The effective potential will be:

\[ U_{\text{eff}} = \frac{l^2}{2\mu r^2} + \frac{1}{2} kr^2. \]

This is plotted in figure (1). In order to find the circular orbit we set \( \ddot{r} = 0 \) which gives us \( \frac{dU_{\text{eff}}}{dr} = 0 \). For this we find:

\[ U_{\text{eff}}' = -\frac{l^2}{\mu r^3} + kr = 0. \]

Or:

\[ \frac{l^2}{\mu r^3} = kr. \]
Which leads to:
\[ r_0 = \sqrt[4]{\frac{l^2}{\mu k}}. \]

To find the Taylor expansion we want to find:
\[ U_{\text{eff}}(r) = U_{\text{eff}}(r_0) + U'_{\text{eff}}(r_0)(r - r_0) + \frac{1}{2}U''_{\text{eff}}(r_0)(r - r_0)^2 + \ldots \]

So we have:
\[ U_{\text{eff}}(r_0) = \frac{l^2}{2\mu r_0} + \frac{1}{2}kr_0^2. \]

If we plug in:
\[ r_0^2 = \frac{l}{\sqrt{\mu k}}. \]

We have:
\[ U_{\text{eff}}(r_0) = \sqrt{\frac{k}{\mu}l}. \]

By definition (equation 1) \( U'_{\text{eff}}(r_0) = 0 \), so we need to find the second derivative:
\[ U''_{\text{eff}}(r_0) = \frac{3l^2}{\mu r_0^4} + k. \]

If we plug in:
\[ r_0^4 = \frac{l^2}{\mu k}. \]

We have:
\[ U''_{\text{eff}}(r_0) = 4k. \]

So our Taylor expansion is:
\[ U_{\text{eff}}(r) \approx \sqrt{\frac{k}{\mu}l + \frac{1}{2}4k(r - r_0)^2 + \ldots} \]

Equation 8.29 in the text gives the equation of motion:
\[ \mu \ddot{r} = -\frac{dU_{\text{eff}}(r)}{dr}. \]

With this our equation of motion will be:
\[ \ddot{r} = -\frac{4k}{\mu}(r - r_0). \]

Which if we take \( r = r_0 + \epsilon(t) \) gives us:
\[ \ddot{\epsilon} = -\frac{4k}{\mu}\epsilon. \]
So this leads to an oscillator frequency of:

\[ \omega = \sqrt{\frac{4k}{\mu}}. \]

\[ (2) \text{ Taylor 8.14} \]

![Figure 2: Plot of \( U_{\text{eff}} \) vs. \( r \) for \( U = \frac{k}{r} \) where \( \frac{k\mu}{l^2} = -10 \).](image)

The effective potential will be:

\[ U_{\text{eff}} = \frac{l^2}{2\mu r^2} + k r^n. \]

This is plotted in figure (1) and (2) for values \( n = 2, -1, \) and \( -3 \). (Note: since \( kn > 0 \) if \( n < 0 \) \( k < 0 \) as well). In order to find the circular orbit we set \( \ddot{r} = 0 \) which gives us \( \frac{dU_{\text{eff}}}{dr} = 0 \). For this we find:

\[ U'_{\text{eff}} = -\frac{l^2}{\mu r^3} + nk r^{n-1} = 0. \]

\[ (2) \]
Figure 3: Plot of $U_{eff}$ vs. $r$ for $U = \frac{k}{r^n}$ where $\frac{k}{\mu} = -0.1$.

Or:

$$\frac{l^2}{\mu r^3} = nkr^{n-1}. $$

Which leads to:

$$r_0 = \frac{n+2}{\sqrt{n\mu k}}. $$

To determine which are stable orbits we need a second derivative test. We will have a stable equilibrium when $U_{eff}''(r_0) > 0$.

So:

$$U_{eff}''(r_0) = \frac{3l^2}{\mu r_0^4} + n(n - 1)kr_0^{n-2}. $$

Now from above we have:

$$r_0 = \frac{n+2}{\sqrt{n\mu k}}. $$

Or:

$$r_0^{n+2} = \frac{l^2}{n\mu k}. $$
Or:

\[ r_0^{n-2} = \frac{l^2}{n\mu kr_0^4} \]

Where I divided both sides by \( r_0^4 \).

Inserting this into equation (3) we have:

\[ U''_{\text{eff}}(r_0) = \frac{3l^2}{\mu r_0^4} + n(n-1)k\frac{l^2}{n\mu kr_0^4} \]

Or:

\[ U''_{\text{eff}}(r_0) = \frac{3l^2}{\mu r_0^4} + (n-1)\frac{l^2}{\mu r_0^4} \]

Or finally:

\[ U''_{\text{eff}}(r_0) = \frac{(n+2)l^2}{\mu r_0^4} \]  \hspace{1cm} (4)

Where we see \( U''_{\text{eff}}(r_0) > 0 \) for all \( n > -2 \). This is in agreement with our plots. The plots of \( n = 2 \) and \( n = -1 \) have a stable minimum point where the plot of \( n = -3 \) does not.

We can find the period by finding the angular frequency from the Taylor expansion:

\[ U_{\text{eff}}(r) = U_{\text{eff}}(r_0) + U'_{\text{eff}}(r_0)(r - r_0) + \frac{1}{2}U''_{\text{eff}}(r_0)(r - r_0)^2 + ... \]

with equations (2) and (4) we have:

\[ U_{\text{eff}}(r) \approx U_{\text{eff}}(r_0) + \frac{1}{2} \frac{(n + 2)l^2}{\mu r_0^4} (r - r_0)^2. \]

Equation 8.29 in the text gives the equation of motion:

\[ \mu \ddot{r} = -\frac{dU_{\text{eff}}(r)}{dr}. \]

With this our equation of motion will be:

\[ \ddot{r} = -\frac{(n + 2)l^2}{\mu^2 r_0^4} (r - r_0). \]

Which if we take \( r = r_0 + \epsilon(t) \) gives us:

\[ \ddot{\epsilon} = -\frac{(n + 2)l^2}{\mu^2 r_0^4} \epsilon. \]

So this leads to a oscillator frequency of:

\[ \omega = \frac{\sqrt{n + 2l}}{\mu r_0^2}. \]
So we see:

\[
\tau_{osc} = \frac{2\pi \mu v_0^2}{\sqrt{n + 2l}} = \frac{\tau_{orb}}{\sqrt{n + 2}}.
\]

(Note: check the first sentence of the Professor’s notes in section 9.4.4 for the definition of \(\tau_{orb}\)).

Now if \(\sqrt{n + 2}\) is rational we have:

\[
\frac{\tau_{osc}}{\tau_{orb}} = \frac{A}{B},
\]

Where \(A\) and \(B\) are integers. This implies that the orbit will indeed repeat itself, therefore being a closed orbit.

(3) Taylor 8.17

If \(G = r \cdot p\) then:

\[
\dot{G} = \dot{r} \cdot p + r \cdot \dot{p}.
\]

We can write this a little different:

\[
\dot{G} = v \cdot p + r \cdot F.
\]

Integrating both sides over \(t\) we have:

\[
\int_0^t dt' \dot{G} = \int_0^t dt' [v \cdot p + r \cdot F] .
\]

Which can be rewritten as:

\[
G(t) - G(0) = 2 \int_0^t dt' \frac{1}{2} mv^2 + \int_0^t dt' F \cdot r.
\]

Dividing both sides by \(t\) we have:

\[
\frac{G(t) - G(0)}{t} = \frac{1}{t} \left[ 2 \int_0^t dt' T + \int_0^t dt' F \cdot r \right].
\]

Or:

\[
\frac{G(t) - G(0)}{t} = 2 < T > + < F \cdot r > .
\]

Then we have:

\[
0 = 2 < T > + < -nk^{-n-1}r \cdot r > .
\]

Or:

\[
0 = 2 < T > - n < k r^n > .
\]

And:

\[
0 = 2 < T > - n < U > .
\]

So we have:

\[
< T > = n < U > / 2.
\]
(4) Taylor 8.19

Here we have equations for $r_{\text{min}}$ and $r_{\text{max}}$ for an ellipse:

![Figure 4: Figure for 8.19.](image)

$$r_{\text{max}} = \frac{c}{1 - \epsilon},$$
and:

$$r_{\text{min}} = \frac{c}{1 + \epsilon}.$$  

Now remember $r_{\text{min}} = 6400 \text{ km} + 300 \text{ km} = 6700 \text{ km}$, and $r_{\text{max}} = 6400 \text{ km} + 3000 \text{ km} = 9400 \text{ km}$. Where 6400 km is the radius of the Earth.

So solving the above two equations for $\epsilon$ we get $\epsilon = 0.17$.

At this point it’s easy to plug back in for $c = 7802 \text{ km}$. Subtracting the radius of the Earth we get $d = 1400 \text{ km}$ which is the satellites distance to the surface of the Earth when it crosses the y-axis.

(5) Taylor 8.29

By the virial theorem we see that for a circular orbit under the influence of a power law potential $U = kr^n$:

$$< T > = -n < U > /2.$$  

Which since the gravitational potential has $n = 1$ our kinetic energy is:

$$T = -U/2.$$  

Where I dropped the average sign.

So our total energy would be:

$$E = -U_0/2 + U_0.$$  

Now if the sun lost half of it’s mass the potential energy would drop by a half, but the kinetic would not change. So we would have:

$$E = -U_0/2 + U_0/2 = 0.$$
We know that for $E = 0$ we have $\epsilon = 1$ and a parabolic orbit, so the earth would eventually leave the sun.

(6) Taylor 8.35

This is similar to example 8.6 in the text except run backwards. Initially we have $r_{max} = r_{max}$ for the initial circular orbit $R_i$ and the elliptical path which will transfer the craft between circular orbits.

Our equation will be:

$$\frac{c_1}{1 - \epsilon_1} = \frac{c_2}{1 - \epsilon_2}.$$  

However $\epsilon_1 = 0$ because it’s a circular orbit so we have:

$$c_1 = \frac{c_2}{1 - \epsilon_2}.$$  \hspace{1cm} (5)

Now the relation between $c$ constants is:

$$c_1 = \lambda^2 c_2.$$  \hspace{1cm} (6)

Again this is derived from:

$$v_1 = \lambda v_2,$$

and the fact that $v \propto l$ and $l^2 \propto c$.

Solving for $\epsilon_2$ we get:

$$\epsilon_2 = 1 - \lambda^2.$$  

Now we also want the $r_{min}$ of this ellipse to match with our final radius $R_f$. For this to happen we need:

$$c_3 = \frac{c_2}{1 + \epsilon_2}.$$  

Or:

$$R_f = \frac{\lambda^2 R_i}{1 + \epsilon_2}.$$
If we plug in our value for $\epsilon_2$ we get:

$$\lambda = \sqrt{\frac{2R_f}{R_i + R_f}} = \sqrt{\frac{2}{5}}.$$  

For the second thrust we want to switch from the elliptical orbit into a circular one. So we want to have the same $r_{\text{min}}$ and we will have a relationship between $c$ constants of:

$$c_3 = \lambda'^2 c_2.$$  

(7)

So in order to have the same $r_{\text{min}}$ we have:

$$c_3 = \frac{c_2}{1 + \epsilon_2}.$$  

(8)

Solving these for the thrust factor we get:

$$\lambda = \frac{1}{2 - \lambda^2} = \frac{\sqrt{5}}{8}.$$  

Similarly to the example we use angular momentum to solve for the overall gain in speed:

$$v_3 = \lambda \frac{v_{2\text{(per)}}}{v_{2\text{(apo)}}} v_1 = \frac{v_1}{2}.$$