PHYSICS 2D
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QUIZ 5

FALL QUARTER 2000
OCTOBER 27, 2000

Schrodinger equation : $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+U(x)\Psi(x,t)=i\hbar\frac{\partial\Psi}{\partial t}$; $\Psi(x,t)=\Psi(x)e^{-i\frac{E}{\hbar}t}$

Time – independent Schrodinger equation : $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+U(x)\Psi(x)=E\Psi(x)$

Square well (box): $E_n=\frac{\pi^2\hbar^2n^2}{2mL^2}$; $\Psi_n(x)=\sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$

Harmonic oscillator: $E_n=(n+\frac{1}{2})\hbar\omega$; $\Psi_0(x)=Ce^{-\frac{m\omega}{2\hbar}x^2}$; $E=\frac{p^2}{2m}+\frac{1}{2}\kappa x^2$; $\omega=\sqrt{\frac{\kappa}{m}}$

Operators: $[p_x]=\frac{\hbar}{i}\frac{\partial}{\partial x}$; $[x]=x$; $[H]=\frac{[p^2]}{2m}+[U(x)]$; eigenvalue: $[Q]\Psi=Q\Psi$

Observables: $<Q>=\int dx\Psi^*(x)[Q]\Psi(x)$; uncertainty: $\Delta Q=\sqrt{<Q^2>-<Q>^2}$

Free particle (U = 0) : $\Psi(x)=e^{i(kx-\omega t)}$; $E=\frac{\hbar^2k^2}{2m}=\hbar\omega$; $p=\hbar k$

Transmission probability : $T(E)=e^{-\frac{2}{\hbar}\sqrt{2m}\int_{x_1}^{x_2}dx\sqrt{U(x)-E}}$

Problem 1 (10 pts) Consider the following 5 wave functions:

$\Psi_1(x)=\cos x$

$\Psi_2(x)=e^{-2i\beta x}$

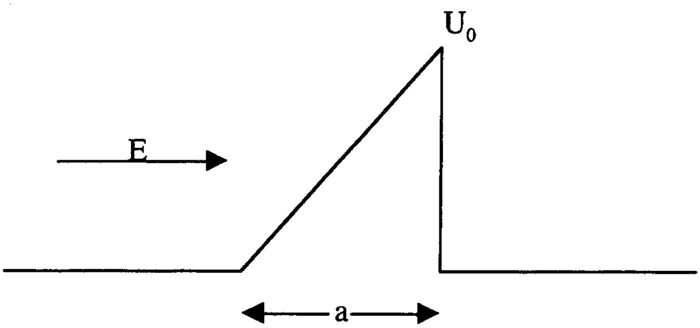
$\Psi_3(x)=e^{-\alpha x^2}$

$\Psi_4(x)=\cos kx+\sin kx$

$\Psi_5(x)=\cos kx+i\sin kx$

(a) Which are eigenfunctions of p? What are the eigenvalues?
(b) Which are eigenfunctions of p^2 ? What are the eigenvalues?
(c) Do you know any other operator for which Ψ_3 is an eigenfunction? If there is none, explain why. If there is, write down its form.
(d) What are the uncertainties in momentum (Δp) and in position (Δx) for Ψ_2?

Problem 2 (10 pts)

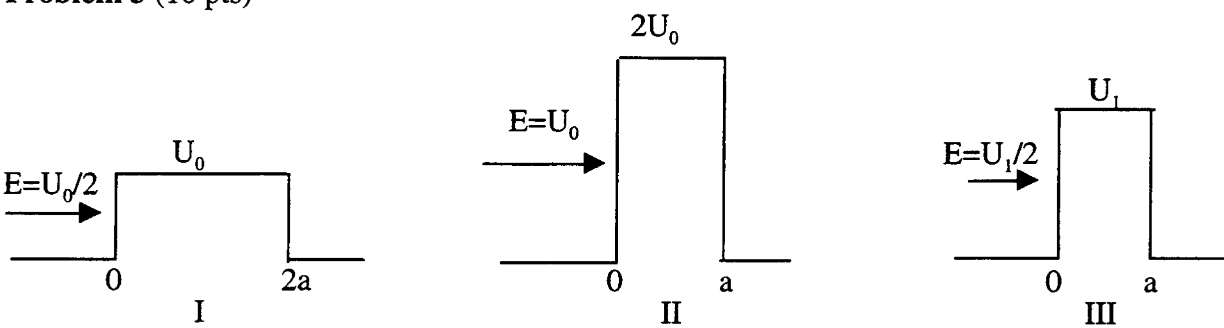


Electrons of energy $E=U_0/2$ are incident on the triangular barrier of height $U_0=128$eV shown in the figure. The width of the barrier is a=1A.

(a) Find $I=\int_{x_1}^{x_2} dx\sqrt{U(x)-E}$, where x_1 and x_2 are the classical turning points that enter in the expression for the transmission probability. I should be in units $\sqrt{eV}\times A$.

(b) If 10,000 electrons are incident on this barrier with this energy, estimate how many electrons will be transmitted and how many will be reflected.

Problem 3 (10 pts)



In the three barriers shown, electrons are incident with energy **half** the barrier height. The transmission probability for barrier I is $T_I=10^{-10}$, and $U_0=1.25$ eV. Barrier II is twice as high as barrier I. Note that barrier I is twice as wide as the other two barriers.
(a) Find the transmission probability for barrier II, T_{II}.
(b) Find the value of U_1 (in eV) for which the transmission in barrier III is the same as that in barrier I.
Hint: It is not necessary to find the value of a.

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Problem 1

$$\Psi_1 = \cos kx$$
$$\Psi_2 = e^{-2i\beta x}$$
$$\Psi_3 = e^{-\alpha x^2}$$
$$\Psi_4 = \cos kx + \sin kx$$
$$\Psi_5 = \cos kx + i \sin kx = e^{ikx}$$

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$p\Psi_1 = -\frac{\hbar k}{i}\sin kx$ no

$p\Psi_2 = -2\beta\hbar\,\Psi_2$ yes

$p\Psi_3 = \frac{\hbar}{i}\left(-\alpha e^{-\alpha x^2}\cdot 2x\right)$ no

$p\Psi_4 = \frac{\hbar k}{i}\left(-\sin kx + \cos kx\right)$ no

$p\Psi_5 = +\frac{\hbar}{i}k\left(-\sin kx + i\cos kx\right) = \hbar k\,\Psi_5$ yes

(a) $\boxed{\Psi_2,\ \Psi_5 \text{ are eigenfunctions of } p. \text{ Eigenvalues } (-2\beta\hbar),\ \hbar k \text{ respectively}}$

(b) Since $\frac{\partial^2 \sin kx}{\partial x^2} = -k^2 \sin kx$, and $\frac{\partial^2 \cos kx}{\partial x^2} = -k^2\cos kx$, $\Rightarrow$

$\boxed{\Psi_1,\ \Psi_2,\ \Psi_4,\ \Psi_5 \text{ are eigenfunctions of } p^2.}$

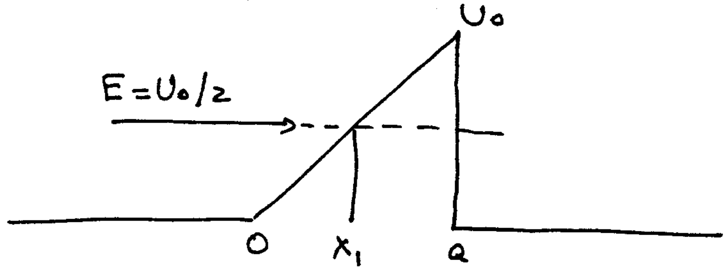
$\boxed{\Psi_3 \text{ is not,}}$ because $\frac{\partial^2 e^{-\alpha x^2}}{\partial x^2} = \frac{\partial}{\partial x}\left(-2\alpha x e^{-\alpha x^2}\right) = \left(4\alpha^2 x^2 - 2\alpha\right)e^{-\alpha x^2} \neq c\,\Psi_3$

(c) Ψ_3 has the form of the ground state wave function of harmonic oscillator

$\Rightarrow$ it is eigenfunction of $H = \frac{p^2}{2m} + \frac{1}{2}Kx^2$, if $\alpha = \frac{m\omega}{2\hbar} = \frac{m}{2\hbar}\sqrt{\frac{K}{m}}$

(d) Ψ_2 is eigenfunction of p $\Rightarrow$ $\boxed{\Delta p = 0 \Rightarrow \Delta x = \infty \quad (\text{since } \Delta x\,\Delta p \sim \hbar)}$

Problem 2



$\boxed{U(x) = U_0 \frac{x}{a}}$, since $U(0) = 0$, $U(a) = U_0$

$\boxed{x_2 = a}$, x_1 is point where $U(x_1) = E = \frac{U_0}{2} \Rightarrow \frac{U_0 x_1}{a} = \frac{U_0}{2} \Rightarrow$

$\boxed{x_1 = a/2}$

$$I = \int_{x_1}^{x_2} dx \sqrt{U(x) - E} = \int_{a/2}^{a} dx \sqrt{\frac{U_0 x}{a} - \frac{U_0}{2}} = \sqrt{U_0} \int_{a/2}^{a} dx \sqrt{\frac{x}{a} - \frac{1}{2}} =$$

$$= \frac{2}{3} \sqrt{U_0}\, a \left(\frac{x}{a} - \frac{1}{2}\right)^{3/2} \Bigg]_{a/2}^{a} = \frac{2}{3} \sqrt{U_0}\, a \cdot \left(\frac{1}{2}\right)^{3/2} = \frac{1}{3} \sqrt{\frac{U_0}{2}}\, a$$

$$\Rightarrow \boxed{I = \frac{1}{3} \sqrt{64\,eV} \cdot 1\,\text{Å} = \frac{8}{3} \sqrt{eV}\ \text{Å}}$$

(b) Transmission probability:

$$2 \frac{\sqrt{2m}}{\hbar} \cdot I = \frac{2\sqrt{2mc^2}}{\hbar c} \cdot I = 2 \times \frac{\sqrt{2 \times 0.511 \times 10^6}}{1973} \times \frac{8}{3} = 2.73$$

$$T = e^{-2.73} = 0.065$$

$\Rightarrow$ $\boxed{\text{650 electrons get transmitted, 9350 get reflected}}$

<u>Problem 3</u>

$$T = e^{-2\frac{\sqrt{2m}}{\hbar}\int_{x_1}^{x_2} dx\sqrt{U(x)-E}} \equiv e^{-2\frac{\sqrt{2m}}{\hbar} I} \equiv e^{-c\cdot I}$$

$$I = \int_{x_1}^{x_2} dx\sqrt{U(x)-E} = \sqrt{U-E}\cdot L \quad \text{for square barrier, with } L = x_2 - x_1$$

Barrier I:

$$I_I = \sqrt{U_0 - \frac{U_0}{2}}\cdot 2a = \frac{\sqrt{U_0}}{\sqrt{2}}\cdot 2a = \sqrt{2}\sqrt{U_0}\,a$$

Barrier II:

$$I_{II} = \sqrt{2U_0 - U_0}\cdot a = \sqrt{U_0}\cdot a$$

Hence:

$$T_{II} = e^{-c\sqrt{U_0}a}, \quad T_I = e^{-c\sqrt{2}\sqrt{U_0}a} = (T_{II})^{\sqrt{2}}$$

(a)

$$\Rightarrow \boxed{T_{II} = (T_I)^{\frac{1}{\sqrt{2}}} = (10^{-10})^{\frac{1}{\sqrt{2}}} = 10^{-7.07}}$$

(b) $$I_{III} = \sqrt{U_1 - \frac{U_1}{2}}\cdot a = \frac{\sqrt{U_1}}{\sqrt{2}}\,a$$

If $T_{III} = T_I \Rightarrow I_{III} = I_I \Rightarrow \dfrac{\sqrt{U_1}}{\sqrt{2}}\,a = \sqrt{2}\sqrt{U_0}\,a \Rightarrow$

$$\Rightarrow \sqrt{U_1} = 2\sqrt{U_0} \Rightarrow U_1 = 4U_0$$

if $U_0 = 1.25\,eV \Rightarrow \boxed{U_1 = 5\,eV}$