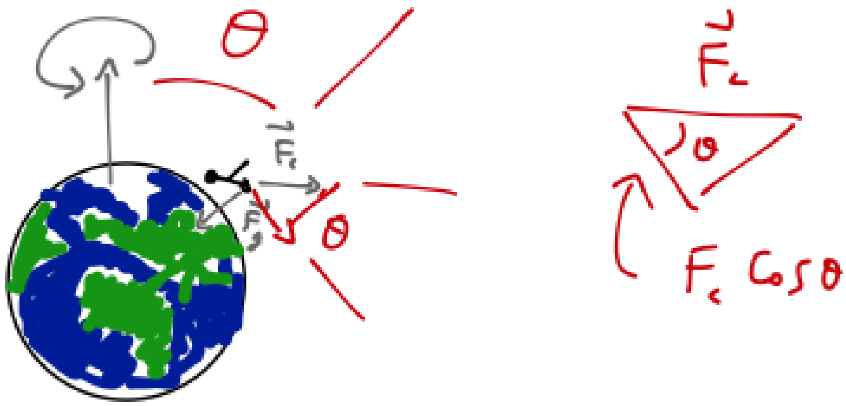
6. **Eötvös**

   What is the optimal latitude at which to perform the Eötvös experiment?

   $\pi/4$. The biggest effect happens when the centrifugal force (proportional to $\sin\theta$ in the horizontal direction, where θ is the polar angle) has the largest component normal

[1] Thanks to Shenglong Xu for helping me clarify this point.

to the gravitational force (towards the center of the earth). The projection onto the surface of the earth of this force is then $\cos\theta \sin\theta \propto \sin 2\theta$ which is maximal at $\theta = \pi/4$.



7. **Poincaré group**

Show that the Poincaré group satisfies all the properties of a group. (That is: it has an identity, it is closed under the group law, it is associative, and every element has an inverse in the group.)

An element of the Poincare group $x^\mu \mapsto \Lambda^\mu_\nu x^\nu + a^\mu$ is labelled by (Λ, a), with

$$\Lambda^\rho_\mu \eta_{\rho\sigma} \Lambda^\sigma_\nu = \eta_{\mu\nu}, \tag{11}$$

and no constraint on a. The group law is composition:

$$x^\mu \mapsto \Lambda^\mu_\nu x^\nu + a^\mu \mapsto \hat{\Lambda}^\mu_\rho \left(\Lambda^\rho_\nu x^\nu + a^\rho\right)$$

The RHS here is another group element because $\hat{\Lambda}^\mu_\rho a^\rho = \hat{a}^\rho$ is a new translation, and

$$\hat{\Lambda}^\mu_\rho \Lambda^\rho_\nu$$

also satisfies the defining relation (11) for a Lorentz transformation. The identity is $(\Lambda, a) = (\mathbb{1}, 0)$ – do nothing. The inverse is $(\Lambda, a)^{-1} = (\Lambda^{-1}, \Lambda^{-1}(-a))$; notice that Λ^{-1} exists by the derivation on page 21-22 of the lecture notes, from the defining relation (11). Also, these rules are associative because they come from matrix multiplication and addition.