Lecture 11:

Diffusion

- Diffusion in real and momentum space;
- Fokker-Planck equation

\[ \langle r' (t) \rangle = 2d D t, \ t \gg \tau \]

\[ D = \frac{1}{d} \frac{v_F^2}{\tau} = \frac{1}{d} \frac{\ell^2}{\tau}, \ \ell = v_F \tau \]

b) Random walk on the lattice

\[ D = \frac{1}{d} \frac{\ell^2}{\tau} \]

c) Random hopping in continuum

\[ D = \frac{1}{d} \int d\ell \ R^2 w (\ell, \ell + \ell) \]

More generally, if \( w (r, \mathbf{r} + \mathbf{R}) = \) transition rate from \( r \) to \( r + \mathbf{R} \), then

\[ D = \frac{1}{d} \int d\mathbf{R} \ R^2 w (\mathbf{r}, \mathbf{r} + \mathbf{R}) \]

Drift-diffusion equation

\[ \frac{\partial n}{\partial t} = D \nabla^2 n \quad \text{- diffusion in an isotropic uniform medium} \]

More generally, \[ \frac{\partial n}{\partial t} = - \nabla \Phi, \quad \Phi = \text{particle flux} \]

\[ \Phi = - D \nabla n + \nabla \cdot \mathbf{J} \quad \text{Both } D \text{ and } \mathbf{J} \text{ can be } r, t \text{- dependent} \]

\[ \frac{\partial n}{\partial t} = \nabla \left[ - n \mathbf{v}_d (r) + D (r) \nabla n \right] \]
Fokker-Planck equation

Diffusion in the space of other coordinates (e.g., in momentum space).

Example: small-angle elastic scattering

\[ |q| = |p' - p| \ll p_F \]

\[ D_p = ? \]

Let's derive the FP equation from kinetic equation

\[ \mathcal{I}[f] = \int (dq) \left[ w(p+q,q)f(p+q) - w(p,q)f(p) \right] \]

\[ w(p+q,q)f(p+q) = w(p,q)f(p) + \frac{q^2}{2}\frac{\partial^2}{\partial p} [w(p,q)f(p)] + \frac{\partial}{\partial p} \sum q \cdot q' \cdot \frac{\partial}{\partial q} [w(p,q)f(p)] \]

\[ \mathcal{I}[f] = \sum_{\alpha} \frac{\partial}{\partial p_{\alpha}} \left[ A_{\alpha} f + \sum_{\beta} B_{\alpha \beta} (B_{\beta \beta} f) \right] \]

\[ A_{\alpha} = \int (dq) q_{\alpha} w(p,q) \quad \text{and} \quad B_{\alpha \beta} = \int (dq) q_{\alpha} q_{\beta} w(p,q) \]

\[ \mathcal{I}[f] = -\frac{\partial}{\partial p} \Phi, \quad \Phi = \overrightarrow{\nabla}_d f - \frac{\partial}{\partial p} (D_p f), \quad (\nabla)_d = -A \frac{\partial}{\partial p} \overrightarrow{B} \]

Diffusion coefficient in momentum space:

\[ D_p = \frac{1}{d} \int (dq) q^2 w(p,q) \quad \text{in full analogy w/ D in real space} \]

Q: What is \( \overrightarrow{v}_d \)?

A: If \( f = f_0 \), we must have \( \Phi = 0 \), and so

\[ \overrightarrow{v}_d = \frac{1}{f_0} \frac{\partial}{\partial p} (D_p f_0) \]

\( \overrightarrow{v}_d \) directed radially inward, keeping particles inside the Fermi surface.
The net result of the drift + diffusion terms is diffusion in the angular space, i.e., along the Fermi surface.

\[ \theta_{\text{rms}} = \left\langle [\theta(t) - \theta(0)]^2 \right\rangle^{1/2} = \sqrt{2D_\theta t} \]

Let's relate \( D_\theta \) to \( D \) in the real space.

Consider the velocity autocorrelation function

\[ C_v(t) = \left\langle \vec{v}(0) \vec{v}(t) \right\rangle = v_F^2 \left\langle \cos \left[ \theta(t) - \theta(0) \right] \right\rangle = ? \]

Let \( \theta(0) = 0 \).

\[ C_v(t) = v_F^2 \text{Re} \left\langle e^{i\theta(t)} \right\rangle = v_F^2 \text{Re} \int d\theta f(\theta,t) e^{i\theta} \]

\[ f(\theta,t) = \frac{1}{\sqrt{2\pi} \theta_{\text{rms}}} \exp \left( -\frac{\theta^2}{2\theta_{\text{rms}}^2} \right) \]

\[ C_v(t) = v_F^2 \text{Re} \ e^{-\frac{1}{2} \theta_{\text{rms}}^2} = v_F^2 e^{-D_\theta t} = v_F^2 e^{-t/\tau} \]

\[ C_s = e^{-4\tau} \quad D = \frac{1}{d} v_F^2 \tau = \frac{1}{2} \quad v_F^2 \tau = \frac{1}{2} \frac{v_F^2}{D_\theta} \]

Example: Motion under action of a random force \( F(t) \)

\[ \dot{v} = F(t) \quad v(t) = v(0) + \int_0^t dt' F(t') \]

\[ \left\langle [v(t)-v(0)]^2 \right\rangle = \int_0^t \int_0^{t'} \langle F(t') F(t'') \rangle \]

\[ T = \frac{t'' - t'}{2}, \quad \Delta t = t'' - t', \quad t = 18 \Delta t \]

\[ \left\langle [v(t)-v(0)]^2 \right\rangle = \int_{-t}^{t} d(\Delta t) \int_0^{\Delta T} \langle F(T - \frac{\Delta t}{2}) F(T + \frac{\Delta t}{2}) \rangle \]
let \( C_{ff}(t) = \langle F(t_0)F(t_0 + t) \rangle \) - function of \( t \) only (ergodicity), rapidly decaying at \( t > \tau \).

\[
-\langle [V(t_1) - V(t_0)]^2 \rangle \approx 2 D_v t ,
\]

\[
D_v = \int_0^\infty dt \, C_{ff}(t) .
\]

In general, if the equation of motion is

\[
v = u(t) + F(t)
\]

\( u(t) \) deterministic, \( F(t) \) random, slowly varying, rapidly varying,

we are led to the Fokker-Planck equation

\[
\frac{\partial}{\partial t} f(v,t) = \frac{\partial}{\partial v} \left[ -u f + D_v \frac{\partial}{\partial v} f \right] .
\]

The equilibrium state satisfies the equation \( \frac{\partial f}{\partial t} = 0 \), so that

\[
-u f + D_v \frac{\partial f}{\partial v} = J = \text{const} .
\]

For example, if \( J = 0 \), we get

\[
f(v) = C \exp \left( -\frac{1}{D_v} \int_0^\infty u(v_1) dv_1 \right) .
\]