Qual Review, Meeting 7: Potpourri 3

1. Basic Techniques in Scattering

1a. Assume the potential off which an incoming wave gets scattered is spherically symmetric, i.e. $V(\mathbf{r}) = V(r)$. Find an expression for the differential cross section in the Born approximation in terms of an integral over r.

1b. The Yukawa potential roughly models the binding force in an atomic nucleus and is given by $V(r) = \beta \frac{e^{-\mu r}}{r}$. Find the differential cross section in the Born approximation for this potential.

1c. Use your result from part b to calculate the differential cross section for scattering from a Coulomb potential (Say the particle creating the potential has charge Q and the scattered particle has charge q). Amazingly, this answer is EXACTLY what you'd get from doing a classical scattering calculation. Calculate the total cross section due to Coulomb scattering using this answer. What does your answer mean?

2. Consider a long chain of indentical simple pendula of mass m and length l coupled with springs of spring constant k. In equilibrium with the pendula all vertical, the springs are unstretched and the separation between adjacent pendula is a. Find the dispersion relation for the propagation of waves of small amplitude. If the Nth pendulum is connected to the first pendulum, find the allowed vibration frequencies.

3. An experimentalist, when not busy staring at shiny objects, notices an ion beam with uniformly distributed charge per unit length λ. Calculate the force on a single beam ion of charge Q located at radius r, assuming that the beam radius R is greater than r and that all the ions have the same velocity v.

4. Find the first term in the expansion of

$$\int_0^{\pi/2} \sqrt{\sin t}\, e^{-x \sin^4 t} dt$$

for large x.

Potpourri 3 Solns

1] $$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \int e^{-i(\underset{\sim}{k}'-\underset{\sim}{k})\cdot \underset{\sim}{r_0}} V(r_0)\, d^3 r_0 \right|^2$$

Take $\underset{\sim}{K} = \underset{\sim}{k}' - \underset{\sim}{k}$, so let's do the integral:

$$\int e^{-i\underset{\sim}{K}\cdot\underset{\sim}{r}} V(r) r^2 \sin\theta\, dr\, d\theta\, d\phi = 2\pi \int e^{-iKr\cos\theta} V(r) r^2 \sin\theta\, dr\, d\theta$$

$\sin\theta\, d\theta = -d(\cos\theta)$.

So $$\int_1^{-1} e^{-iKr\cos\theta} - d(\cos\theta) = \frac{1}{-iKr} e^{-iKr\cos\theta}\Big|_{-1}^{1} = \frac{1}{-iKr}\left(e^{-iKr} - e^{iKr}\right)$$

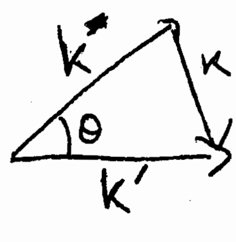
$$= \frac{-2i\sin(Kr)}{-iKr} = \frac{2\sin(Kr)}{Kr}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| 2\pi \cdot 2 \int \frac{\sin(Kr)}{Kr} r^2 V(r)\, dr \right|^2$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{2m}{K\hbar^2}\right)^2 \left| \int r \sin(Kr) V(r)\, dr \right|^2}$$

Note that

$\underset{\sim}{K} = \underset{\sim}{k}' - \underset{\sim}{k}$



$K \Rightarrow |K| = 2k\sin(\theta/2)$

o] $V(r) = \beta \frac{e^{-\mu r}}{r}$

$\Rightarrow$ The integral in question is

$$\int r\sin(\kappa r)\beta\frac{e^{-\mu r}}{r}\,dr = \beta\int_0^\infty \sin(\kappa r)e^{-\mu r}\,dr$$

$$= \beta\int_0^\infty \frac{e^{i\kappa r}-e^{-i\kappa r}}{2i}e^{-\mu r}\,dr$$

$$= \frac{\beta}{2i}\left[\frac{1}{ik-\mu}(-1) + \frac{1}{-ik-\mu}(+1)\right]$$

$$= \frac{\beta}{2i}\left[\frac{+ik+\mu+ik+\mu}{\mu^2+\kappa^2}\right]$$

$$= \frac{\beta\kappa}{\mu^2+\kappa^2}$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \left(\frac{2m}{\kappa\hbar^2}\right)^2\left(\frac{\beta\kappa}{\mu^2+\kappa^2}\right)^2 = \left(\frac{2m\beta}{\hbar^2(\mu^2+\kappa^2)}\right)^2 \text{ with } \kappa = 2k\sin(\theta/2).}$$

c) $V(r) = \beta \frac{e^{-\mu r}}{r} \Rightarrow \beta = qQ, \mu = 0 \Rightarrow V(r) = \frac{qQ}{r}$.

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{(\mu^2+\kappa^2)^2} = \left(\frac{2mqQ}{\hbar^2}\right)^2 \frac{1}{\left(2\sin\left(\frac{\theta}{2}\right)\right)^4}$$

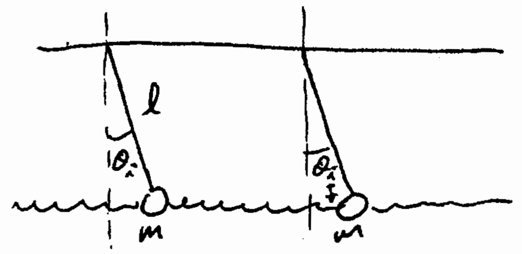
$$= \left(\frac{mqQ}{2\hbar^2}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}.$$

We need $\displaystyle\int_0^{\pi} \frac{\sin\theta\, d\theta}{\sin^4\left(\frac{\theta}{2}\right)} = \int_0^{\pi/2} \frac{\sin(2\theta)\, 2d\theta}{\sin^4\theta} = \int_0^{\pi/2} \frac{4\cos\theta\, d\theta}{\sin^3\theta}$

$$= \int_0^1 \frac{4\, du}{u^3} \Rightarrow \infty \,!$$

So <u>every</u> bit of the incoming wave is affected by the Coulomb potential $\rightarrow$ it's a "long-range" force.

1]



$$L = \sum_i \left(\frac{1}{2} m l^2 \dot{\theta}_i^2 - \frac{1}{2} k \left(l \sin\theta_{i+1} - l \sin\theta_i \right)^2 + m g l \cos\theta_i \right)$$

Small θ's: $\sin\theta \simeq \theta$, $\cos\theta \simeq 1 - \frac{\theta^2}{2}$

$$\Rightarrow L = \sum_i \left(\frac{1}{2} m l^2 \dot{\theta}_i^2 - \frac{1}{2} k l^2 (\theta_{i+1} - \theta_i)^2 - \frac{mgl}{2} \theta_i^2 \right)$$ ignoring the const. term.

OM: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i} \right) = \frac{\partial L}{\partial \theta_i}$

$$\frac{\partial L}{\partial \dot{\theta}_i} = m l^2 \dot{\theta}_i, \quad \frac{\partial L}{\partial \theta_i} = -k l^2 (\theta_{i+1} - \theta_i)(-1) - k l^2 (\theta_i - \theta_{i-1}) - m g l \theta_i$$

$$= k l^2 (\theta_{i+1} - 2\theta_i + \theta_{i-1}) - m g l \theta_i$$

$$\Rightarrow m l^2 \ddot{\theta}_i = k l^2 (\theta_{i+1} - 2\theta_i + \theta_{i-1}) - m g l \theta_i.$$

sing $x_i = l \sin\theta_i \simeq l\theta_i \Rightarrow$

$$m l \ddot{x}_i = k l (x_{i+1} - 2x_i + x_{i-1}) - m g x_i$$

Now Assume $x_n = A e^{iqx} e^{-i\omega t} = A e^{iqna} e^{-i\omega t}$ (plane waves)

$$\Rightarrow -m l \omega^2 e^{iqna} = k l \left(e^{iq(n+1)a} - 2 e^{iqna} + e^{iq(n-1)a} \right) - m g e^{iqna}$$

$$\Rightarrow -ml\omega^2 = kl\left(e^{iqa} + e^{-iqa} - 2\right) - mg$$

$$\Rightarrow -\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right) - \frac{g}{l}$$

$$\Rightarrow \omega^2 = \frac{g}{l} + \frac{k}{m}\left(2 - 2\cos(qa)\right)$$

$$\Rightarrow \omega^2 = \frac{g}{l} + \frac{2k}{m}\left(2\sin^2\left(\frac{qa}{2}\right)\right)$$

$$\Rightarrow \boxed{\omega^2 = \frac{g}{l} + \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)}$$

Periodic b.c. : $X_{N+1} = X_1 \Rightarrow e^{iq(N+1)a} = e^{iqa}$

$$\Rightarrow e^{iqNa} = 1$$

$$\Rightarrow qNa = 0, 2\pi, 4\pi, 6\pi, \ldots$$

$$\Rightarrow q = \frac{2n\pi}{Na}$$

$$\Rightarrow \boxed{\omega_n^2 = \frac{g}{l} + \frac{4k}{m}\sin^2\left(\frac{n\pi}{N}\right)}$$

3] $\mathcal{O}$: Lab frame
$\mathcal{O}'$: Moving w/ beam.

In $\mathcal{O}'$, $j^{\mu'} = (\rho', \underline{0}) \Rightarrow j^{\mu} = (\gamma\rho', \gamma\underline{\beta}\rho')$ in $\mathcal{O}$.

$$\Rightarrow \rho = \gamma\rho' \Rightarrow \lambda = \gamma\lambda'.$$

In $\mathcal{O}'$, get E_r by Gauss's Law: $2\pi r \ell E' = 4\pi \frac{\lambda' \cancel{\ell}}{\cancel{\ell}\pi R^2} \ell \pi r^2$

$$\Rightarrow E_r' = \frac{2\lambda' r}{R^2}, \quad E_{\parallel}' = \underline{B}' = 0.$$

Transform fields to $\mathcal{O}$:

$E_{\parallel}' = E_{\parallel} = 0$. $\quad \underset{\sim}{E_\perp} = \gamma\left(\underset{\sim}{E_\perp'} - \frac{\underset{\sim}{v} \times \underline{B_\perp}}{c}\right) = \frac{2\gamma\lambda' r}{R^2}\hat{r} = \frac{2\lambda r}{R^2}\hat{r}$

$B_{\parallel}' = B_{\parallel} = 0$. $\quad \underset{\sim}{B_\perp^{\varepsilon}} = \gamma(\underline{B_\perp}' + \underset{\sim}{\beta} \times \underset{\sim}{E_\perp'})$

$$= \gamma\left(\frac{2v\lambda' r}{cR^2}\hat{z}\times\hat{r}\right) = \frac{2v\lambda'\gamma r}{cR^2}\hat{\phi} = \frac{2v\lambda r}{cR^2}\hat{\phi}.$$

So force on Q: $\quad \underset{\sim}{F} = Q\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underline{B}}{c}\right)$

$$= Q\left(\frac{2\lambda r}{R^2}\hat{r} + \frac{v^2}{c^2}\frac{2\lambda r}{R^2}\hat{z}\times\hat{\phi}\right) = Q\left(\frac{2\lambda r}{R^2}\right)\left(1 - \frac{v^2}{c^2}\right)\hat{r}$$

$$\Rightarrow \boxed{\underset{\sim}{F} = \frac{2\lambda Q r}{\gamma^2 R^2}\hat{r}}$$

1 $\displaystyle\int_0^{\pi/2} \sqrt{\sin t}\, e^{-x\sin^4 t}\, dt$

For large x, the phase is stationary @ $t=0$, so expand round there. $\sin^4 t \approx t^4$, $\sqrt{\sin t} \approx t^{1/2}$

$$\Rightarrow \int_0^{\pi/2} t^{1/2} e^{-xt^4}\, dt \Rightarrow \int_0^{\infty} t^{1/2} e^{-xt^4}\, dt$$

Make sub $u = xt^4$, so $t = u^{1/4} x^{-1/4}$, $dt = \frac{u^{-3/4}}{4} x^{-1/4}\, du$

$$\Rightarrow \int_0^{\infty} x^{-\frac{1}{8}} u^{\frac{1}{8}} e^{-u} \frac{u^{-3/4}}{4} x^{-\frac{1}{4}}\, du = \frac{x^{-3/8}}{4} \int_0^{\infty} u^{-5/8} e^{-u}\, du$$

$$= \boxed{\frac{x^{-3/8}}{4} \Gamma\left(\frac{3}{8}\right)}$$