Variational principle for conductivity of inhomogeneous media

\[ G = G(r); \ \vec{j} = \vec{j}(r) = ? \] for given \( \vec{j}(r) \) at the boundaries.

Joule heating

\[ P[\vec{j}] = \int dV \frac{\vec{j}^2(r)}{2} \vec{V}, \ \delta_m = \text{effective conductivity of the medium}. \]

Alternatively, we can work with

\[ P_2 = \int dV \left\{ \frac{\vec{j}^2}{2} - 2\phi \vec{j} \nabla \phi \right\}, \ \phi(r) = \text{Lagrange multiplier that enforces} \ \vec{j} = 0. \]

Check:

\[ \frac{\delta P}{\delta j} = \frac{\delta \phi}{\delta \phi} = 0 \Rightarrow \delta G = \delta \phi \phi \Rightarrow \vec{j} = -\nabla \phi \phi \text{ - Ohm's law.} \]

Since the equation \( \nabla \phi = \nabla \phi \phi \) is elliptic, it has a unique solution for suitable boundary conditions. The extremum of \( P[\vec{j}, \phi] \) is unique \( \Rightarrow \) it is the minimum.

(Note that \( \delta \phi > 0 \Rightarrow \) the quadratic form is positive definite). Therefore, the variational principle is

\[ \frac{\delta j^2}{2} / \delta m = \min P[\vec{j}] \text{ subject to} \ \vec{j} = 0. \]

Lower bound for \( \delta m \):

\[ \frac{\delta j^2}{2} / \delta m = \int dV \frac{\delta j^2}{\delta \phi} \Rightarrow \frac{1}{\delta m} = \int dV \frac{1}{G(r)} \Rightarrow \delta m \geq \frac{1}{\langle 1/6(r) \rangle} \]

Attained for serial connection \( G \text{ is minimum} \) or \( G(r) = 6(r), \ \vec{j} \parallel \vec{x} \)

Upper bound for \( \delta m \):

Another variational principle:

\[ P_2 = \int dV \left\{ G(r) \left( \nabla \phi \right) \cdot \left( -2\vec{j} \cdot \nabla \phi \right) \right\} \]

Lagrange multiplier enforcing \( \delta (\vec{\phi} \phi) = 0 \).

\[ \nabla \delta \phi \phi = \min P_2 [\delta \phi, \phi] \]

Trial state \( \phi = \langle \delta \phi \rangle \): calculate

\[ \delta m \leq \langle G(r) \rangle. \]

Attained for parallel connection \( \vec{j} \parallel \vec{x} \).

Weak inhomogeneity

\[ G = \langle G \rangle + \delta G; \ \nabla \left( \phi \nabla \phi \right) = 0 \Rightarrow \nabla \phi = -\frac{1}{\delta \phi} \nabla \left( \delta G \phi \right). \]

Note that this charge density has no net charge, i.e., it is made of dipole, quadrupole, etc., terms.

\[ \phi = \phi_0 + \delta \phi, \ \phi_0 = -\nabla \times \vec{E}(r), \ \delta \phi = +\nabla \times \vec{E}(r) \]

Integrating by parts, we find:

\[ \delta \phi = +\nabla \times \vec{E}(r) \delta G(r) dx + 0(\delta^2). \]

Here \( G(r) \) is Green's function (e.g., \( G(r) = 1/(4\pi r) \) in \( d = 3 \)).
\[ \delta E_x = - \partial_x \delta \phi = - \frac{E_x}{\langle \delta \rangle} \int \text{d}r' \partial_x \delta \phi(r') \delta \phi(r) \quad \text{Note that} \quad \langle \delta E_x(r) \rangle = 0 \]

\[ \delta j_x = \delta E_x(r) \delta \phi(r) + E_x \delta \phi(r) + \langle \delta \rangle \delta E_x(r) \]

\[ \langle \delta j_x \rangle = \langle \delta E_x(r) \delta \phi(r) \rangle \]

\[ = - \frac{E_x}{\langle \delta \rangle} \langle \int \text{d}r' \partial_x \delta \phi(r') \delta \phi(r) \rangle \]

\[ \delta m = \langle \delta \rangle + \frac{\langle \delta j_x \rangle}{E_x} = \langle \delta \rangle - \frac{1}{\langle \delta \rangle} \int \text{d}r' \partial_x \delta \phi(r') \langle \delta \phi(r) \delta \phi(r') \rangle \]

For isotropic random system \( \langle \delta \phi(r) \delta \phi(r') \rangle = \text{function of} \ |r-r'| \)

In this case we can replace \( \partial_x \delta \phi \) by \( \frac{1}{d} \nabla^2 \delta \phi = \frac{1}{d} \delta(r-r') \)

\[ \delta m = \langle \delta \rangle - \frac{1}{d} \frac{\langle \delta \phi^2 \rangle}{\langle \delta \rangle} \]

Compare with the rigorous bounds:

\[ \langle \frac{1}{\langle \delta \rangle} \rangle^{-1} = \left( \frac{\langle \delta \phi \rangle}{\langle \delta \rangle} - \frac{1}{\langle \delta \rangle^2} + \frac{\langle \delta \phi^2 \rangle}{2 \langle \delta \rangle^3} + \ldots \right)^{-1} = \left( \frac{\langle \delta \phi \rangle}{\langle \delta \rangle} + \frac{\langle \delta \phi^2 \rangle}{2 \langle \delta \rangle^3} \right)^{-1} = \langle \delta \rangle - \frac{\langle \delta \phi^2 \rangle}{2 \langle \delta \rangle^3} \]

\[ \langle \delta \phi(r) \rangle = \langle \delta \rangle - 0 \cdot \frac{\langle \delta \phi^2 \rangle}{\langle \delta \rangle} \]

So, the bounds are obeyed.
Effective medium theory for resistor networks


Kirkpartrick, Rev. Mod. Phys. 45, 574 (1973):
Consider a network where all conductances are identical \( = G_m \) except \( G_{AB} \).
Far away from \( A \) and \( B \) the current and electric field are uniform, e.g., directed vertically downward, let \( V_m \) is the voltage drop per row. One can show that the voltage across \( G_{AB} \) is

\[ V_m + \delta V_{AB}, \text{ where } \delta V_{AB} = \frac{G_m - G_{AB}}{G_{AB} + \left( \frac{z}{2} - 1 \right) G_m} \]

\( z \) is the coordination number (\# of connections to a given site).
Voltages across all other bonds are also perturbed. It is easy to see that this perturbation looks like an electric dipole in the far field.
Indeed, at large \( r \) we can use continuum description:
From the previous pages

\[ \nabla^2 \phi = -\frac{1}{G_m} \nabla \cdot (\delta G \cdot \nabla \phi) = -4\pi \rho(r). \]
Clearly, \( \int \rho \, dr = 0 \Rightarrow \) the leading term in the multipole expansion is the dipolar one, with

\[ 4\pi \rho \text{ (dipole moment) } = -\frac{1}{G_m} \Delta G \cdot (V_m + \delta V_{AB}) = -\frac{3}{G_m} \delta G \cdot \nabla V_m \rightarrow \delta G \times G_m - G_m. \]

For finite but small concentration of perturbed resistors, this dipoles give a net correction to electric field:

\[ \delta E_x = E_x \left\langle -\frac{\delta G}{\delta G + \frac{3}{2} G_m} \right\rangle \Rightarrow \frac{\delta R}{R} = -\left\langle \frac{\frac{3}{2} \delta G}{\delta G + \frac{3}{2} G_m} \right\rangle. \]

To get the effective-medium approximation we require that even when the density of perturbed bonds is not small, we still can use this formula.
It entails the self-consistency condition \( \langle \delta R/R \rangle = 0 \), which determines the effective \( G_m \).
Hence, the defining equation of Bruggemann-type effective-medium theory is
\[
\left\langle \frac{G_m - G_j}{G_j + (\frac{q}{2} - 1)G_m} \right\rangle = 0.
\]
In continuum, it reads
\[
\left\langle \frac{G_m - G(r)}{G(r) + (d-1)G_m} \right\rangle = 0.
\]

Example
Consider binary distribution $G = 1$ with probability $p$ and zero otherwise. The EMT equation reads
\[
P(G) = p \delta(G-1) + (1-p) \delta(G), \quad i.e.,
\]
\[
P \left( \frac{G_m - 1}{1 + \left( \frac{q}{2} - 1 \right)G_m} + (1-p) \frac{G_m}{\left( \frac{q}{2} - 1 \right)G_m} \right) = 0.
\]
The solution is
\[
G_m = 1 - \frac{1-p}{1 - \left( \frac{q}{2} \right)}
\]
Of course, we should use it only if it gives $G_m \geq 0$; otherwise, $G_m = 0$.

- The EMT is not rigorous but it has many correct features:
  1) reproduces the perturbation theory for weak inhomogeneity (near $p=1$)
  2) correctly predicts the existence of the percolation point. However, the behavior near $p_c$ is not captured—it is characterized by power-laws different from a single linear increase.

Here $G_m = 0$ because there is no continuous path through $G=1$ bonds.

Here such a path does exist.