Today's Lecture

Review

Solar System

Kepler's Laws

Equivalence Principle

"The effects of a uniform gravitational field are equivalent to a uniform acceleration".

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Equivalence Principle

"The effects of a uniform gravitational field are equivalent to a uniform acceleration".

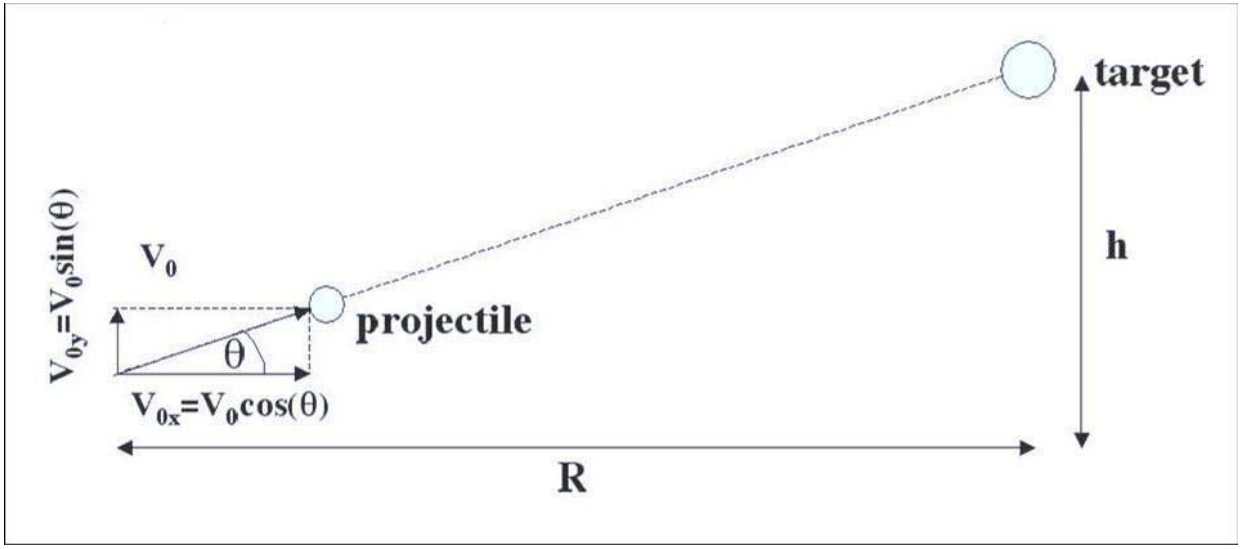
It is important to note here that a freely falling observer in a gravitational field would measure the same laws of physics as an observer would far from any massive objects (no gravitational field). The force between electrical charges is unchanged, between nucleons in the nucleus etc. Newton's second law still applies for forces outside of gravitational forces.

This is interesting in that all mention of mass and acceleration due to the gravitational field has disappeared!

We now define a reference frame as a coordinate system in which you describe the motion of the objects in that frame. In interstellar space a frame that has a uniform velocity is defined as an "inertial frame". In the absence of any acceleration, in this frame you will feel weightless. So in a very real sense a freely falling frame in a gravitational field is an "inertial frame"!

Example: Targeting

In a typical introductory physics class a projectile accurately aims at a target as shown in the figure. The moment the projectile leaves the barrel (or blow gun) the target is released (monkey lets go) and the student is asked to determine if the projectile (dart) hits the target (monkey).



Given what we now know about the equivalence principle, does the projectile hit the target or should the projectile have originally led the target by aiming low?

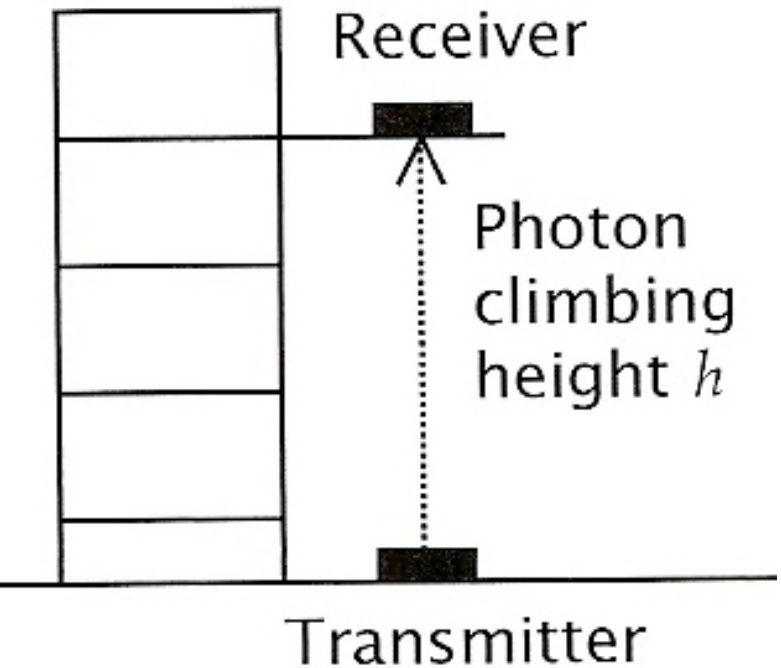
Doppler Shift

As long as the velocity of the observer, ***v***, is much smaller than the speed of light, ***c***, (for the case of sound waves much smaller than the speed of sound) then the expression that we derived is a very good approximation. Taking into account ***v*** may be in the opposite direction

$$f' = f(1 \pm v/c)$$

At this point you might ask why the shift in direction from the discussion of the equivalence principle. Soon, as we shall see, we can put this together with the equivalence principle to derive the gravitational redshift of light!

Gravitational Redshift of Light



In 1960 Pound and Rebka and later, 1965, with an improved version Pound and Snider measured the gravitational redshift of light using the Harvard tower, ***h* = 22.6*m***.

From the equivalence principle, at the instant the light is emitted from the transmitter, only a freely falling observer will measure the same value of ***f*** that was emitted by the transmitter.

But the stationary receiver is not free falling. During the time it takes light to travel to the top of the tower, ***t* = *h/c***, the receiver is traveling at a velocity, ***v* = *gt***, away from a free falling receiver.

Hence the measured frequency is:

$$f' = f(1 - v/c) = f(1 - gh/c^2)$$

$$f' = f\left(1 - \frac{9.8 \times 22.6}{(3 \times 10^8)^2}\right) = f(1 - 2.46 \times 10^{-15})$$

Gravity Slows Time

Now assume that the receiver at the top of the tower has an identical light source that emits light at $\mathbf{10^{15}}\boldsymbol{Hz}$. Then the light coming from the bottom of the tower will be losing approximately **2.5** beats every second when compared to the light source at the top.

$$f' = (10^{15} - 2.46)Hz$$

If both observers have clocks that tick $\mathbf{10^{15}}$ times per second in phase with the wave crests emitted by their own light source, then the observer at the top of the tower will be forced to conclude that the clock at the bottom of the tower must be losing **2.5** ticks every second when compared to a clock at the top! Hence clocks are running slower (albeit by a minuscule amount) at the bottom when compared to the top.

What would be the measured frequency if the transmitter were at the top of the tower and the receiver at the bottom?

In this case does the observer at the bottom think clocks are running slower or faster at the top than his clocks?

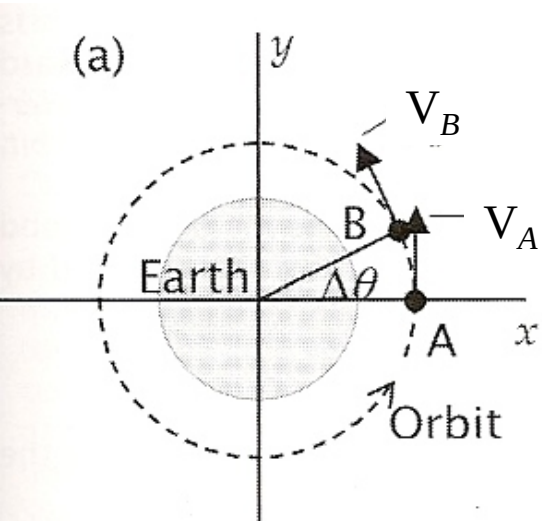
Gravity Slows Time

Given two identical clocks, if we place one of them (for a while) higher up in a gravitational field, then at a later time bring it back down to the lower clock we will find the clock at the higher elevation has gone faster!

This is not just an abstract point! GPS satellites must be synchronized to a high degree of accuracy with those on the ground for them to know where they are over the surface of the Earth. Due to their height above the surface they measure about $\mathbf{3\mu s}$ more per day than those on the ground. In $\mathbf{3\mu s}$ light travels $\mathbf{1000m}$! This means that a GPS satellite will lose $\mathbf{1000m}$ of accuracy each day they are in orbit due to the gravitational redshift. Hence a redshift correction must be built into a GPS satellite.

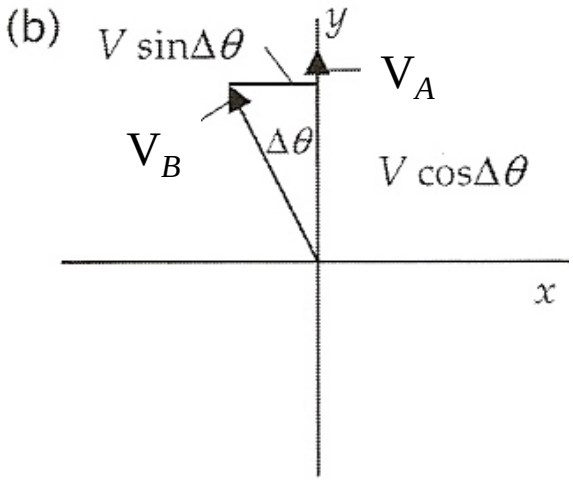
This effect is a direct result of the equivalence principle! Which lead to the conclusion that “To remove the effects (actually almost all) of gravity you have to be in a freely falling frame” which is as close as possible to an inertial frame in a gravitational field.

Satellites



Last time we found that the magnitude of the "radial" acceleration in a circular orbit is:

$$a = \lim_{\Delta t \to 0} \frac{V\Delta\theta}{\Delta t} = V \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = V\frac{d\theta}{dt}$$

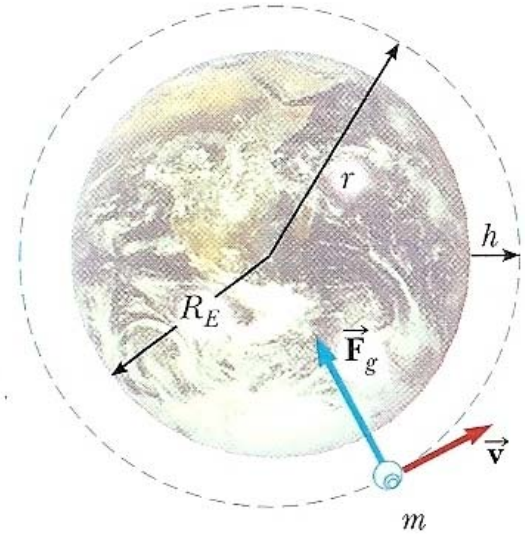


From our definition of angular measurement (radians), the arc length is $\Delta \boldsymbol{s} = \boldsymbol{r}\Delta\boldsymbol{\theta}$. Multiplying the numerator and denominator by the radius, $\boldsymbol{r}$, we find:

$$a = \frac{V}{r}\frac{dr\theta}{dt} = \frac{V}{r}\frac{ds}{dt} = \frac{V^2}{r}$$

We also found that this acceleration pointed toward the center of the circular orbit.

Satellites



From Newton's second law of motion and his law of gravity we can write:

$$\frac{GmM_E}{r^2} = ma_r = m\frac{V_{orb}^2}{r}$$

$$V_{orb}^2 = GM_E/r \rightarrow V_{orb} = \sqrt{GM_E/r}$$

Since the orbital velocity can be written as $\boldsymbol{V_{orb} = 2\pi r/T}$ we see that:

$$V_{orb}^2 = (2\pi r/T)^2 = GM_E/r$$

Simplifying:

$$\boxed{\frac{r^3}{T^2} = \frac{GM_E}{4\pi^2}}$$

When we know the radius of the orbit, ***r***, along with its period, ***T***, then to the accuracy that we know ***G***, we can determine the mass of the Earth, $\boldsymbol{M_E}$.

Satellite - Moon

Newton was well aware of Kepler's math modeling of the planetary orbits. For a circular orbit Newton knew that the radial acceleration was given by

$$\boldsymbol{a = v^2/r}.$$

For a planet moving in at a constant speed in a circular orbit the velocity is

$$\boldsymbol{v = 2\pi r/T}.$$

This means that in terms of the orbital period and radius, the acceleration is

$$\boldsymbol{a = 4\pi^2 r/T^2}.$$

According to Kepler's third law the quantity $\boldsymbol{r^3/T^2}$ is a constant. This means

$$\boldsymbol{r^2 a/4\pi^2 = r^3/T^2 = \text{const}}.$$

From this the acceleration of the planets in their orbits must obey:

$$a = \frac{const.}{r^2}$$

Satellite - Moon

Additionally Newton knew the distance to the Moon, $\boldsymbol{r_{Moon}}$ **= 3.84x10^5*km*** and its period ~27days (not 29.5 days?), $\boldsymbol{T_{Moon}}$ **= 2.36x10^{6}sec**.

This data enabled Newton determine the acceleration of the Moon in its orbit;

$$a_{Moon} = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 \times 3.84 \times 10^8}{(2.36 \times 10^6)^2} = .00272 m/s$$

The ratio of the Earth's radius to the distance to the Moon is;

$$\frac{r}{R_E} = \frac{3.84 \times 10^8}{6.38 \times 10^6} = 60.2$$

The ratio of the acceleration of the Moon to that at the surface of the Earth is:

$$\frac{a_{Moon}}{g} = \frac{.00272}{9.80} = 2.776 \times 10^{-4} = \frac{1}{3603}$$

Satellite - Moon

The ratio of the Earth's radius to the distance to the Moon is:

$$\frac{r}{R_E} = \frac{3.84 \times 10^8}{6.38 \times 10^6} = 60.2$$

While the ratio of the acceleration of the Moon to that at the surface of the Earth is:

$$\frac{a_{Moon}}{g} = \frac{.00272}{9.80} = 2.776 \times 10^{-4} = \frac{1}{3603}$$

From these results:

$$\boxed{\frac{a_{Moon}}{g} \simeq \left(\frac{R_E}{r}\right)^2}$$

To Newton, the fact that the acceleration of the Moon relative to the acceleration on Earth is equal to the square of the ratio of Earth's radius relative to the distance to the Moon was NOT a coincidence!

These results combined with Newton's 2nd law tell us that the force of gravity must satisfy an inverse square law for circular orbits.

Solar System

Nicholas Copernicus – To do a better job of predicting the location of the planets, he adopted the concept of the planets orbiting the Sun in circular orbits. He also discovered geometrical relations that allowed him to calculate the orbital periods of the planets. He was hesitant to publish his work due to the ridicule that he feared would follow. He finally did publish his work and received his first printed copy while he lay on his death bed. http://en.wikipedia.org/wiki/Nicolaus_Copernicus

Tycho Brahe – The greatest naked-eye astronomer of all time! He and his assistants complied observations of the planets to an accuracy better than one arc-minute, 1/60 of a degree. He was convinced that the planets must orbit the Sun, but never had a satisfactory explanation. Additionally he could not detect stellar parallax for the angular position of the stars across the diameter of the Earth's orbit. http://en.wikipedia.org/wiki/Tycho_Brahe

Solar System

Johannes Kepler – Like Copernicus, Kepler believed that the planets orbited the Sun, albeit in circular orbits. He was an assistant to Tycho. As Tycho lay on his death bed, he implored that Kepler make some sense of his observations so, "that it may not appear I have lived my life in vain".
http://en.wikipedia.org/wiki/Johannes_Kepler
Kepler was successful. Kepler's Three Laws of Planetary Motion are:

Kepler's Three Laws

1. The orbit of each planet around the Sun is an ellipse with the Sun at one focus.

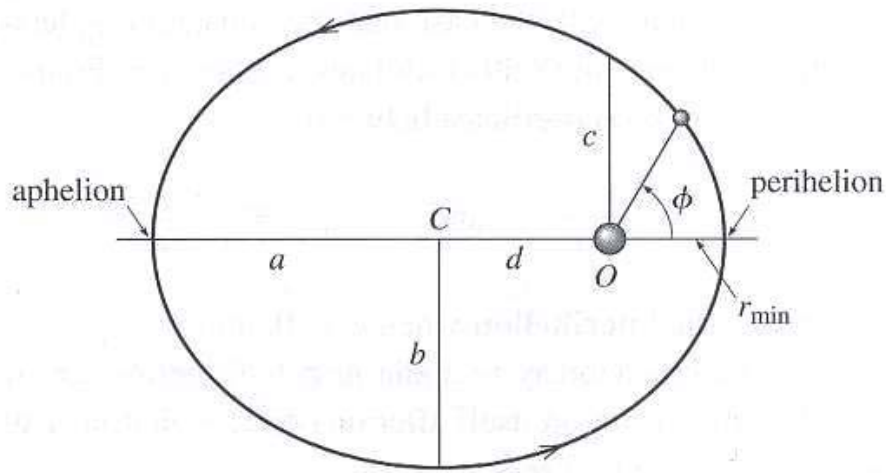
2. As a planet moves in its orbit, it sweeps out equal areas in equal times.

3. The more distant planets orbit the Sun at slower average speeds, obeying the precise relation $T^2 = r^3$, where T is the planet's orbital period measured in years and r is the average distance to the planet measured in astronomical units (AU) the average distance from the Earth to the Sun.

$$1\text{AU} = 1.496\text{x}10^{10} m = 150\text{x}10^{6}\ km$$

Kepler's 1st – Elliptical Orbits

Using Newton's second law combined with the inverse square law for gravity, it is relatively straightforward to determine all of the possible orbits. However, to go through the necessary math is beyond the scope of this class. Suffice it to say that the most general form for a closed orbit satisfies:



$$r(\phi) = \frac{r_o}{1 + \epsilon \cos\phi}$$

Here ***r*** is measured from one of the foci, the parameter, **ε**, is the eccentricity of the ellipse, and $\boldsymbol{\phi}$ is the angle as measured from the perihelion (point of closest approach).

The perihelion and aphelion distances correspond to $\boldsymbol{\phi} = \mathbf{0}$ and $\boldsymbol{\phi} = \boldsymbol{\pi}$ respectively.

$$\textit{perihelion}: \quad r_{\min} = r_o/(1+\epsilon)$$

$$\textit{aphelion}: \quad r_{\max} = r_o/(1-\epsilon)$$

Kepler's 1st – Elliptical Orbits

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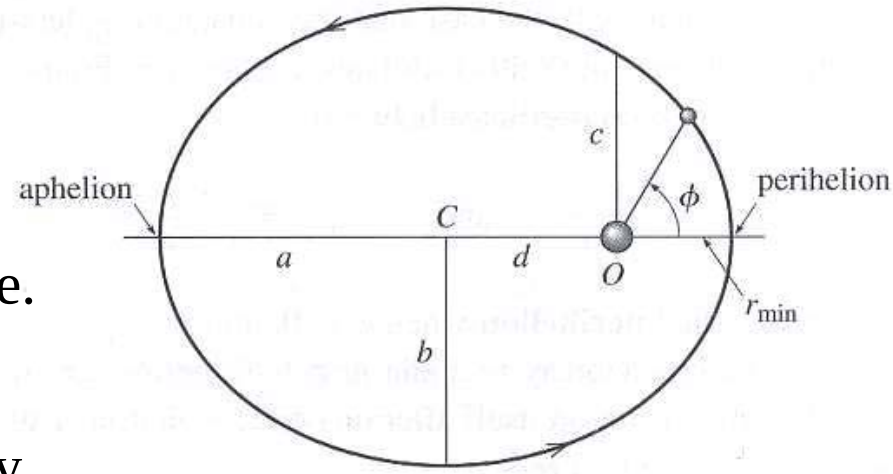


Note that when $\boldsymbol{\varepsilon = 0}$, $\boldsymbol{r = r_o}$, i.e. a circle.

The perihelion and aphelion distances correspond to $\boldsymbol{\phi = 0}$ and $\boldsymbol{\phi = \pi}$ respectively.

$$\textit{perihelion}: \quad r_{\min} = r_o/(1+\epsilon)$$

$$\textit{aphelion}: \quad r_{\max} = r_o/(1-\epsilon)$$

The standard equation for an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(d + r\cos\phi)^2}{a^2} + \frac{r^2 \sin^2\phi}{b^2} = 1$$

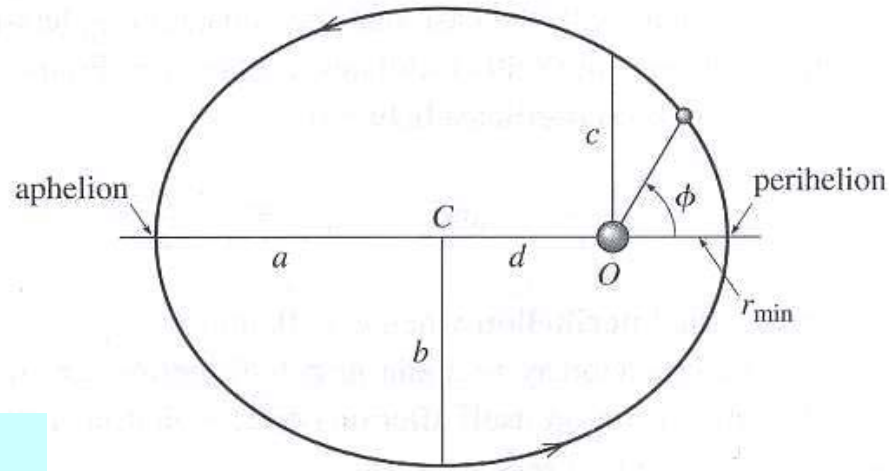
The semi-major axis is found from:

$$a = \frac{1}{2}(r_{\min} + r_{\max}) = r_o/(1-\epsilon^2)$$

Kepler's 1st – Elliptical Orbits

The most general form for a closed orbit satisfies:

$$r(\phi) = \frac{r_o}{1 + \epsilon \cos\phi}$$



The standard equation for an ellipse is:

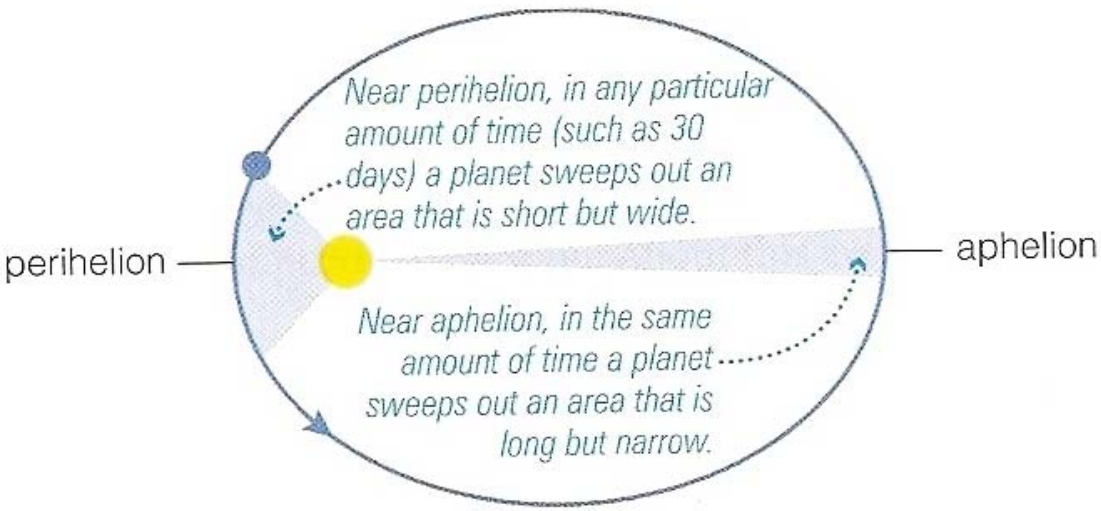
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(d + r\cos\phi)^2}{a^2} + \frac{r^2 \sin^2\phi}{b^2} = 1$$

The semi-minor axis is found from $\boldsymbol{dy/d\phi = d(r\sin\phi)/d\phi = 0}$, with the result:

$$\cos\phi_o = -\epsilon \text{ and } b = r(\phi_o)\sin\phi_o = r_o/\sqrt{1-\epsilon^2}$$

Finally ***d*** is given by: $d = a - r_{\min} = \epsilon a$

Kepler's 2nd – *dA/dt* = const



Kepler's second law states a planet sweeps out equal areas in equal time. This is equivalent to stating "the rate at which a planet sweeps out an area is constant" or ***dA/dt* = const**.

Consider the area being swept out when the planet is at aphelion. During a small time ***dt*** the planet moves a distance ***ds=vdt***. The differential area of the shaded triangle is ***dA=1/2 rvdt***. Since ***dA/dt*** is constant, then so must the product ***rv* = const.**

The angular momentum (we will discuss in more detail later) for a planet of mass ***m*** in a circular orbit is ***L* = *mvr***. The planet's mass is a constant, thus Kepler's second law is stating that the angular momentum is conserved. It turns out angular momentum is conserved for central forces including gravity!

Kepler's 3rd – $T^2=r^3$

Since the law of gravity satisfies an inverse square law, based on Newton's 3rd law of motion, he decided it must have the form: $\boldsymbol{F_{grav} = G\, m_1 m_2 / r^2}$. This form is consistent with the force of $\boldsymbol{m_1}$ on $\boldsymbol{m_2}$ being equal (and opposite) of the force of $\boldsymbol{m_2}$ on $\boldsymbol{m_1}$.

When we discussed the acceleration of an orbiting object in a circular orbit we noted the radial acceleration is $\boldsymbol{a_r = v^2/r}$.

For a circular orbit the orbital velocity is $\boldsymbol{C/T = 2\pi r/T}$. Using Newton's second law for a planet orbiting the Sun we find:

$$G\frac{m_{planet}M_\odot}{r^2} = m_{planet}\frac{v^2}{r} = m_{planet}\frac{4\pi^2 r}{T^2}$$

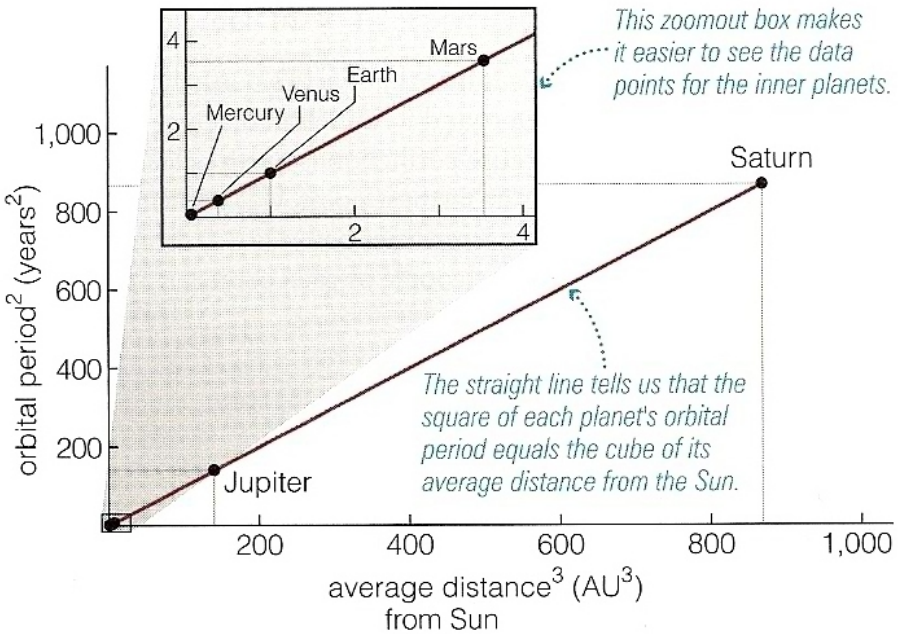
$$\frac{GM_\odot}{4\pi^2} = \frac{r^3}{T^2} = const$$

This is identical to that for a satellite orbiting the Earth, only now the "satellites" are the planets and the source of the gravitational field is the Sun!

Kepler's 3rd – $T^2 = r^3$

For a circular orbit we find:

$$\frac{GM_\odot}{4\pi^2} = \frac{r^3}{T^2} \rightarrow T^2 = \frac{4\pi^2}{GM_\odot} r^3$$



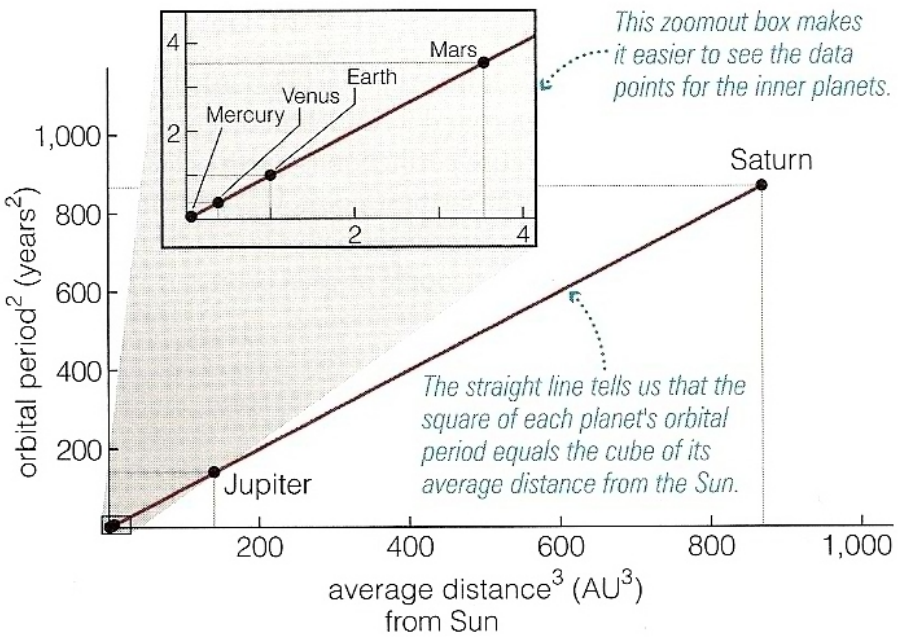
(i) This expression is independent of the planet's mass. (ii) This expression is in terms of SI units (***kg***, ***s***, ***m***), whereas Kepler's expression uses *AU*'s for the unit of distance and one Earth year for the unit of time. (iii) Knowing the average distance of the planet from the Sun and its orbital period enables us to determine the $GM_\odot$ product.

Note the amazingly good fit of Kepler's third law for all of the planets (out to Saturn)!

Kepler's 3rd – $T^2 = r^3$

For a circular orbit we find:

$$\frac{GM_\odot}{4\pi^2} = \frac{r^3}{T^2} \rightarrow T^2 = \frac{4\pi^2}{GM_\odot} r^3$$



As examples:

The distance from the Sun to Mars is $r_{\text{Mars}} = 1.52\text{AU}$.

Hence a Martian year is:

$T_{\text{Mars}} = (1.52)^{3/2} = 1.87\text{yrs}$.

The distance from the Sun to Jupiter is $r_{\text{Jup}} = 5.2\text{AU}$.

Hence a year on Jupiter is:

$T_{\text{Jup}} = (5.2)^{3/2} = 11.9\text{yrs}$.

Kepler's 3rd – $T^2=r^3$

The Sun also satisfies Kepler's laws. Jupiter has enough mass to cause the Sun to rotate in its own little circular orbit. The center of this circular orbit lies on the line between Jupiter and the Sun. We have already seen that the radial acceleration is given by:

$$a_r = \frac{v^2}{r} = \frac{4\pi^2 r^2}{T^2}\frac{1}{r} = \frac{4\pi^2 r}{T^2}$$

From Newton's third law (action-reaction) the radial accelerations of Jupiter and the Sun must satisfy:

$$M_\odot a_\odot = M_{Jup} a_{Jup} \rightarrow M_\odot \frac{4\pi^2 r_\odot}{T_\odot^2} = M_{Jup}\frac{4\pi^2 r_{Jup}}{T_{Jup}^2}$$

Since the center of the Sun's circular orbit lies on a line between Jupiter and the Sun, the orbital period of the Sun must equal that of Jupiter. Hence:

$$M_\odot r_\odot = M_{Jup} r_{Jup} \rightarrow r_\odot = \frac{M_{Jup}}{M_\odot} r_{Jup}$$

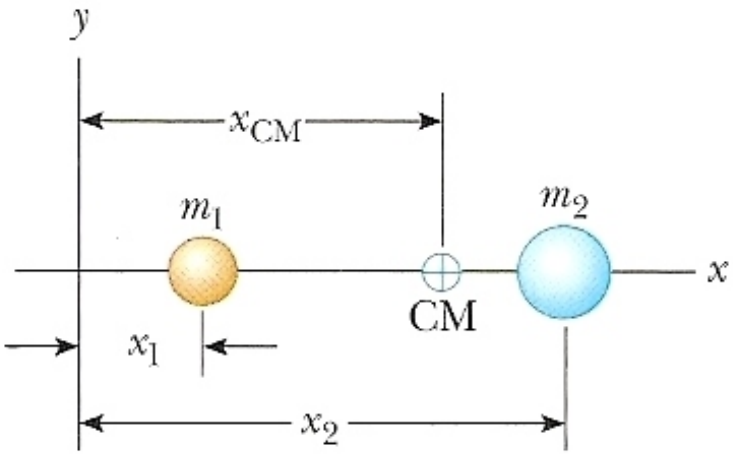
Kepler's 3rd – $T^2=r^3$

From the data: $M_\odot = 1.99 \times 10^{30} kg,\ M_{Jup} = 1.90 \times 10^{27} kg,\ \ r_{Jup} = 779 \times 10^9 m$

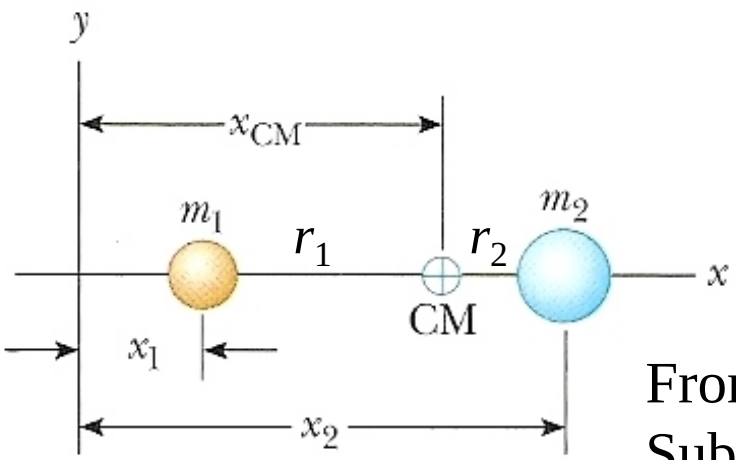
The radius of the Sun's circular orbit is:

$$r_\odot = \frac{M_{Jup}}{M_\odot} r_{Jup} = \frac{1.90 \times 10^{27}}{1.99 \times 10^{30}} \times 779 \times 10^9 = 744 \times 10^3 km$$

The radius of the Sun is **696x10^3*km***. This means that the Sun executes an orbit about a point just outside its surface with a period equal to that of Jupiter. This point is the center of mass for the Sun and Jupiter. Where the center of mass for two masses is shown in the figure.



Center of Mass



The center of mass for two objects is defined as:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

From the figure $\boldsymbol{x_1 = x_{CM} - r_1}$ and $\boldsymbol{x_2 = x_{CM} + r_2}$. Substituting these expressions into the equation for $\boldsymbol{x_{CM}}$ yields :

$$(m_1 + m_2)x_{CM} = m_1(x_{CM} - r_1) + m_2(x_{CM} + r_2)$$

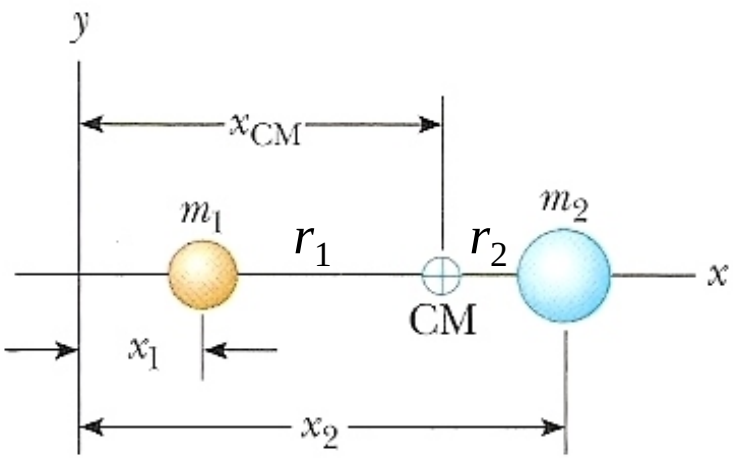
$$m_1 r_1 = m_2 r_2$$

This is exactly the relation that we used to find the center of the Sun's orbit.

$$M_\odot r_\odot = M_{Jup} r_{Jup}$$

As it turns out Jupiter is also rotating about the center of mass between the Sun and Jupiter.

Center of Mass



The center of mass for two objects can be found from:

$$m_1 r_1 = m_2 r_2$$

The mass of the Earth and Moon are $\boldsymbol{M_E = 5.97\text{x}10^{24} kg}$ and $\mathbf{M_{Moon} = 7.36\text{x}10^{22}}\boldsymbol{kg}$. The distance between their CM's is $\boldsymbol{R = 3.84\text{x}10^{5} km}$. Find the location of the CM relative to the Earth CM.

Define $\boldsymbol{r_1}$ as the distance to the CM from the center of the Earth. Then the distance from the CM to the center of the Moon is $\boldsymbol{R - r_1}$. From the data:

$$M_E r_1 = M_{Moon}(R - r_1) \rightarrow r_1 = \frac{M_{Moon}}{M_E + M_{Moon}} R$$

$$r_1 = \frac{7.36 \times 10^{22}}{5.97 \times 10^{24} + 7.36 \times 10^{22}} 3.84 \times 10^5$$

$$r_1 = 4680 km$$

Since the radius of the Earth is $\boldsymbol{R_E = 6360 km}$ the CM of the Earth–Moon system is inside the Earth.

Deflection of Light by the Sun

In simple Newtonian physics light travels in straight lines independent of gravity. However using the equivalence principle, light will travel in a straight line as observed by a freely falling observer.

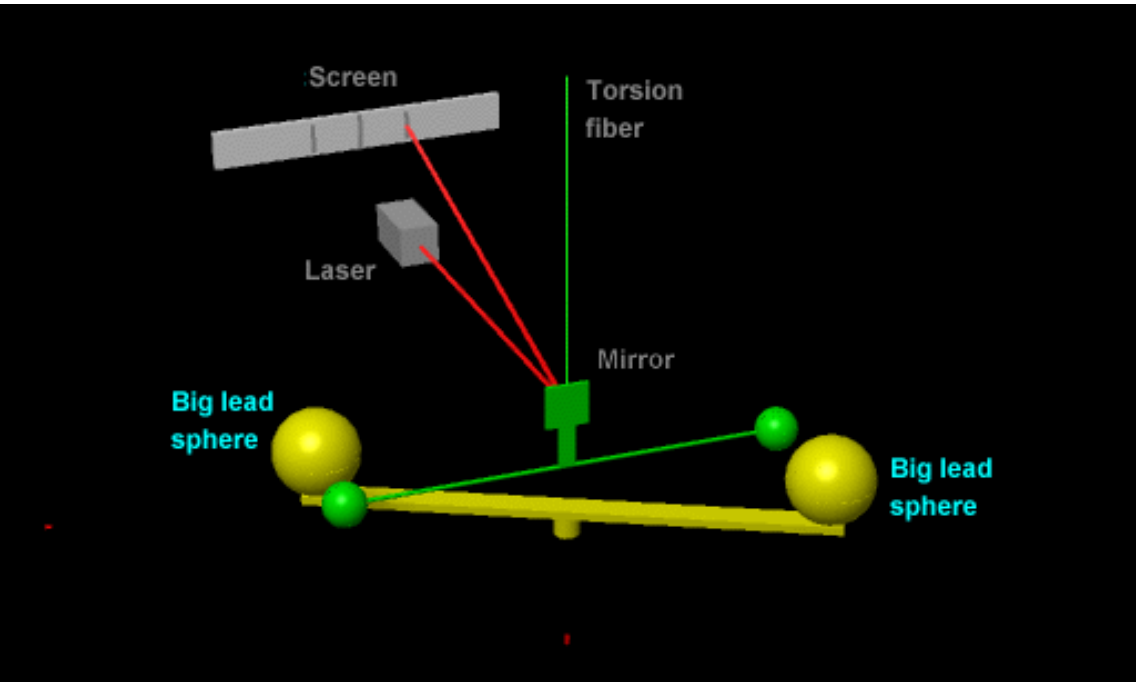
Assume that our free falling observer is at the radius of the Sun, $\boldsymbol{R}$, which has a mass $\boldsymbol{M}$. At this location the acceleration is $\boldsymbol{a = GM/R^2}$. Light traveling at a speed $\boldsymbol{c}$ will experience this acceleration for approximately $\boldsymbol{t = R/c}$ as it approaches the observer. The change in velocity is normal to its original direction. Hence the small angle of deflection is $\boldsymbol{\delta = at/c = GM/Rc^2}$. As the light leaves the observer it will deflect by the same amount. Hence the total angle of deflection will be:

$$2\delta = 2GM/Rc^2$$

This is an extremely small angular deviation from a straight line, approximately **.014** ***arcmin***. As it turns out the theory of General Relativity predicts exactly twice this value, $\boldsymbol{4GM/Rc^2}$. During an eclipse in 1919 Sir Arthur Eddington was able to confirm the general relativistic prediction.

What is The Value of G?

With Kepler's third law, astronomers can determine the value of $GM_{\odot}$. But to determine the mass of the Sun from the orbits of the planets, or the Earth from the orbits of satellites (not to mention the mass of other stars) or from the knowledge of ***g*** we need to know the value of ***G***!



In 1798 **Henry Cavendish** designed a more crude version of the experiment shown. Knowledge of the torsional constant of the fiber, enabled him to determine ***G*** which from his knowledge of ***g*** led to the mass of the Earth. His value for ***G***, ***G*** **= 6.74x10^{-11}** , within 1% of today's accepted value.

Astronomers routinely measure the $GM_{\odot}$ product to 8 significant figures. However physicists can only measure ***G*** to 4 significant figures!