3 Notes on time domain circuit analysis

3.1 Background

The most common first order differential equation in circuit analysis, such as for a RC high-pass or low-pass filter, has constant coefficients and is of the form

\[ \tau \frac{dV(t)}{dt} + V(t) = F(t). \quad (3.1) \]

The full solution is the solution to the homogeneous equation, denied \( V_H(t) \) plus the solution to the inhomogeneous solution, denoted \( V_I(t) \). The homogeneous equation is the part with \( F(t) = 0 \), i.e.,

\[ \tau \frac{dV(t)}{dt} + V(t) = 0 \quad (3.2) \]

for which the homogeneous solution is, to within a constant,

\[ V_H(t) = e^{-t/\tau}. \quad (3.3) \]

The inhomogeneous solution satisfies the full equation

\[ \tau \frac{dV_I(t)}{dt} + V_I(t) = F(t) \quad (3.4) \]

and we take as an ansatz that the inhomogeneous solution is proportion to the homogeneous solution times a yet to be determined function of time \( \alpha(t) \), i.e.,

\[ V_I(t) = V_H(t) \alpha(t) = e^{-t/\tau} \alpha(t) \quad (3.5) \]

and

\[ \frac{dV_I(t)}{dt} = e^{-t/\tau} \left( \frac{d\alpha(t)}{dt} - \frac{1}{\tau} \alpha(t) \right). \quad (3.6) \]

If we plug these back into the original equation, we get

\[ \frac{d\alpha(t)}{dt} = e^{t/\tau} F(t) \quad (3.7) \]

or

\[ \alpha(t) = \int_0^t \frac{dt'}{\tau} e^{t'/\tau} F(t') + \text{Constant} \quad (3.8) \]
and, substituting back for $\alpha(t)$,

$$V_I(t) = e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} F(t') + \text{Constant} \quad (3.9)$$

$$= \int_0^t \frac{dt'}{\tau} e^{(t-t')/\tau} F(t') + \text{Constant.}$$

Thus

$$V(t) = V_I(t) + V_H(t) \quad (3.10)$$

$$= \int_0^t \frac{dt'}{\tau} e^{(t-t')/\tau} F(t') + V(0^-) e^{-t/\tau}$$

and we note that the integral for the driven response is call the ”convolution” integral. The above is just a very special case of the relation between the inoput and the driven response for a linear system, where the response of the linear system to a pulse (“delta function”) is given by $\Phi(t)$ and

$$V_I(t) = \int_0^t \frac{dt'}{\tau} \Phi(t-t') F(t'). \quad (3.11)$$

### 3.2 Example of step input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0$ for $t \geq 0$.

$$V(t) = \int_0^t \frac{dt'}{\tau} e^{(t-t')/\tau} V_0 + V(0^-) e^{-t/\tau} \quad (3.12)$$

$$= V_0 (1 - e^{-t/\tau}) + V(0^-) e^{-t/\tau}$$

### 3.3 Example of sine input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0 \sin(\omega_0 t)$ for $t \geq 0$. For focus only on the driven part and take $V(0^-) = 0$.

$$V(t) = \int_0^t \frac{dt'}{\tau} e^{(t-t')/\tau} \sin(\omega_0 t) \quad (3.13)$$

$$= V_0 e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$$
\[ V(t) = V_0 \frac{e^{-t/\tau}}{2i} \int_0^t dx \left( \frac{e^{(1+i\omega_0 \tau)x} - e^{(1-i\omega_0 \tau)x}}{2i} \right) \]

\[ = V_0 \frac{e^{-t/\tau}}{2i} \left( \frac{e^{t/\tau}e^{i\omega_0 t} - 1}{1 + i\omega_0 \tau} - \frac{e^{t/\tau}e^{-i\omega_0 t} - 1}{1 - i\omega_0 \tau} \right) \]

\[ = V_0 \frac{1}{2i} \left[ \left( \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{1 + i\omega_0 \tau} - 1 - i\omega_0 \tau \right) - e^{-t/\tau} \left( \frac{1}{1 + i\omega_0 \tau} - \frac{1}{1 - i\omega_0 \tau} \right) \right] \]

\[ = V_0 \frac{1}{2i} \left[ \frac{(1 - i\omega_0 \tau)e^{i\omega_0 t} - (1 + i\omega_0 \tau)e^{-i\omega_0 t} + 2i\omega_0 \tau e^{-1/\tau}}{1 + (\omega_0 \tau)^2} \right] \]

\[ = V_0 \frac{1}{2i} \left[ \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} - \omega_0 \tau \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} + \omega_0 \tau e^{-t/\tau} \right] \]

\[ = V_0 \left[ \frac{1}{1 + (\omega_0 \tau)^2} \sin(\omega_0 t) - \frac{\omega_0 \tau}{1 + (\omega_0 \tau)^2} \cos(\omega_0 t) + \frac{\omega_0 \tau}{1 + (\omega_0 \tau)^2} e^{-t/\tau} \right] \]

The exponential represents the transient from turning the signal on at \( t=0 \). At short times, i.e., \( t \ll \tau \),

\[ V(t) \approx \frac{\omega_0 \tau}{1 + (\omega_0 \tau)^2} \frac{t}{\tau} \quad (3.14) \]

and in steady state the solution reduces to

\[ V(t) = V_0 \frac{1}{\sqrt{1 + (\omega_0 \tau)^2}} \left[ \frac{1}{\sqrt{1 + (\omega_0 \tau)^2}} \sin(\omega_0 t) + \frac{\omega_0 \tau}{\sqrt{1 + (\omega_0 \tau)^2}} \cos(\omega_0 t) \right] \quad (3.15) \]

\[ = V_0 \frac{1}{\sqrt{1 + (\omega_0 \tau)^2}} \sin[\omega_0 t - \text{atan}(\omega_0 \tau)] \]