Lecture VIII

Eigenstates in 1D white noise potential model

Problem
Given: \( t^2/2m = 1 \), \( \langle U(x) \rangle = 0 \),
\[ -\psi'' + U(x)\psi = E\psi(x) \]
\[ \langle U(x)\psi(x) \rangle = 2D \delta(x-x') \]

Boundary conditions:
\[ \cos \left( \sin \left( \frac{x}{L} \right) \right), \psi(\pm L/2) - \sin \left( \sin \left( \frac{x}{L} \right) \right) \psi'(-L/2) = 0 \]
e.g. \( \psi(\pm L/2) = 0 \) gives Dirichlet b.c. \( \psi(\pm L/2) = 0 \).

Find: \( \psi(x), E \) or more precisely, statistical properties of \( \psi(x) \) and \( E \), such as:

a) density of states \( D(E) \):
\[ J(E) = \frac{1}{4\pi\alpha} = \frac{1}{2\pi\sqrt{L/\alpha}} \]

b) the rate of spatial decay \( \gamma_\pm(E) \):
\[ \gamma_\pm = -\lim_{L \to \infty} \left[ \frac{\ln v(\pm L/2)}{L/2} \right] \]
\[ \gamma(x) = \sqrt{\psi^2(x) + \psi'^2(x)} \]
(note that \( \psi \) & \( \psi' \) cannot be zero simultaneously)

Expect: \( \gamma_+ = \gamma_- = \frac{1}{\xi} \), \( \xi \) = localization length,
\[ \xi \sim L, \quad L = \text{mean free path} \]
\[ f = \frac{u \xi}{\kappa} \]
\[ \frac{f}{c} = \frac{2\pi c}{\kappa} \cdot \frac{A}{\pi k B} \cdot |\phi(2\kappa)|^2 \]
\[ (\langle \psi(0) \rangle)^2 = \frac{\int \psi^*(x) \psi(x) \, dx}{\int |\psi(x)|^2 \, dx} \]
\[ (\langle \psi(0) \rangle)^2 = \langle \psi(0) \rangle^2 \]
\[ \langle \psi(x) \psi(x') \rangle = 2D \delta(x-x') \]

\[ e^{-\gamma|x|} \]

\[ e^{-\gamma+X} \]

\[ e^{-\gamma-x} \]
Mapping onto a classical harmonic oscillator:
\[
\begin{align*}
\frac{d}{dx} \psi &= \psi', \\
\frac{d}{dx} \psi' &= -[E-U(x)] \psi.
\end{align*}
\]
\(x\) \(\rightarrow\) time, \(\psi\) \(\rightarrow\) coordinate, \(\psi'\) \(\rightarrow\) momentum, \(E-U(x)\) \(\rightarrow\) elastic constant

Phase space (coordinate-momentum) diagram

\[x = \arccos z + \pi n, \quad z = \frac{\psi'}{\psi}, \quad r = \sqrt{\psi^2 + \psi'^2} = |\psi| \sqrt{1+z^2}.\]

\[\frac{dl}{dx} = \cos l(x) + [E-U(x)] \sin^2 l(x) \quad \text{if } r(x) > 0.\]

For \(U\) \(-\text{const.}\), the trajectory is a closed ellipse.

The eigenenergy \(E_n\) is found from the condition that given \(l(-\frac{L}{2}) = 0\), we have \(l(+\frac{L}{2}) = n\pi\) (assuming Dirichlet b.c., as an example).

Alternatively, it is found from the condition that if we follow the evolution of \(l(x)\) in two ways:

1) From the left end, starting with \(l(-\frac{L}{2}) = 0\) and 2) from the right end, starting with \(l(+\frac{L}{2}) = n\pi\), then at \(x=0\) they match.

In this "two-end-meet" formulation, \(\psi(x)\) exponentially increases rather than decreases (except for small number of states whose central gravity in within \(1/\psi\) near the ends; a negligible minority in the limit \(L \rightarrow \infty\)).
The rate of growth: we arrive at the new definition

\[ y = \lim_{X \to -\infty} \left[ \ln \left( \frac{r(x)}{r(x_0)} \right) \right] \]

Next,

\[ \ln \frac{r(x)}{r(x_0)} = \ln \frac{\sqrt{1+2z^2(x)}}{\sqrt{1+2z^2(x_0)}} + \ln \left| \frac{z'(x)}{z(x_0)} \right| \]

Cartoon of \( z(x) \):

\[ z = \ln \frac{\sqrt{1+2z^2(x)}}{\sqrt{1+2z^2(x_0)}} + \int_{x_0}^{x} dx' z(x') \quad (\text{principal value}) \]

We will show that the fluctuations of \( z(x) \) are sufficiently modest so that a well-defined probability distribution function \( P(z) \) exists in the limit \( x, x_0 \to \pm \infty \), such that \( P(0) \sim \frac{1}{2z} \), \( z \to \pm \infty \).

In this case

\[ \langle z \rangle = \int_{-\infty}^{\infty} dz \, z \, P(z) \quad \text{and} \quad \langle \ln \sqrt{1+z^2} \rangle = \int_{-\infty}^{\infty} dz \, \frac{1}{\sqrt{1+z^2}} \, P(z) \] are finite.

\[ y = \lim_{\Delta x \to \infty} \left\{ \frac{\langle \ln \sqrt{1+z^2} \rangle}{\Delta x} + \frac{1}{\Delta x} \int_{0}^{\Delta x} dx' \langle z \rangle \right\} = \langle z \rangle. \]
Equation of motion for \( z(x) \):
\[
\frac{dz}{dx} = -(z^2 + E) + U(x),
\]

For \( P(z,x) \) we expect the following evolution:

\[ P(z,x_0) \rightarrow P(z,x) \]

\( P(z,x) \approx P(z,0) \), independent of \( x \) (steady state)

Fokker-Planck equation (see Lecture III) reads
\[
\frac{\partial P}{\partial x} = -\frac{\partial}{\partial z} J, \quad J = -(z^2 + E) P - D \frac{\partial^2 P}{\partial z^2}, \quad D = \frac{1}{2} \int_{-\infty}^{\infty} \langle U(x) U(0) \rangle \, dx.
\]

Steady state: \( \partial P/\partial x = 0 \Rightarrow J = \text{const} \),

\[ P(z) = -\frac{J}{D} e^{-\Phi(z)} \int_{-\infty}^{z} dt e^{\Phi(t)} \]

\( \Phi(t) = \frac{t^3}{6D} + \frac{Et}{D} \).

1) \( E > 0 \)

At \( z \to \pm \infty \) where \( P(z), \frac{\partial P}{\partial z} \to \infty \) we have

\[ J = -(z^2 + E) P, \]

\[ P(z) \to -\frac{J}{z^2 + E} = \frac{1}{2} \frac{1}{z^2 + E}. \]

\( J \) is determined from normalization condition

\[ \int_{-\infty}^{\infty} dz P(z) = 1. \]
\[ \gamma = \langle z \rangle = \int_{-\infty}^{\infty} dz \ z \ p(z) = \frac{|J|}{D} \int_{-\infty}^{\infty} dz \ z \ e^{-\frac{\varphi(z)}{2}} \int_{-\infty}^{\infty} dt \ e^{\varphi(t)} \]

where

\[ \frac{D}{|J|} = \int_{-\infty}^{\infty} dz \ e^{-\frac{\varphi(z)}{2}} \int_{-\infty}^{\infty} dt \ e^{\varphi(t)} \]

Using Laplace (steepest descent) method, one can find asymptotic expressions:

a) \( E > 0 \), \( E \gg D^{2/3} \)

\[ |J| \sim \frac{\sqrt{E}}{\pi} \]

\[ \gamma \sim \frac{D}{4E} \]

\[ \xi \sim \frac{4E}{D} \sim 4\xi_0 \]

\underline{Physics:} Bragg mirrors

b) \( E < 0 \), \( |E| \gg D^{2/3} \)

\[ |J| \sim \frac{|E|^{1/2}}{\pi} \exp \left( -\frac{4}{3} \frac{|E|^{3/2}}{D} \right) \]

\[ \gamma \sim |E|^{1/2} \]

\[ \xi \sim \frac{1}{|E|^{1/2}} = \frac{1}{\sqrt{\frac{2m}{\hbar^2} |E|}} \]

\underline{Physics:} rare negative fluctuating of \( U(x) \) (potential traps).

\[ V(E) = \frac{d}{dE} |J(E)| \]

\[ \gamma_0 \sim \frac{1}{2\pi|\xi|} \]

\[ \sim \exp \left( -\frac{|E|^2}{D} \right) \]

\[ \xi \sim \frac{4E}{D} \sim 4\xi_0 \]

\[ \xi \sim \frac{1}{|E|^{1/2}} \]

\underline{Summary}

**Local length**

\[ 1 \leftarrow \xi \leftarrow \xi \leftarrow \xi \]

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One can show that \( V(E) = \frac{d}{dE} |J(E)| \).

see Lifshitz, Gredeskul, Pastor.