1 Notes on time domain circuit analysis

1.1 Background

The most common first-order differential equation in circuit analysis, such as for a RC high-pass or low-pass filter, has constant coefficients and is of the form

$$\tau \frac{dV(t)}{dt} + V(t) = F(t). \quad (1.1)$$

The full solution is the solution to the homogeneous equation, denoted $V_H(t)$ plus the solution to the inhomogeneous solution, denoted $V_I(t)$. The homogeneous equation is the part with $F(t) = 0$, i.e., the equation for the decay from $V(0)$ for which the homogeneous solution is, to within a constant,

$$V_H(t) = V(0)\Phi(t) \quad (1.2)$$

with

$$\Phi(t) = e^{-t/\tau}. \quad (1.3)$$

The inhomogeneous solution satisfies the full equation

$$\tau \frac{dV_I(t)}{dt} + V_I(t) = F(t) \quad (1.4)$$

and we take as an ansatz that the inhomogeneous solution is proportion to the homogeneous solution times a yet to be determined function of time $\alpha(t)$, i.e.,

$$V_I(t) = \Phi(t)\alpha(t) \quad (1.5)$$

and

$$\tau \frac{dV_I(t)}{dt} = \tau e^{-t/\tau} \frac{d\alpha(t)}{dt} - e^{-t/\tau} \frac{1}{\tau} \alpha(t). \quad (1.6)$$

If we plug these back into the original equation, we get

$$\frac{d\alpha(t)}{dt} = e^{t/\tau} F(t) \quad (1.7)$$

or

$$\alpha(t) = \int_0^t \frac{dt'}{\tau} e^{t'/\tau} F(t') + \text{Constant} \quad (1.8)$$
and, substituting back for $\alpha(t)$ gives

$$V_I(t) = e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} F(t') + \text{Constant} \quad (1.9)$$

$$= \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} F(t') + \text{Constant}.$$  

Thus

$$V(t) = V_I(t) + V_H(t) \quad (1.10)$$

$$= \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} F(t') + V(0^-)e^{-t/\tau}$$

The integral for the driven response is called the "convolution" integral. The above relation is a special case, for constant coefficients, between the input and the driven response for a linear system. In general, the response of such a linear system to a pulse ("delta function") is given by $\Phi(t)$ and

$$V_I(t) = \int_0^t \frac{dt'}{\tau} \Phi(t - t') F(t'). \quad (1.11)$$

1.2 Example of step input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0$ for $t \geq 0$.

$$V(t) = \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} V_0 + V(0^-)e^{-t/\tau} \quad (1.12)$$

$$= V_0 (1 - e^{-t/\tau}) + V(0^-)e^{-t/\tau}$$

1.3 Example of sine input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0 \sin(\omega_0 t)$ for $t \geq 0$. For focus only on the driven part and take $V(0^-) = 0$.

$$V(t) = \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} V_0 \sin(\omega_0 t') \quad (1.13)$$

$$= V_0 e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$$

$$= V_0 \frac{e^{-t/\tau}}{2i} \int_0^t dx \left( e^{i\omega_0 \tau x} - e^{-i\omega_0 \tau x} \right)$$
\[ V(t) = V_0 e^{-t/\tau} \int_0^{t/\tau} dx \left( e^{(1+i\omega_0\tau)x} - e^{(1-i\omega_0\tau)x} \right) \]

\[ = V_0 e^{-t/\tau} \left( \frac{e^{t/\tau} e^{i\omega_0 t} - 1}{1+i\omega_0\tau} - \frac{e^{t/\tau} e^{-i\omega_0 t} - 1}{1-i\omega_0\tau} \right) \]

\[ = V_0 \frac{1}{2i} \left[ \left( \frac{e^{i\omega_0 t}}{1+i\omega_0\tau} - \frac{e^{-i\omega_0 t}}{1-i\omega_0\tau} \right) - e^{-t/\tau} \left( \frac{1}{1+i\omega_0\tau} - \frac{1}{1-i\omega_0\tau} \right) \right] \]

\[ = V_0 \frac{1}{2i} \left[ \frac{(1-i\omega_0\tau)e^{i\omega_0 t} - (1+i\omega_0\tau)e^{-i\omega_0 t} + 2i\omega_0\tau e^{-1/\tau}}{1 + (\omega_0\tau)^2} \right] \]

\[ = V_0 \frac{1}{1 + (\omega_0\tau)^2} \left[ \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} - \omega_0\tau \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} + \omega_0\tau e^{-t/\tau} \right] \]

\[ = V_0 \left[ \frac{1}{1 + (\omega_0\tau)^2} \sin(\omega_0 t) - \frac{\omega_0\tau}{1 + (\omega_0\tau)^2} \cos(\omega_0 t) + \frac{\omega_0\tau}{1 + (\omega_0\tau)^2} e^{-t/\tau} \right] \]

The sine term represents a faithful transmission of the input and the cosine terms represents a phase suited version, while the exponential represents the transient from turning the signal on at t=0. With a bit more algebra, recalling that \( \sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b) \), we can simply this.

\[ V(t) = \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \left[ \frac{1}{\sqrt{1 + (\omega_0\tau)^2}} \sin(\omega_0 t) - \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} \cos(\omega_0 t) + \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} e^{-t/\tau} \right] \]

\[ = \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \left[ \sin(\omega_0 t - \text{atan}(\omega_0\tau)) + \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} e^{-t/\tau} \right] \quad (1.14) \]

At short times, i.e., \( t << \tau \),

\[ V(t) \approx \frac{\omega_0\tau}{1 + (\omega_0\tau)^2} \frac{t}{\tau} \quad (1.15) \]

and in steady state, i.e., \( t >> \tau \),

\[ V(t) = \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \sin(\omega_0 t - \text{atan}(\omega_0\tau)) \quad (1.16) \]