Lect 18 Doppler effect, 4-vectors

§ non-relativistic sound wave Doppler effect. (sound velocity ω)

∘ source moving velocity v_s.

receiver

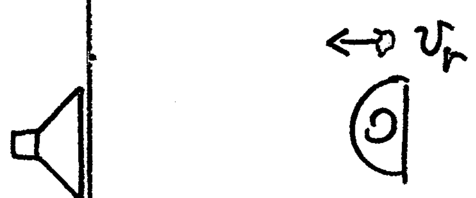
The two successive sound pulse generated at source at time interval $\Delta T = \frac{1}{\nu_0}$. The time interval for the reciever to receive them is

$$\Delta T - \frac{v_s \Delta T}{\omega} = \Delta T \left(1 - \frac{v_s}{\omega}\right)$$

$$\Rightarrow \nu_D = \frac{1}{\Delta T\left(1 - \frac{v_s}{\omega}\right)} = \frac{\nu_0}{1 - \frac{v_s}{\omega}}.$$

source

∘ receiver moving:



The time interval for the receiver to receive two successive pulse $\Delta T'$

$$\Delta T - \frac{\Delta T' v_r}{\omega} = \Delta T'$$

$$\Rightarrow \Delta T' = \Delta T \Big/ \left(1 + \frac{v_r}{\omega}\right) \Rightarrow \nu_D = \nu_0 \left(1 + \frac{v_r}{\omega}\right)$$

receiver

$\Rightarrow$ Combine together

$$\boxed{\nu_D = \frac{1 + \frac{v_r}{\omega}}{1 - \frac{v_s}{\omega}} \nu}$$

receiver/source motions are non-equivalent

source motion can has singularity.

§ Relativistic Doppler effect for light

- A light source flashes at $\tau_0 = \frac{1}{\nu_0}$ in its own rest frame. Now the light source is moving at v toward the receiver, which is at rest.

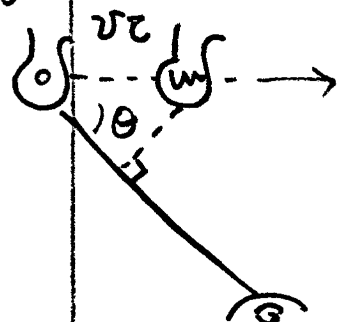
In the receiver's frame, the period of light pulse's is $\tau = \gamma \tau_0$ due to time-dilation. The period at the receiver

$$\tau - \frac{v}{c}\tau = \gamma \tau_0 (1 - \frac{v}{c})$$

$\Rightarrow$ receiver's frequency $f = \frac{1}{\tau} = \frac{1}{\tau_0}\sqrt{1-\frac{v^2}{c^2}}/(1-\frac{v}{c}) = f_0\sqrt{\frac{1+\frac{v}{c}}{1-v/c}}$.

- only the relative motion between the light source and the receiver's matters, because there's no the 3rd frame of media like in the case of sound wave.

☆ if the observer is not at the direction of motion of the light source



Similarly, the propagation distance between two successive pulses is shortened by $\frac{v\tau\cos\theta}{c}$

$$\Rightarrow \tau - \frac{v\tau\cos\theta}{c} = \gamma\tau_0 (1 - \frac{v}{c}\cos\theta),$$ transverse Doppler

- $\Rightarrow$ receiver's frequency

$$f = \frac{1}{\tau} = f_0 \frac{\sqrt{1-v^2/c^2}}{1-\frac{v}{c}\cos\theta}.$$

$\boxed{\text{even if } \theta = 90^\circ\text{, we have } f = f_0\sqrt{1-\frac{v^2}{c^2}}}$

Example: light source red $\lambda_0 = 650nm$, how large the speed is for a driver to see it is green $\lambda = 500nm$?

$$\frac{f}{f_0} = \sqrt{\frac{1+v/c}{1-v/c}} = \frac{\lambda_0}{\lambda} = \frac{650}{500} \Rightarrow \frac{v}{c} = \frac{1-(\lambda/\lambda_0)^2}{1+(\frac{\lambda}{\lambda_0})^2} = 0.26c.$$

§ 4-vector

we have
$$\begin{pmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & 0 \\ \hline & & 1 & \\ & 0 & & 1 \end{array}\right) \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} \leftarrow \text{4-vecter}$$

$$(cdt)^2 - [(dx)^2 + (dy)^2 + (dz)^2] = (cdt')^2 - [(dx')^2 + (dy')^2 + (dz')^2]$$

For a particle with the rest mass m_0, ~~~~ and velocity $\vec{u}$, we define its proper time $d\tau = \sqrt{1-u^2/c^2}\, dt$

$d\tau$: time interval measured in the particle rest frame.

dt: time interval measured in the lab frame

$d\tau$ and m_0 is independent of frame transformation.

Define $\left(\frac{m_0 c\, dt}{d\tau}, \frac{m_0\, dx}{d\tau}, \frac{m_0\, dy}{d\tau}, \frac{m_0\, dz}{d\tau}\right)$ ← multiply $m_0/d\tau$ to the space-time interval

$$= [\gamma m_0 c, \gamma m_0 u_x, \gamma m_0 u_y, \gamma m_0 u_z] = \left[\frac{E}{c}, P_x, P_y, P_z\right]$$

$\Rightarrow$ energy and mometum follow the same transformation of $(cdt/d\tau$ ...

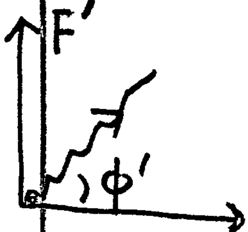
relativistic

$\Rightarrow \left(\frac{E}{c}\right)^2 - (\vec{P})^2$ is invariant, which equals to $(m_0 c)^2$.

we can also apply it to light. For light $E = \frac{hf}{c}$, and $p = \frac{hf}{c}$.

and $\left(\frac{E}{c}\right)^2 - p^2 \equiv 0$

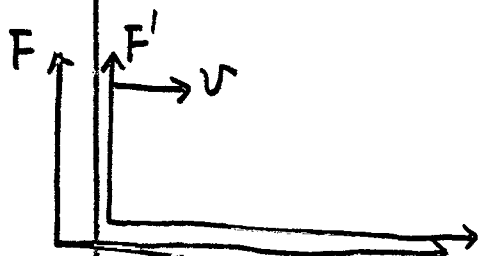
Rederive Doppler effect.



source rest frame

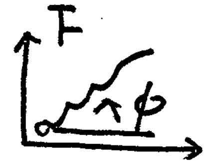
$$\left(\frac{hf_0}{c},\ \frac{hf_0}{c}\cos\phi',\ \frac{hf_0}{c}\sin\phi',\ 0\right).$$

in the lab frame (receive's)



in F frame, the energy-momentum vector

$$\left(\frac{hf}{c},\ \frac{hf}{c}\cos\phi,\ \frac{hf_0}{c}\sin\phi,\ 0\right)$$



$$\Rightarrow \underbrace{\begin{pmatrix} \frac{hf_0}{c} \\ hf_0\cos\phi' \end{pmatrix}}_{F'} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} \frac{hf}{c} \\ \frac{hf}{c}\cos\phi \end{pmatrix}}_{F}$$

$$\Rightarrow \frac{hf_0}{c} = \gamma\frac{hf}{c} - \beta\gamma\frac{hc}{c}\cos\phi$$

$$\Rightarrow f_0 = \gamma(1-\beta\cos\phi)f \Rightarrow \boxed{f = \frac{\sqrt{1-v^2/c^2}}{1-\frac{v}{c}\cos\phi} f_0}$$