Thouless number: \( G = \frac{\hbar}{e^2} e L^{d-2} = \frac{E\tau}{\Delta} \)

\[ E_{th} = \frac{\hbar}{\Delta t}, \quad \Delta t = \frac{L^2}{D} \quad \text{Thouless energy} \]

\[ \Delta = \frac{e^2}{\hbar D} \quad \text{Energy level spacing (per spin)} \]

This formula combines quantum quantities (such as \( \nu \)) and classical ones (\( D \)).

Is it possible to make consistently quantum?

Thouless idea: \( E_{th} \) plays the role of hopping matrix element \( t \) in the coarse-grained Anderson model. Now, if all "sites" are identical,

we get

\[ + (r+l) = + (r)e^{ikL} \]
\[ E(k) = E_0 - 2t \cos kL \]

\[ k=0 : \quad \psi_+(r) = + \psi_+(r+l) \]
\[ \xi_+ = E_0 - 2t \]

\[ k=\frac{4\pi}{L} : \quad \psi_-(r) = - \psi_-(r+l) \]
\[ \xi_- = E_0 + 2t \]

\[ \xi \sim \Delta \quad \text{shift of the energy levels due to switching left from sym/asym.} \]

A key advantage is: only eigenenergies need to be calculated, not the wavefunctions, not the transmission coefficients.

Thouless number: \( G_{th} \approx \frac{\langle \xi_- - \xi_+ \rangle}{\Delta} \)

Thouless vs. Landauer conductance: let \( \frac{\hbar e^2}{L} = 1 \); \( \xi \sim k^d L \), \( L = \text{mean-free-path} \).

1) \[ G = \sum_{i=1}^{N} T_i(\xi_i) = N \langle \xi \rangle \] Since \( N \sim (kL)^{d-1} \) and \( G \sim L^{d-2} \Rightarrow \langle \xi \rangle \sim \frac{L}{L} \ll 1 \). Why?

Typically, there is no energy level exactly equal to \( N \xi \)

\( \Delta \)-splitting \( \sim E_F \) in an isolated box.

\( \xi_1, \xi_2, \ldots, E_F \ldots, E \quad \text{The nearest level} \)

is \( \sim \Delta \) away. In contrast, in the transmission calculation (w/o leads) the spectrum is continuous.
This continuum appears due to coupling to the leads. We can imagine that this coupling is turned on gradually, e.g., by connecting the leads via intermediate tunneling barriers. Then each energy \( E_j \) gives rise to a transmission resonance

\[
T_j(E) = \frac{T_0}{1 + \frac{(E - E_j)^2}{\Gamma_j^2}}.
\]

As the barriers become more and more open, the width of the resonance increases.

For perfect leads (no barrier), \( \Gamma_j \) is determined by the escape rate \( \Gamma_j \), i.e., \( \Gamma_j \) reaches the max. value of \( \sim \frac{E}{h} \). Correspondingly, only \( \sim \frac{E}{h} \) levels of the isolated box actively contribute to transmitting channels: these \( N_{\text{tot}}/L \) channels have \( T_i \sim 1 \), the rest have \( T_i \ll 1 \). This picture is confirmed by rigorous calculations, which give the \( T_i \) density distribution

\[
\nu_i(T) = \frac{1}{2} \frac{N_i}{L} \frac{1}{\Gamma_i \sqrt{1 - T}} e^{-\frac{2T}{\Gamma_i}} \quad \forall T_i \ll T < 1.
\]

(Dorokhov, 1984; Imry, 1986)

Another useful physical picture: Quasiclassically, we deal with the interplay of two times: diffusion time \( \Delta T_{\text{th}} \sim \frac{D}{L^2} \) and the Ehrenfest time \( \Delta t_{\text{e}} = \frac{h}{\Delta} = \text{time to close the trajectory, i.e., arrive to the original point with accuracy } \Delta r, \Delta \phi \text{ such that } \Delta r, \Delta \phi \sim \Delta T_{\text{th}}. \)

\[
G_{\text{th}} \sim \frac{\Delta t_{\text{e}}}{\Delta T_{\text{th}}}
\]

b) Thouless number \( G \) for localised systems:

Here \( T \sim e^{-L^2/\xi} \), \( G_{\text{th}} \sim \frac{\nu}{\Delta} \sim e^{-L^2/\xi} \)

On the other hand

\( G \sim T \sim e^{-2L^2/\xi} \Rightarrow G_{\text{th}} \sim \sqrt{G}. \)

Not equal to \( G \) but still a definite function of \( G \) only!
2. 1-parameter scaling theory of localization

$G_0$ measures the change of $G$ after doubling the system size. Main assumption: this change depends only on $G$ itself:

$$\frac{G(2L) - G(L)}{G(L)} = f(G(L)).$$

Instead of this finite-difference equation, we can work with a differential equation

$$\frac{d}{d \log_2 L} G(L) = f(G),$$

or

$$\frac{d \ln G}{d \ln L} = \beta(G) \approx \beta_{\text{function}} \text{ or scaling function.}$$

$\beta(G)$ should be a universal function which may depend only on $d$ and some global symmetry properties of the random system (e.g., presence or absence of time invariance).

1. $G \gg 1$

$G = G_0 L^{d-2} \Rightarrow G(L) = L^{d-2}$ (metal)

$$\beta(G) \approx d-2 \quad \text{(more accurately, } \beta = d-2 - \frac{\pi}{2} + O(\frac{1}{d})) \quad \text{for systems with time-reversal invar. and no spin-orbit interaction})$$

2. $G \ll 1$

$G \sim e^{-U/\xi} \Rightarrow \beta(G) = \ln G + \text{const}$ (insulator)

Interpolating between these limits, we expect the following:

(insulator) $\beta$  (metal)

\[ \begin{array}{c}
\text{Fixed point} \\
\log \xi
\end{array} \]

If $d \leq 2$, then all the states are localized in the $L \to \infty$ limit.

Arrows on the curve indicate how $G$ changes as $L$ increases.
3. **Near the critical point** (which exists if \( d > 2 \))

We expect \( G_c \) = universal number of the order of unity. For \( G - G_c \ll G_c \) we have the linear expansion

\[
\beta(G) = \beta(G_c) \cdot (G - G_c) \Rightarrow \frac{d \ln G}{d \ln L} = \beta' (G - G_c),
\]

and we find an anomalous power-law behavior

\[
G(L) - G_c = [G(L_0) - G_c] \cdot \left( \frac{L}{L_0} \right)^{1/\delta}, \quad S = \frac{1}{G_c \beta'(G_c)} \sim 1.
\]

This behavior is restricted to \( G - G_c \ll G_c \), i.e.,

\[
L \ll L_0 \cdot \left| \frac{G_c}{G(L_0) - G_c} \right|^S.
\]

At larger \( L \), \( G(L) \) should cross over to the previous limits: \( G \sim L^{d-2} \) (for \( G > G_c \)) or \( G \sim e^{-L^S} \) (for \( G < G_c \)). This enables us to conclude that \( S \) has a power-law divergence near the transition: \( S \sim L/18^1, \quad S \ll 1 \).

Insulating side

\[ \xi \]

Metallic side

\[ \sigma \]

At \( L \)=0 (insulating side) we should have

\[ G(L_0) = (k_F L)^{d-1} = G_c + A \cdot \sigma, \quad A \sim 1. \]

\[ \therefore \xi \sim L \cdot \left( \frac{G_c}{G_c \sigma} \right)^{1/\delta} \sim \frac{L}{18^{1/\delta}}. \]

**Numerical calculations** for \( d=3 \) Anderson model gave \( S \approx 0.9 \).

**Analytical calculations** can be done only for \( (d-2) \ll 1 \). They give \( S = \frac{1}{d-2} \).

The exponent \( S \) is similar to the critical exponent for the correlation length in other 2nd-order phase transitions. Typically, the theory of such transitions is difficult at low \( d \) but becomes mean-field-like above a certain \( d_c \) (e.g., \( d_c=4 \) for ferromagnetic). It is not known whether mean-field value \( S=1/2 \) ever applies to Anderson transition.